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The Modified Simple Equation Method and the Exact Solutions for the sine-Gordon Equation and the Generalized Variable-Coefficient KdV-mKdV Equation

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Abstract

The modified simple equation method is used to construct the exact solutions for the sine-Gordon equation and the generalized variable-coefficient KdV-mKdV equation. Some exact solutions of the hyperbolic function form for the sine-Gordon equation and the generalized variable-coefficient KdV-mKdV equation are derived by the method. When taking special values of the parameters, the exact traveling wave solutions of the equations are derived from the exact solutions.

Keywords

The Modified Simple Equation Method, sine-Gordon Equation, Variable-Coefficient KdV-mKdV Equation, Exact Solutions

修正的简单方程法与sine-Godon方程和广义的变系数KdV-mKdV方程的精确解

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摘 要

本文用修正的简单方程法对sine-Gordon方程和广义的变系数KdV-mKdV方程进行求解，并给出了它们的行波解，当给参数取特殊值时，便可以得到相应的精确行波解。

关键词

修正的简单方程法，sine-Gordon方程，变系数KdV-mKdV方程，精确解

1. 引言

求解非线性发展方程是非线性理论的一个重要课题，迄今为止，学者们已经利用 Painlevé 分析法[1] [2]、Bäcklund 变换法[3]、辅助方程法[4]、G'/G—展开法[5]等对非线性发展方程进行了求解。受 G'\G—展开法的启发，近来学者们又提出“修正的简单方程法[6] [7]”并对 Fitzhugh-Nagumo 方程和 Sharma-Tasso-Olver 方程等许多常系数非线性发展方程进行精确求解。但目前为止未见用此方法来求解该方法无法直接运用的方程，如 sine-Gordon 方程[8] [9]以及变系数非线性发展方程。故本文旨在利用该方法给出 sine-Gordon 方程和广义的变系数 KdV-mKdV 方程[10]的精确行波解。

sine-Gordon 方程在非线性光学、等离子物理、固体物理等自然科学领域中有着广泛的应用；在流体力学和等离子体中，广义变系数非等谱 KdV-mKdV 方程用来刻画弱非线性长波在 KdV 介质中传播，因此这两个方程都是有重要物理背景的模型方程。因此寻找 sine-Gordon 方程和广义的变系数 KdV-mKdV 方程的精确解在理论和实际应用中有着至关重要的意义。

2. sine-Gordon 方程

下面考虑用修正的简单方程法求解 sine–Gordon 方程

$$u_{tt} - u_{xx} + \sin u = 0 \tag{2.1}$$

的问题。为此作变换

$$v = e^{iu}，\quad 则\ u = -i \ln v \tag{2.2}$$

则可将方程(2.1)转化为

$$2vv_{tt} - 2v_t^2 - 2vv_{xx} + 2v_x^2 + v^3 - v = 0 \tag{2.3}$$

再作行波变换

$$v(x,t) = v(\xi)，\quad \xi = kx + wt \tag{2.4}$$

并将(2.4)代入(2.3)得到

$$vv'' - v'^2 + \frac{1}{2\left(k^2 - w^2\right)}\left(v - v^3\right) = 0 \tag{2.5}$$

利用齐次平衡原则，平衡(2.5)中的最高阶导数项 v^3和非线性项 vv'' 和 v'^2，知 $n = 2$，故可设

$$v(\xi) = A_0 + A_1 \frac{\psi'(\xi)}{\psi(\xi)} + A_2 \left(\frac{\psi'(\xi)}{\psi(\xi)}\right)^2，\quad A_2 \neq 0 \tag{2.6}$$

把(2.6)代入(2.5)后比较ψ^0，ψ^{-1}，ψ^{-2}，ψ^{-3}，ψ^{-4}，ψ^{-5}，ψ^{-6}的系数可得

$$\frac{1}{2\left(k^2-w^2\right)}\left(A_0-A_0^3\right)=0 \tag{2.7}$$

$$\frac{1}{2\left(k^2-w^2\right)}\left(A_1-3A_0^2A_1\right)\psi'+A_0A_1\psi'''=0 \tag{2.8}$$

$$\begin{aligned}&\frac{1}{2\left(k^2-w^2\right)}\left(A_2-3A_0A_1^2-3A_0^2A_2\right)\psi'^2-3A_0A_1\psi'\psi''\\&+\left(2A_0A_2-A_1^2\right)\psi''^2+\left(A_1^2+2A_0A_2\right)\psi'\psi'''=0\end{aligned} \tag{2.9}$$

$$\begin{aligned}&\left(2A_0A_1-\frac{A_1^3}{2\left(k^2-w^2\right)}-\frac{3A_0A_1A_2}{k^2-w^2}\right)\psi'^2-\left(A_1^2+10A_0A_2\right)\psi'\psi''\\&-2A_1A_2\psi''^2+3A_1A_2\psi'\psi'''=0\end{aligned} \tag{2.10}$$

$$\left(A_1^2+6A_0A_2-\frac{3A_1^2A_2+3A_0A_2^2}{2\left(k^2-w^2\right)}\right)\psi'^2-5A_1A_2\psi'\psi''-2A_2^2\psi''^2+2A_2^2\psi'\psi'''=0 \tag{2.11}$$

$$\left(4A_1A_2-\frac{3A_1A_2^2}{2\left(k^2-w^2\right)}\right)\psi'-2A_2^2\psi''=0 \tag{2.12}$$

$$\left(2A_2^2-\frac{A_2^3}{2\left(k^2-w^2\right)}\right)\left(\psi'\right)^6=0 \tag{2.13}$$

由(2.7)可得：$A_0=0,1,-1$；由(2.13)可得：$A_2=4\left(k^2-w^2\right)$

1) 当$A_0=0$时，(2.5)无(2.6)形式的解

2) 当$A_0=1$时，(2.8)~(2.12)化为常微分方程组

$$-2\lambda\psi'+\psi'''=0 \tag{2.14}$$

$$\left(-4-3\lambda A_1^2\right)\psi'^2-3A_1\psi'\psi''+\left(\frac{4}{\lambda}-A_1^2\right)\psi''^2+\left(A_1^2+\frac{4}{\lambda}\right)\psi'\psi'''=0 \tag{2.15}$$

$$-\left(10A_1+\lambda A_1^3\right)\psi'^2-\left(A_1^2+\frac{20}{\lambda}\right)\psi'\psi''-\frac{4A_1}{\lambda}\psi''^2+\frac{6A_1}{\lambda}\psi'\psi'''=0 \tag{2.16}$$

$$-5A_1^2\psi'^2-\frac{10A_1}{\lambda}\psi'\psi''-\frac{8}{\lambda^2}\psi''^2+\frac{8}{\lambda^2}\psi'\psi'''=0 \tag{2.17}$$

$$A_1\psi'+\frac{2}{\lambda}\psi''=0 \tag{2.18}$$

其中：$\lambda=\frac{1}{2\left(k^2-w^2\right)}$，$A_2=\frac{2}{\lambda}$

由(2.18)可得

$$\psi''=-\frac{\lambda A_1}{2}\psi'，\text{则}\ \psi'''=-\frac{\lambda A_1}{2}\psi'' \tag{2.19}$$

把(2.19)代入(2.14)可得

$$\psi'' = -\frac{4}{A_1}\psi', \quad 则 \psi''' = -\frac{4}{A_1}\psi'' \tag{2.20}$$

把(2.20)代入(2.15)可得

$$\psi'' = m_1\psi', \quad 则 \psi''' = m_1\psi'', \quad m_1 = \frac{-8\lambda A_1^2 - 3\lambda^2 A_1^4 + 64}{4\lambda A_1^3 + 16A_1} \tag{2.21}$$

把(2.21)代入(2.16)可得

$$\psi'' = n_1\psi', \quad 则 \psi''' = n_1\psi'' \tag{2.22}$$

$$n_1 = \frac{10\lambda A_1 + \lambda^2 A_1^3 + m\lambda A_1^2 + 20m + 4m^2 A_1}{6mA_1} \tag{2.23}$$

把(2.22)代入(2.17)可得

$$\frac{\psi''}{\psi'} = l_1, \quad l_1 = \frac{5\lambda^2 A_1^2 + 10n\lambda A_1 + 8n^2}{8n} \tag{2.24}$$

对ξ积分两次有

$$\psi(\xi) = c_2 + \frac{c_1 e^{l_1\xi}}{l_1} \tag{2.25}$$

$$v(\xi) = 1 + l_1 A_1 \cdot \frac{c_1 e^{l_1\xi}}{l_1 c_2 + c_1 e^{l_1\xi}} + \frac{2l_1^2}{\lambda} \cdot \left(\frac{c_1 e^{l_1\xi}}{l_1 c_2 + c_1 e^{l_1\xi}}\right)^2 \tag{2.26}$$

因此

$$v(x,t) = 1 + l_1 A_1 \cdot \frac{c_1 e^{l_1(kx+wt)}}{l_1 c_2 + c_1 e^{l_1(kx+wt)}} + \frac{2l_1^2}{\lambda} \cdot \left(\frac{c_1 e^{l_1(kx+wt)}}{l_1 c_2 + c_1 e^{l_1(kx+wt)}}\right)^2 \tag{2.27}$$

从而

$$u(x,t) = -i\ln\left[1 + l_1 A_1 \cdot \frac{c_1 e^{l_1(kx+wt)}}{l_1 c_2 + c_1 e^{l_1(kx+wt)}} + \frac{2l_1^2}{\lambda} \cdot \left(\frac{c_1 e^{l_1(kx+wt)}}{l_1 c_2 + c_1 e^{l_1(kx+wt)}}\right)^2\right] \tag{2.28}$$

取$c_1 = 1$，$c_2 = \pm\frac{1}{l_1}$，得到 sine-Gordon 方程的精确孤波解

$$u_1(x,t) = -i\ln\left\{1 + \frac{l_1 A_1}{2}\left[1 + \tanh\frac{l_1}{2}(kx+wt)\right] + \frac{l_1^2}{2\lambda}\left[1 + \tanh\frac{l_1}{2}(kx+wt)\right]^2\right\} \tag{2.29}$$

$$u_2(x,t) = -i\ln\left\{1 + \frac{l_1 A_1}{2}\left[1 + \coth\frac{l_1}{2}(kx+wt)\right] + \frac{l_1^2}{2\lambda}\left[1 + \coth\frac{l_1}{2}(kx+wt)\right]^2\right\} \tag{2.30}$$

3) 当$A_0 = -1$时，(2.8)~(2.12)化为常微分方程组

$$2\lambda\psi' + \psi''' = 0 \tag{2.31}$$

$$\left(-4 + 3\lambda A_1^2\right)\psi'^2 + 3A_1\psi'\psi'' - \left(\frac{4}{\lambda} + A_1^2\right)\psi''^2 + \left(A_1^2 - \frac{4}{\lambda}\right)\psi'\psi''' = 0 \tag{2.32}$$

$$\left(10A_1 - \lambda A_1^3\right)\psi'^2 - \left(A_1^2 - \frac{20}{\lambda}\right)\psi'\psi'' - \frac{4A_1}{\lambda}\psi''^2 + \frac{6A_1}{\lambda}\psi'\psi''' = 0 \tag{2.33}$$

$$-5A_1^2\psi'^2-\frac{10A_1}{\lambda}\psi'\psi''-\frac{8}{\lambda^2}\psi''^2+\frac{8}{\lambda^2}\psi'\psi'''=0 \tag{2.34}$$

$$A_1\psi'+\frac{2}{\lambda}\psi''=0 \tag{2.35}$$

完全类似于方程组(2.14)~(2.18)的求解过程,我们得到

$$\psi(\xi)=c_2+\frac{c_1e^{l\xi}}{l}，\quad l_2=\frac{5\lambda^2A_1^2+10n\lambda A_1+8n_2^2}{8n_2} \tag{2.36}$$

$$n_2=\frac{4m_2^2A_1-20m_2+m_2\lambda A_1^2+\lambda^2A_1^3-8\lambda A_1}{6m_2A_1}，\quad m_2=\frac{-8\lambda A_1^2+3\lambda^2A_1^4-64}{16A_1-4\lambda A_1^3} \tag{2.37}$$

$$v(\xi)=-1+l_2A_1\cdot\frac{c_1e^{l_2\xi}}{l_2c_2+c_1e^{l_1\xi}}+\frac{2l_2^2}{\lambda}\cdot\left(\frac{c_1e^{l_2\xi}}{l_2c_2+c_1e^{l_1\xi}}\right)^2 \tag{2.38}$$

因此

$$v(x,t)=-1+l_2A_1\cdot\frac{c_1e^{l_2(kx+wt)}}{l_2c_2+c_1e^{l_2(kx+wt)}}+\frac{2l_2^2}{\lambda}\cdot\left(\frac{c_1e^{l_2(kx+wt)}}{l_2c_2+c_1e^{l_2(kx+wt)}}\right)^2 \tag{2.39}$$

从而

$$u(x,t)=-i\ln\left[-1+l_2A_1\cdot\frac{c_1e^{l_2(kx+wt)}}{l_2c_2+c_1e^{l_2(kx+wt)}}+\frac{2l_2^2}{\lambda}\cdot\left(\frac{c_1e^{l_2(kx+wt)}}{l_2c_2+c_1e^{l_2(kx+wt)}}\right)^2\right] \tag{2.40}$$

取 $c_1=1$，$c_2=\pm\frac{1}{l_2}$，得到 sine-Gordon 方程的精确孤波解

$$u_1(x,t)=-i\ln\left\{-1+\frac{l_2A_1}{2}\left[1+\tanh\frac{l_2}{2}(kx+wt)\right]+\frac{l_2^2}{2\lambda}\left[1+\tanh\frac{l_2}{2}(kx+wt)\right]^2\right\} \tag{2.41}$$

$$u_2(x,t)=-i\ln\left\{-1+\frac{l_2A_1}{2}\left[1+\coth\frac{l_2}{2}(kx+wt)\right]+\frac{l_2^2}{2\lambda}\left[1+\coth\frac{l_2}{2}(kx+wt)\right]^2\right\} \tag{2.42}$$

3. 广义变系数 KdV-mKdV 方程

广义变系数 KdV-mKdV 方程的形式为

$$u_t-6h_0(t)uu_x-6h_1(t)u^2u_x+h_2(t)u_{xxx}-h_3(t)u_x+h_4(t)(Au+xu_x)=0 \tag{3.1}$$

利用齐次平衡原则，平衡(3.1)中的最高阶导数项和最高幂次的非线性项，得 $n=1$。于是，(3.1)有如下形式的形式解

$$u(x,t)=u(\xi)=A_0(t)+A_1(t)\frac{\psi'(\xi)}{\psi(\xi)}，\quad A_1(t)\neq0，\quad \xi=p(t)x+q(t) \tag{3.2}$$

把(3.2)代入(3.1)，比较 ψ^0，ψ^{-1}，$\psi^{-1}x$，ψ^{-2}，$\psi^{-2}x$，ψ^{-3}，ψ^{-4} 的系数得

$$AA_0h_4+A_0'=0 \tag{3.3}$$

$$(AA_1h_4+A_1')\psi'-(6pA_0A_1h_0+6pA_0^2A_1h_1+pA_1h_3-A_1q')\psi''+p^3A_1h_2\psi^{(4)}=0 \tag{3.4}$$

$$(pA_1h_4+A_1p')\psi''=0 \tag{3.5}$$

$$\begin{aligned}&\left(6pA_0A_1h_0+6pA_0^2A_1h_1+pA_1h_3-A_1q'\right)\psi'^2-\left(6pA_1^2h_0+12pA_0A_1^2h_1\right)\psi'\psi''\\&-3p^3A_1h_2\psi''^2-4p^3A_1h_2\psi'\psi'''=0\end{aligned}\tag{3.6}$$

$$\left(pA_1h_4+A_1p'\right)\psi'^2=0\tag{3.7}$$

$$\left(6pA_1^2h_0+12pA_0A_1^2h_1\right)\psi'-\left(6pA_1^3h_1-12p^3A_1h_2\right)\psi''=0\tag{3.8}$$

$$\left(6pA_1^3h_1-6p^3A_1h_2\right)\left(\psi'\right)^4=0\tag{3.9}$$

由(3.3)可得：$A_0=c_1(t)\mathrm{e}^{-\int Ah_4\mathrm{d}t}$，由(3.5)和(3.7)可得：$p(t)=c_2(t)\mathrm{e}^{-\int h_4(t)\mathrm{d}t}$

由(3.9)可得：$A_1(t)=\pm p(t)\sqrt{\dfrac{h_2(t)}{h_1(t)}}=\pm c_2(t)\sqrt{\dfrac{h_2(t)}{h_1(t)}}\mathrm{e}^{-\int h_4(t)\mathrm{d}t}$

由(3.8)可得：

$$\psi''=\frac{A_1h_0+2A_0A_1h_1}{A_1^2h_1-2p^2h_2}\psi'\tag{3.10}$$

把(3.10)代入(3.6)得到：

$$\psi'''=\frac{m(t)}{n(t)}\psi''\tag{3.11}$$

其中：$m=m(t)=\left(6pA_0A_1h_0+6pA_0^2A_1h_1+pA_1h_3-A_1q'\right)\left(A_1^2h_1-2p^2h_2\right)^2$

$$-\left(6pA_1^2h_0+12pA_0A_1^2h_1\right)\left(A_1h_0+2A_0A_1h_1\right)\left(A_1^2h_1-2p^2h_2\right)$$

$$-3p^3A_1h_2\left(A_1h_0+2A_0A_1h_1\right)^2$$

$n(t)=4p^3A_1h_2\left(A_1^2h_1-2p^2h_2\right)^2$

把(3.11)代入(3.4)可得：

$$\frac{\psi''}{\psi'}=\frac{E(t)}{F(t)}\tag{3.12}$$

其中：$E(t)=nAA_1h_4+nA_1'$，$F(t)=6npA_0A_1h_0+6npA_0^2A_1h_1+npA_1h_3-nA_1q'-mp^3A_1h_2$

给(3.12)的两端对ξ积分两次可得：

$$\psi(\xi)=c_4(t)+\frac{c_3(t)F(t)}{E(t)}\mathrm{e}^{\frac{E(t)}{F(t)}\xi}\tag{3.13}$$

从而：

$$u(\xi)=c_1(t)\mathrm{e}^{\int-Ah_4\mathrm{d}t}\pm c_2(t)\sqrt{\frac{h_2(t)}{h_1(t)}}\mathrm{e}^{\int-Ah_4\mathrm{d}t}\cdot\frac{c_3(t)\mathrm{e}^{\frac{E(t)}{F(t)}\xi}}{c_4(t)+c_3(t)\mathrm{e}^{\frac{E(t)}{F(t)}\xi}}\tag{3.14}$$

于是

$$u(x,t)=c_1(t)\mathrm{e}^{\int-Ah_4\mathrm{d}t}\pm c_2(t)\sqrt{\frac{h_2(t)}{h_1(t)}}\mathrm{e}^{\int-Ah_4(t)\mathrm{d}t}\cdot\frac{c_3(t)\mathrm{e}^{\frac{E(t)}{F(t)}\left(xc_2(t)\mathrm{e}^{\int-h_4(t)\mathrm{d}t}+q(t)\right)}}{c_4(t)+\dfrac{c_3(t)F(t)}{E(t)}\mathrm{e}^{\frac{E(t)}{F(t)}\left(xc_2(t)\mathrm{e}^{\int-h_4(t)\mathrm{d}t}+q(t)\right)}}\tag{3.15}$$

若令 $c_3(t)=\dfrac{E(t)}{F(t)}$， $c_4(t)=\pm1$，得到广义变系数 KdV-mKdV 方程的精确行波解如下

$$u_1(x,t)=c_1e^{\int -Ah_4dt}\pm c_2\sqrt{\frac{h_2(t)}{h_1(t)}}\frac{E(t)}{2F(t)}e^{\int -Ah_4dt}\left[1+\tanh\frac{E(t)}{2F(t)}\left(x\cdot c_2e^{-\int h_4(t)dt}+q(t)\right)\right] \tag{3.16}$$

$$u_2(x,t)=c_1e^{\int -Ah_4dt}\pm c_2\sqrt{\frac{h_2(t)}{h_1(t)}}\frac{E(t)}{2F(t)}e^{\int -Ah_4dt}\left[1+\coth\frac{E(t)}{2F(t)}\left(x\cdot c_2e^{-\int h_4(t)dt}+q(t)\right)\right] \tag{3.17}$$

本文把修正的简单方程法推广到求解 sine-Gordon 方程和变系数方程中，得到 sine-Gordon 方程和广义变系数 KdV-mKdV 方程的精确孤波解。鉴于修正的简单方程法是受 G'/G 展开法的启发产生的一种新方法，因此，如果能够类似于 G'/G-展开法的推广将此方法进行推广就有可能给出非线性发展方程的其他类型的精确解。

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