Advances in Applied Mathematics 应用数学进展, **2018, 7(6), 714-722**
Published Online June 2018 in Hans. http://www.hanspub.org/journal/aam
https://doi.org/10.12677/aam.2018.76086

Exact Solution of a Type of Fractional Order Generalized KdV Equation

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Received: May 30th, 2018; accepted: Jun. 15th, 2018; published: Jun. 22nd, 2018

Abstract

Fractional sub-equation method is an effective method to explore the exact solution of fractional partial differential equations. In this paper, we use this method to obtain new exact solutions of a type of fractional generalized KdV equation. Moreover, we show the graphs of these exact solutions. Such solutions may provide potential support for the wide applications in mathematics, physics and related fields.

Keywords

Fractional Sub-Equation Method, Fractional Generalized KdV Equation, Exact Solutions

一类分数阶广义KdV方程精确解

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收稿日期：2018年5月30日；录用日期：2018年6月15日；发布日期：2018年6月22日

摘　要

分数阶子方程法是探寻分数阶偏微分方程精确解的一种行之有效的方法。本文利用该方法获得了一类分数阶广义KdV方程的精确解，并描绘精确解的图像。这些解能够对该方程在数学、物理等领域的广泛应用提供潜在的理论支持。

文章引用: 李国放. 一类分数阶广义 KdV 方程精确解[J]. 应用数学进展, 2018, 7(6): 714-722.
DOI: 10.12677/aam.2018.76086

关键词

分数阶子方程法，分数阶广义KdV方程，精确解

1. 引言

随着对微分方程研究的不断深入，人们对微分方程的研究逐渐从整数阶发展到分数阶。分数阶微分方程是对整数阶微分方程的推广，由它构造的系统广泛应用于物理、工程、系统识别、控制理论、信号处理、分数动力学等各个方面，其最大的优点是这种方程构造的系统的下一个状态不仅取决于它的当前状态，还取决于它的所有历史形态。近几十年间，许多数学、物理、生物等领域的学者积极投入到了对分数阶微分方程的研究中，为获得分数阶微分方程近似解和精确解提出了很多行之有效的方法，如变分迭代法[1] [2]、分数阶微分变换法[3] [4]、分数阶子方程法[5] [6] [7]、Adomian 分解法[8] [9] [10]、同伦扰动法[11]、分数阶 G'/G 展开法[12] [13]、指数函数法[14]、有限差分法[15]等多种方法。然而，截至目前，尚未找到一种能够求解所有分数阶微分方程的方法，因此，寻找合适的方法求解微分方程仍然是一项十分有意义的研究。

本文主要研究如下方程的精确解

$$D_t^{\alpha}u + u_{xxxxx} - uu_{xxx} - u_x u_{xx} - u^2 u_x = 0. \tag{1.1}$$

该方程衍生于广义五阶 KdV 方程，其具体形式如下所示[16]：

$$u_t + \omega u_{xxxxx} + \alpha uu_{xxx} + \beta u_x u_{xx} + \gamma u^2 u_x = 0. \tag{1.2}$$

方程(1.2)既可以表示在重力作用下浅水中的长波运动，同时也是研究量子力学和非线性光学的一个重要模型。本文利用分数阶子方程法研究方程(1.1)的精确解，并利用数学软件 mathematica 求解计算过程中产生的超定方程组，最后获得了方程(1.1)的新的精确解。

本文的内容安排如下：第二节先介绍了修正 Riemann-Liouville 导数定义以及其具有的重要性质，然后介绍了本文所使用的分数阶子方程法；第三节则使用分数阶子方程法求解方程(1.1)的精确解，并给出了方程的具体图像。第四节为对本文内容的总结。

2. 方法概述

2.1. 修正 Riemann-Liouville 导数

修正 Riemann-Liouville 导数具有如下形式[17]：

$$D_t^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(-\alpha)} \int_0^t (t-s)^{-\alpha-1} \left(f(s) - f(0)\right) \mathrm{d}s, \ \alpha < 0, \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-s)^{-\alpha} \left(f(s) - f(0)\right) \mathrm{d}s, \ 0 < \alpha < 1, \\ \left(f^{(n)}(t)\right)^{(\alpha-n)}, \ n < \alpha < n+1, n \ge 1. \end{cases} \tag{2.1}$$

其具有的部分重要性质如下：

$$D_t^{\alpha} t^m = \frac{\Gamma(1+m)}{\Gamma(1+m-\alpha)} t^{m-\alpha},\ m>0, \tag{2.2}$$

$$D_t^{\alpha}\left(u(t)v(t)\right) = v(t)D_t^{\alpha}u(t) + u(t)D_t^{\alpha}v(t), \tag{2.3}$$

$$D_t^{\alpha}\left(u\left(v(t)\right)\right) = u_v'\left(v(t)\right)D_t^{\alpha}\left(v(t)\right) = D_v^{\alpha}\left(u\left(v(t)\right)\right)\left(v'(t)\right)^{\alpha}. \tag{2.4}$$

2.2. 分数阶子方程法

本文以如下分数阶偏微分方程

$$F\left(u,u_t,u_x,D_t^{\alpha}u,D_x^{\alpha}u,D_t^{2\alpha}u,D_x^{2\alpha}u,\cdots\right)=0,\ 0<\alpha\le 1. \tag{2.5}$$

为例来简要介绍分数阶子方程法。其中，$u=u(t,x)$ 是一个未知函数，$D_t^{\alpha}u, D_x^{\alpha}u, D_t^{2\alpha}u, D_x^{2\alpha}u$ 是关于 u 的改进 Riemann-Liouville 导数，F 是一个包含有非线性项和最高阶导数项的多项式。该方法主要分为以下几步：

1) 对方程(2.5)进行行波变换

$$u(t,x)=U(\theta),\ \theta=mt+nx. \tag{2.6}$$

其中 m,n 为待定常数，则方程(2.5)可转化为

$$G\left(U(\theta),mU'(\theta),nU'(\theta),m^{\alpha}D_{\theta}^{\alpha}U(\theta),n^{\alpha}D_{\theta}^{\alpha}U(\theta),\cdots\right)=0. \tag{2.7}$$

2) 假设方程具有如下形式的解：

$$U(\theta)=\sum\nolimits_{i=0}^{n} a_i\left(\Phi(\theta)\right)^i, \tag{2.8}$$

其中，a_i 为待定常数，$\Phi=\Phi(\theta)$ 满足如下的分数阶黎卡提微分方程

$$D_{\theta}^{\alpha}\Phi=\sigma+\Phi^2, 0<\alpha\le 1. \tag{2.9}$$

其中，σ 为待定常数，方程(2.9)的一些解如下所示：

$$\Phi(\theta)=\begin{cases} -\sqrt{-\sigma}\tanh_{\alpha}\left(\sqrt{-\sigma}\theta\right), & \sigma<0, \\ -\sqrt{-\sigma}\coth_{\alpha}\left(\sqrt{-\sigma}\theta\right), & \sigma<0, \\ \sqrt{\sigma}\tan_{\alpha}\left(\sqrt{\sigma}\theta\right), & \sigma>0, \\ -\sqrt{\sigma}\cot_{\alpha}\left(\sqrt{\sigma}\theta\right), & \sigma>0, \\ \dfrac{\Gamma(1+\alpha)}{\theta^{\alpha}+\omega}, & \omega\text{为常数}, \sigma=0. \end{cases} \tag{2.10}$$

广义的三角函数和双曲函数如下所示[17]：

$$\sin_{\alpha}(\theta)=\frac{E_{\alpha}\left(i\theta^{\alpha}\right)-E_{\alpha}\left(-i\theta^{\alpha}\right)}{2i},$$

$$\cos_{\alpha}(\theta)=\frac{E_{\alpha}\left(i\theta^{\alpha}\right)+E_{\alpha}\left(-i\theta^{\alpha}\right)}{2i},$$

$$\tan_{\alpha}(\theta)=\frac{\sin_{\alpha}(\theta)}{\cos_{\alpha}(\theta)},$$

$$\cot_\alpha(\theta)=\frac{\cos_\alpha(\theta)}{\sin_\alpha(\theta)},$$

$$\sinh_\alpha(\theta)=\frac{E_\alpha(\theta^\alpha)-E_\alpha(-\theta^\alpha)}{2},$$

$$\cosh_\alpha(\theta)=\frac{E_\alpha(\theta^\alpha)+E_\alpha(-\theta^\alpha)}{2},$$

$$\tanh_\alpha(\theta)=\frac{\sinh_\alpha(\theta)}{\cosh_\alpha(\theta)},$$

$$\coth_\alpha(\theta)=\frac{\cosh_\alpha(\theta)}{\sinh_\alpha(\theta)}. \tag{2.11}$$

其中， $E_\alpha(\theta^\alpha)$ 是 Mittag-Leffler 方程，具体表现形式为：

$$E_\alpha(\theta^\alpha)=\sum\nolimits_{k=0}^{\infty}\frac{\theta^k}{\Gamma(1+k\alpha)},\ \alpha>0. \tag{2.12}$$

特例：当 $\alpha=\frac{1}{2}$ 时，Mittag-Leffler 方程可转化为：

$$E_{\frac{1}{2}}\left(\theta^{\frac{1}{2}}\right)=\mathrm{e}^\theta Erfc\left(\theta^{\frac{1}{2}}\right),\ Erfc\ \text{为误差函数}. \tag{2.13}$$

3) 将方程(2.8)和(2.9)代入方程(2.7)中，之后平衡方程的非线性项和最高阶导数项，即可确定方程(2.8)中的 n。

4) 将已经确定 n 的方程(2.8)以及方程(2.9)代入方程(2.7)中，可以得到关于 Φ^i 的方程。之后，令这个方程组的系数为零，能够得到一系列关于 a_i,m,n,σ,α 的方程组，求解这些方程组，即可求得方程(2.5)的解。

3. 方程求解

本节将采用上述分数阶子方程法求解方程(1.1)，对于该方程，我们令

$$u(t,x)=u(\theta),\theta=x+ct. \tag{3.1}$$

则方程(1.1)转化为

$$c^\alpha D_t^\alpha u+u^{(5)}-uu^{(3)}-u'u''-u^2u'=0. \tag{3.2}$$

假设方程具有(2.8)形式的解，即

$$u(\theta)=\sum\nolimits_{i=0}^{\infty}a_i(\Phi(\theta))^i. \tag{3.3}$$

将方程(3.3)代入方程(3.2)中，平衡非线性项 u^2u' 和最高阶导数项 $u^{(5)}$ 可得 $n=2$ ，即

$$u(\theta)=a_0+a_1\Phi(\theta)+a_2(\Phi(\theta))^2. \tag{3.4}$$

将方程(3.4)代入(3.2)中，令 $\Phi^i(\theta)=0$ ，可以得到一系列关于 $a_0,a_1,a_2,c,\alpha,\sigma$ 代数方程，具体方程如下所示：

$$(\Phi(\theta))^0:c^\alpha\sigma a_1+16\sigma^3a_1-2\sigma^2a_0a_1-\sigma a_0^2a_1-2\sigma^3a_1a_2=0,$$

$$(\Phi(\theta))^1:-4\sigma^2a_1^2-2\sigma a_0a_1^2+2c^\alpha\sigma a_2+272\sigma^3a_2-16\sigma^2a_0a_2-2\sigma a_0^2a_2-4\sigma^3a_2^2=0,$$

$$\left(\Phi(\theta)\right)^{2}: c^{\alpha}a_{1}+136\sigma^{2}a_{1}-8\sigma a_{0}a_{1}-a_{0}^{2}a_{1}-\sigma a_{1}^{3}-32\sigma^{2}a_{1}a_{2}-6\sigma a_{0}a_{1}a_{2}=0,$$

$$\begin{aligned}\left(\Phi(\theta)\right)^{3}: &-12\sigma a_{1}^{2}-2a_{0}a_{1}^{2}+2c^{\alpha}a_{2}+1232\sigma^{2}a_{2}-40\sigma a_{0}a_{2}\\ &-2a_{0}^{2}a_{2}-4\sigma a_{1}^{2}a_{2}-36\sigma^{2}a_{2}^{2}-4\sigma a_{0}a_{2}^{2}=0,\end{aligned}$$

$$\left(\Phi(\theta)\right)^{4}: 240\sigma a_{1}-6a_{0}a_{1}-a_{1}^{3}-70\sigma a_{1}a_{2}-6a_{0}a_{1}a_{2}-5\sigma a_{1}a_{2}^{2}=0$$

$$\left(\Phi(\theta)\right)^{5}: -8a_{1}^{2}+1680\sigma a_{2}-24a_{0}a_{2}-4a_{1}^{2}a_{2}-68\sigma a_{2}^{2}-4a_{0}a_{2}^{2}-2\sigma a_{2}^{3}=0,$$

$$\left(\Phi(\theta)\right)^{6}: 120a_{1}-40a_{1}a_{2}-5a_{1}a_{2}^{2}=0,$$

$$\left(\Phi(\theta)\right)^{7}: 720a_{2}-36a_{2}^{2}-2a_{2}^{3}=0. \tag{3.5}$$

之后，借助 mathematica 软件求解这些方程，可以得到如下几种形式的解。

情形一：此时 $a_0=a_0, a_1=0, a_2=-30, c=\left(\frac{11a_0^2}{100}\right)^{\frac{1}{\alpha}}, \sigma=-\frac{a_0}{20}$。通过这个解，我们可以得到如下精确解：

$$u_1(t,x)=a_0-30\left(-\sqrt{\frac{a_0}{20}}\tanh_\alpha\left(\sqrt{\frac{a_0}{20}}\left(x+\left(\frac{11a_0^2}{100}\right)^{\frac{1}{\alpha}}t\right)\right)\right)^2, a_0>0,$$

$$u_2(t,x)=a_0-30\left(-\sqrt{\frac{a_0}{20}}\coth_\alpha\left(\sqrt{\frac{a_0}{20}}\left(x+\left(\frac{11a_0^2}{100}\right)^{\frac{1}{\alpha}}t\right)\right)\right)^2, a_0>0,$$

$$u_3(t,x)=a_0-30\left(\sqrt{-\frac{a_0}{20}}\tan_\alpha\left(\sqrt{-\frac{a_0}{20}}\left(x+\left(\frac{11a_0^2}{100}\right)^{\frac{1}{\alpha}}t\right)\right)\right)^2, a_0<0,$$

$$u_4(t,x)=a_0-30\left(-\sqrt{-\frac{a_0}{20}}\cot_\alpha\left(\sqrt{-\frac{a_0}{20}}\left(x+\left(\frac{11a_0^2}{100}\right)^{\frac{1}{\alpha}}t\right)\right)\right)^2, a_0<0. \tag{3.6}$$

在精确解 $u_1(t,x)$ 中，我们令 $a_0=10, \alpha=\frac{2}{3}, c=11\sqrt{11}, \sigma=-\frac{1}{2}$，并将公式(2.11)和(2.12)代入 $u_1(t,x)$ 中，借助于 mathematica 软件，可得

$$u_1(t,x)=10-15\left(\frac{\sum_{k=0}^{\infty}\frac{\left(\frac{x+11\sqrt{11}t}{\sqrt{2}}\right)^k}{\Gamma\left(1+\frac{2}{3}k\right)}-\sum_{k=0}^{\infty}\frac{\left(-\frac{x+11\sqrt{11}t}{\sqrt{2}}\right)^k}{\Gamma\left(1+\frac{2}{3}k\right)}}{\sum_{k=0}^{\infty}\frac{\left(\frac{x+11\sqrt{11}t}{\sqrt{2}}\right)^k}{\Gamma\left(1+\frac{2}{3}k\right)}+\sum_{k=0}^{\infty}\frac{\left(-\frac{x+11\sqrt{11}t}{\sqrt{2}}\right)^k}{\Gamma\left(1+\frac{2}{3}k\right)}}\right)^2 \tag{3.7}$$

其图像如图 1 所示。

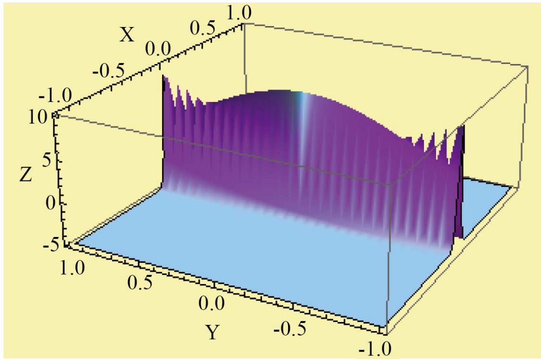

Figure 1. Figure of solution (3.7)
图 1. 解(3.7)的图

在精确解 $u_3(t,x)$ 中，我们令 $a_0=-10,\alpha=\frac{4}{5},c=11^{\frac{5}{4}},\sigma=\frac{1}{2}$，并将公式(2.11)和(2.12)代入 $u_3(t,x)$ 中，借助于 mathematica 软件，可得

$$u_3(t,x)=-10-15\left(\frac{\sum_{k=0}^{\infty}\frac{\left(\frac{x+11^{\frac{5}{4}}t}{\sqrt{2}}i\right)^k}{\Gamma\left(1+\frac{4}{5}k\right)}-\sum_{k=0}^{\infty}\frac{\left(-\frac{x+11^{\frac{5}{4}}t}{\sqrt{2}}i\right)^k}{\Gamma\left(1+\frac{4}{5}k\right)}}{\sum_{k=0}^{\infty}\frac{\left(\frac{x+11^{\frac{5}{4}}t}{\sqrt{2}}i\right)^k}{\Gamma\left(1+\frac{4}{5}k\right)}+\sum_{k=0}^{\infty}\frac{\left(-\frac{x+11^{\frac{5}{4}}t}{\sqrt{2}}i\right)^k}{\Gamma\left(1+\frac{4}{5}k\right)}}\right)^2 \tag{3.8}$$

其图像如图 2 所示。

情形二：此时 $a_0=a_0,a_1=0,a_2=12,c=\left(\frac{a_0^2}{4}\right)^{\frac{1}{\alpha}},\sigma=\frac{a_0}{8}$。通过这个解，我们可以得到如下精确解：

$$u_5(t,x)=a_0+12\left(-\sqrt{-\frac{a_0}{8}}\tanh_\alpha\left(\sqrt{-\frac{a_0}{8}}\left(x+\left(\frac{a_0^2}{4}\right)^{\frac{1}{\alpha}}t\right)\right)\right)^2,a_0<0,$$

$$u_6(t,x)=a_0+12\left(-\sqrt{-\frac{a_0}{8}}\coth_\alpha\left(\sqrt{-\frac{a_0}{8}}\left(x+\left(\frac{a_0^2}{4}\right)^{\frac{1}{\alpha}}t\right)\right)\right)^2,a_0<0,$$

$$u_7(t,x)=a_0+12\left(\sqrt{\frac{a_0}{8}}\tan_\alpha\left(\sqrt{\frac{a_0}{8}}\left(x+\left(\frac{a_0^2}{4}\right)^{\frac{1}{\alpha}}t\right)\right)\right)^2,a_0>0,$$

$$u_8(t,x)=a_0+12\left(-\sqrt{\frac{a_0}{8}}\cot_\alpha\left(\sqrt{\frac{a_0}{8}}\left(x+\left(\frac{a_0^2}{4}\right)^{\frac{1}{\alpha}}t\right)\right)\right)^2,a_0>0. \tag{3.9}$$

在精确解 $u_5(t,x)$ 中，我们令 $a_0=-2\sqrt{2},\alpha=\dfrac{1}{2},c=4,\sigma=-\dfrac{2\sqrt{2}}{8}$，并将公式(2.11)和(2.13)代入 $u_5(t,x)$ 中，借助于 mathematica 软件，可得

$$u_5(t,x)=-2\sqrt{2}+3\sqrt{2}\left(\frac{e^{\frac{(x+4t)^2}{2\sqrt{2}}}Erfc\left[-\frac{x+4t}{2^{\frac{3}{4}}}\right]-e^{\frac{(x+4t)^2}{2\sqrt{2}}}Erfc\left[\frac{x+4t}{2^{\frac{3}{4}}}\right]}{e^{\frac{(x+4t)^2}{2\sqrt{2}}}Erfc\left[-\frac{x+4t}{2^{\frac{3}{4}}}\right]+e^{\frac{(x+4t)^2}{2\sqrt{2}}}Erfc\left[\frac{x+4t}{2^{\frac{3}{4}}}\right]}\right)^2 \tag{3.10}$$

其图像如图 3 所示。

在精确解 $u_8(t,x)$ 中，我们令 $a_0=2,\alpha=\dfrac{2}{3},c=1,\sigma=\dfrac{1}{4}$，并将公式(2.11)和(2.12)代入 $u_8(t,x)$ 中，借助于 mathematica 软件，可得

$$u_8(t,x)=2+3\left(\frac{\sum_{k=0}^{\infty}\frac{\left(\frac{x+t}{2}i\right)^k}{\Gamma\left(1+\frac{2}{3}k\right)}-\sum_{k=0}^{\infty}\frac{\left(-\frac{x+t}{2}i\right)^k}{\Gamma\left(1+\frac{2}{3}k\right)}}{\sum_{k=0}^{\infty}\frac{\left(\frac{x+t}{2}i\right)^k}{\Gamma\left(1+\frac{2}{3}k\right)}+\sum_{k=0}^{\infty}\frac{\left(-\frac{x+t}{2}i\right)^k}{\Gamma\left(1+\frac{2}{3}k\right)}}\right)^2 \tag{3.11}$$

其图像如图 4 所示。

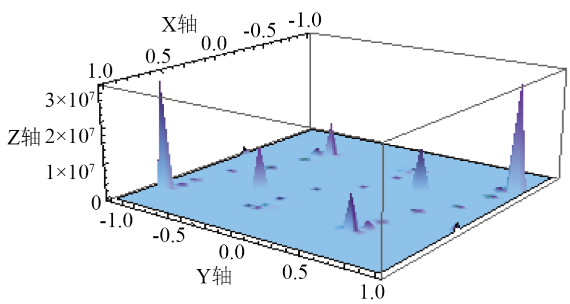

Figure 2. Figure of solution (3.8)
图 2. 解(3.8)的图

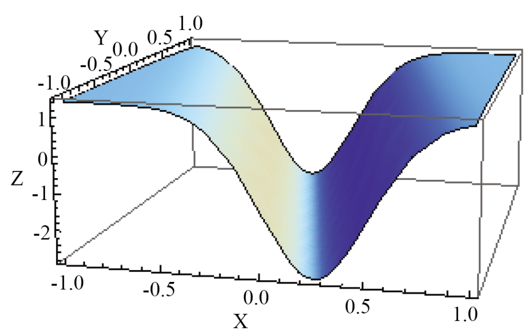

Figure 3. Figure of solution (3.10)
图 3. 解(3.10)的图

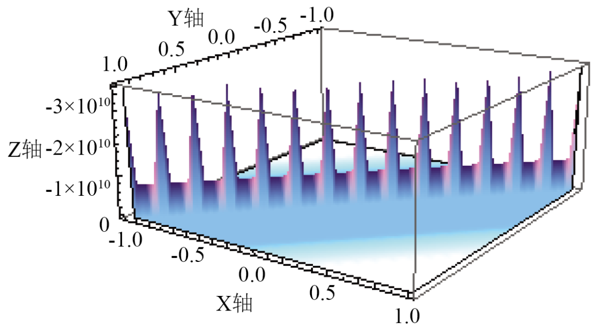

Figure 4. Figure of solution (3.11)
图 4. 解(3.11)的图

4. 总结

本文主要应用分数阶子方程法寻找分数阶广义 KdV 方程(1.1)的精确解，最后给出了方程新的广义双曲函数和三角函数形式的解，画出了解的图像。这些解能够加深人们对分数阶 KdV 方程的理解，从而促进该方程在非线性系统和孤子理论等领域的应用。

致　谢

本文得到北京市教委面上项目(编号：KM201710009011)和北京市教委基础科研计划项目资助。

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