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Singular Perturbation for Boundary Value Problem of a Second-Order Semi-Linear System with Discontinuous Source Terms

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Abstract

In this paper, a singularly perturbed boundary value problem of second-order semi-linear system with discontinuous source terms is discussed. Firstly, we use the boundary function method and sewing method to construct formal asymptotic solution to the original problem. Then, the uniformly validity of the solution obtained is proved by the lower and upper solutions theorem. Finally, an example is given as an illustration.

Keywords

Boundary Function Method, Sewing Method, Discontinuous Source Term, Singular Perturbation

具有不连续源项的二阶半线性系统边值问题的奇摄动

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摘 要

本文讨论了一类带有不连续源项的二阶半线性系统边值问题。首先，用边界函数法和缝接法构造出原问题的形式渐近解；然后运用上下解定理证明形式解的一致有效性。最后给出一个例子验证了结果的有效性。

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关键词

边界函数法，缝接法，不连续源项，奇摄动

1. 引言

奇异摄动边值问题因来源于物理、力学、生物等领域而受到广泛的关注和研究[1] [2] [3] [4]，奇异摄动理论和方法已经成为研究非线性问题的有力工具之一。近几年，在流固耦合[5]、对流扩散[6]、界面交互[7]等问题中出现了大量具有不连续系数和源项的奇摄动边值问题。例如，2012 年，丁海云等在文[8]中考虑了一类带有不连续项的二阶半线性奇摄动边值问题，运用边界函数法和缝接法获得了一致有效的渐近解。2017 年，包立平在文[9]中研究了具有不连续源的半线性微分方程组边值问题，用 Aumann 介值定理对解在间断点处进行了光滑地连接，并证明了解的一致有效性。

最近，Rao 和 Chawla 讨论了带有边界层和内部层的奇摄动半线性反应-扩散方程[10]：

$$\begin{cases}-E\boldsymbol{u}''+\boldsymbol{a}(x,\boldsymbol{u})=\boldsymbol{f},\ x\in\Omega_1\cup\Omega_2,\\ \boldsymbol{u}(0)=\boldsymbol{p},\ \boldsymbol{u}(1)=\boldsymbol{q}.\end{cases}$$

其中，$E=diag(\varepsilon_1,\cdots,\varepsilon_m)$，且 $0<\varepsilon_1<\cdots<\varepsilon_m\le 1$，$\boldsymbol{a}(x,\boldsymbol{u})=(a_1(x,\boldsymbol{u}),\cdots,a_m(x,\boldsymbol{u}))^{\mathrm{T}}$，$\boldsymbol{f}=(f_1,\cdots,f_m)^{\mathrm{T}}$，他们建立了一种基于 Shishkin 网格的数值计算方法。

受上述工作的启发，本文在更一般的条件下讨论如下具有不连续源项的二阶半线性系统的奇摄动边值问题：

$$\begin{cases}-\varepsilon^2y_1''+a_1(x,y_1,y_2)=f_1(x),\\ -\varepsilon^2y_2''+a_2(x,y_1,y_2)=f_2(x),\\ y_1(0)=p_1,\ y_2(0)=p_2,\\ y_1(1)=q_1,\ y_2(1)=q_2.\end{cases}\tag{1.1}$$

其中 $a_1(x,y_1,y_2),a_2(x,y_1,y_2)$ 为 $[0,1]\times R^2$ 上的充分光滑的函数，ε 为 $0<\varepsilon\ll 1$ 的参数，$p_i,q_i\ (i=1,2)$ 为给定常数，函数 $f_1(x),f_2(x)$ 在 $x=c$ 处间断。问题(1.1)可以看作纯量情形[8] [11]的自然推广。

令 $\varepsilon y'=z$，则(1.1)式等价于如下的四维吉洪诺夫系统：

$$\begin{cases}\varepsilon y_1'=z_1,\ \varepsilon y_2'=z_2\\ \varepsilon z_1'=a_1(x,y_1,y_2)-f_1(x),\\ \varepsilon z_2'=a_2(x,y_1,y_2)-f_2(x),\\ y_1(0)=p_1,\ y_2(0)=p_2,\\ y_1(1)=q_1,\ y_2(1)=q_2.\end{cases}\tag{1.2}$$

作如下假设。

(H_1)函数 $f_1(x),f_2(x)$ 定义如下：

$$f_i(x)=\begin{cases}f_{i1}(x), & x\in[0,c),\\ f_{i2}(x), & x\in(c,1],\end{cases}$$

其中，函数 $f_i(x)$ 在 $x\in[0,c)\cup(c,1]$，$i=1,2$ 上充分光滑，且 $f_{i1}(c)\neq f_{i2}(c),i=1,2$。

(H_2)方程组 $a_i(x,y_1,y_2)=f_{i1}(x),x\in[0,c)$，$a_j(x,y_1,y_2)=f_{j2}(x),x\in(c,1]$，分别存在唯一解 $\varphi_i(x)\in C^2[0,c],\psi_j(x)\in C^2[c,1](i,j=1,2)$。

(H_3)存在 $\sigma_1>0$ 和 $\sigma_2>0$ 使得对任意 $(x,y_1,y_2)\in[0,1]\times R^2$ 成立

$$\frac{\partial a_1}{\partial y_1}(x,y_1,y_2)\geq\sigma_1,\ \frac{\partial a_2}{\partial y_2}(x,y_1,y_2)\geq\sigma_2,$$

$$\frac{\partial a_1}{\partial y_2}(x,y_1,y_2)\leq\frac{\sigma_1}{2},\ \frac{\partial a_2}{\partial y_1}(x,y_1,y_2)\leq\frac{\sigma_2}{2}.$$

由假设(H_3)可知，4 阶矩阵

$$\begin{pmatrix}0 & E_2\\ A(x) & 0\end{pmatrix}$$

的特征根有 2 个正实部和 2 个负实部，式中 E_2 为 2×2 单位矩阵。因此，辅助系统

$$\begin{cases}y_1'=z_1,\ y_2'=z_2,\\ z_1'=a_1(0,\varphi_1+z_1,\varphi_2+z_2),\\ z_2'=a_2(0,\varphi_1+z_1,\varphi_2+z_2),\end{cases}\tag{1.3}$$

和

$$\begin{cases}y_1'=z_1,\ y_2'=z_2,\\ z_1'=a_1(1,\psi_1+z_1,\psi_2+z_2),\\ z_2'=a_2(1,\psi_1+z_1,\psi_2+z_2),\end{cases}\tag{1.4}$$

的平衡点(0，0)皆具有两维的稳定流形和两维的不稳定流形。记系统(1.3)的稳定流形为 $W^s=\{(y,z)|y=\Phi(z),y\in G^-\}$；记系统(1.4)的不稳定流形为 $W^u=\{(y,z)|y=\Psi(z),y\in G^+\}$。

存在可逆矩阵

$$B(x)=\begin{pmatrix}B_{11}(x) & B_{12}(x)\\ B_{21}(x) & B_{22}(x)\end{pmatrix},$$

使得

$$B^{-1}(x)\cdot\begin{pmatrix}0 & E_2\\ A & 0\end{pmatrix}\cdot B(x)=\begin{pmatrix}C_-(x) & 0\\ 0 & C_+(x)\end{pmatrix}.$$

其中，$B_{ij}(x),C_-(x),C_+(x)$ 均为 2 阶矩阵，且 $C_-(x)$ 的特征值的实部为负值，$C_+(x)$ 的特征值的实部为正值。

(H_4) $\det B_{11}(0)\neq0,\det B_{22}(1)\neq0$ 且 $(p_1-\varphi_1(0),p_2-\varphi_2(0))\in G^-,(q_1-\psi_1(1),q_2-\psi_2(1))\in G^+$。

注：若(1.1)为线性系统，即若 $a_i(x,y_1,y_2)(i=1,2)$ 是 y_1 和 y_2 的线性函数，则条件(H_3)自然蕴含(H_4)成立。

2. 构造形式渐近解

考虑到源项 $f(x)$ 在 $x=c$ 处间断，把问题(1.1)看成如下左、右问题的耦合。

左问题(L)

$$\begin{cases}-\varepsilon^2\overline{y}_1''+a_1\left(x,\overline{y}_1,\overline{y}_2\right)=f_{11}\left(x\right),\\ -\varepsilon^2\overline{y}_2''+a_2\left(x,\overline{y}_1,\overline{y}_2\right)=f_{21}\left(x\right),\ \ x\in\left[0,c\right),\\ \overline{y}_1\left(0\right)=p_1,\ \ \overline{y}_2\left(0\right)=p_2,\\ \overline{y}_1\left(c\right)=\gamma_1\left(\varepsilon\right),\ \ \overline{y}_2(c)=\gamma_2\left(\varepsilon\right).\end{cases} \tag{2.1}$$

右问题(R)

$$\begin{cases}-\varepsilon^2\tilde{y}_1''+a_1\left(x,\tilde{y}_1,\tilde{y}_2\right)=f_{12}\left(x\right),\\ -\varepsilon^2\tilde{y}_2''+a_2\left(x,\tilde{y}_1,\tilde{y}_2\right)=f_{22}\left(x\right),\ \ x\in\left(c,1\right],\\ \tilde{y}_1\left(1\right)=q_1,\ \ \tilde{y}_2\left(1\right)=q_2,\\ \tilde{y}_1\left(c\right)=\gamma_1\left(\varepsilon\right),\ \ \tilde{y}_2\left(c\right)=\gamma_2\left(\varepsilon\right).\end{cases} \tag{2.2}$$

其中 $\gamma_1\left(\varepsilon\right)=\gamma_{10}+\varepsilon\gamma_{11}+\cdots$， $\gamma_2\left(\varepsilon\right)=\gamma_{20}+\varepsilon\gamma_{21}+\cdots$ 为依赖于 ε 的待定参数。

按照缝接法思想，我们首先构造出左问题(L)和右问题(R)的形式渐近解，然后在间断点 $x=c$ 处对形式解进行光滑地连接，由此确定待定参数 $\gamma_1\left(\varepsilon\right)$ 和 $\gamma_2\left(\varepsilon\right)$

先考虑左问题(L)。由边界函数法[4]可设其形式解为：

$$\begin{cases}\overline{y}_1=\overline{u}_1\left(x\right)+\overline{v}_1\left(\tau\right)+\overline{w}_1\left(\eta\right),\\ \overline{y}_2=\overline{u}_2\left(x\right)+\overline{v}_2\left(\tau\right)+\overline{w}_2\left(\eta\right),\end{cases} \tag{2.3}$$

式中

$$\overline{u}_j\left(x\right)=\overline{u}_{j0}\left(x\right)+\varepsilon\overline{u}_{j1}\left(x\right)+\varepsilon^2\overline{u}_{j2}\left(x\right)+\cdots, \tag{2.3a}$$

$$\overline{v}_j\left(\tau\right)=\overline{v}_{j0}\left(\tau\right)+\varepsilon\overline{v}_{j1}\left(\tau\right)+\varepsilon^2\overline{v}_{j2}\left(\tau\right)+\cdots, \tag{2.3b}$$

$$\overline{w}_j\left(\eta\right)=\overline{w}_{j0}\left(\eta\right)+\varepsilon\overline{w}_{j1}\left(\eta\right)+\varepsilon^2\overline{w}_{j2}\left(\eta\right)+\cdots, \tag{2.3c}$$

其中 $x\in\left[0,c\right),\tau=\dfrac{x}{\varepsilon},\eta=\dfrac{x-c}{\varepsilon},j=1,2$。

将式(2.3)代入式(2.1)中，再根据不同的尺度分离可得：

$$\begin{cases}-\varepsilon^2\overline{u}_1''+a_1\left(x,\overline{u}_1,\overline{u}_2\right)=f_{11}\left(x\right),\\ -\varepsilon^2\overline{u}_2''+a_2\left(x,\overline{u}_1,\overline{u}_2\right)=f_{21}\left(x\right),\end{cases} \tag{2.4}$$

$$\begin{cases}-\overline{v}_1''\left(\tau\right)+a_1\left(x,\overline{u}_1+\overline{v}_1,\overline{u}_2+\overline{v}_2\right)-a_1\left(x,\overline{u}_1,\overline{u}_2\right)=0,\\ -\overline{v}_2''\left(\tau\right)+a_2\left(x,\overline{u}_1+\overline{v}_1,\overline{u}_2+\overline{v}_2\right)-a_2\left(x,\overline{u}_1,\overline{u}_2\right)=0,\end{cases} \tag{2.5}$$

$$\begin{cases}-\overline{w}_1''\left(\eta\right)+a_1\left(x,\overline{u}_1+\overline{v}_1+\overline{w}_1,\overline{u}_2+\overline{v}_2+\overline{w}_1\right)-a_1\left(x,\overline{u}_1+\overline{v}_1,\overline{u}_2+\overline{v}_2\right)=0,\\ -\overline{w}_2''\left(\eta\right)+a_2\left(x,\overline{u}_1+\overline{v}_1+\overline{w}_1,\overline{u}_2+\overline{v}_2+\overline{w}_1\right)-a_2\left(x,\overline{u}_1+\overline{v}_1,\overline{u}_2+\overline{v}_2\right)=0.\end{cases} \tag{2.6}$$

对 $a_1\left(x,y_1,y_2\right),a_2\left(x,y_1,y_2\right)$ 进行 ε 幂级数展开：

$$\begin{cases}a_1\left(x,\overline{u}_1,\overline{u}_2\right)=a_1\left(x,\varphi_1,\varphi_2\right)+\varepsilon\left(\dfrac{\partial a_1}{\partial\overline{y}_1}\left(x,\varphi_1,\varphi_2\right)\cdot\overline{u}_{11}\left(x\right)+\dfrac{\partial a_1}{\partial\overline{y}_2}\left(x,\varphi_1,\varphi_2\right)\cdot\overline{u}_{21}\left(x\right)+\overline{g}_1\left(x\right)\right)+\cdots,\\ a_2\left(x,\overline{u}_1,\overline{u}_2\right)=a_2\left(x,\varphi_1,\varphi_2\right)+\varepsilon\left(\dfrac{\partial a_2}{\partial\overline{y}_1}\left(x,\varphi_1,\varphi_2\right)\cdot\overline{u}_{11}\left(x\right)+\dfrac{\partial a_2}{\partial\overline{y}_2}\left(x,\varphi_1,\varphi_2\right)\cdot\overline{u}_{21}\left(x\right)+\overline{h}_1\left(x\right)\right)+\cdots,\end{cases} \tag{2.7}$$

其中， $\overline{g}_1\left(x\right),\overline{h}_1\left(x\right)$ 为 $x,\varphi_1\left(x\right),\varphi_2\left(x\right)$ 的函数。

将式(2.3a)和(2.7)代入式(2.4)中，得

$$\varepsilon^0:\begin{cases}a_1\left(x,\varphi_1,\varphi_2\right)=f_{11}\left(x\right),\\ a_2\left(x,\varphi_1,\varphi_2\right)=f_{21}\left(x\right),\end{cases}\tag{2.8}$$

$$\varepsilon^i:\begin{cases}\bar{u}_{1i}''=\dfrac{\partial a_1}{\partial y_1}\left(x,\varphi_1,\varphi_2\right)\cdot\bar{u}_{1(i+2)}\left(x\right)+\dfrac{\partial a_1}{\partial y_2}\left(x,\varphi_1,\varphi_2\right)\cdot\bar{u}_{2(i+2)}\left(x\right)+\bar{g}_{i+2}\left(x\right),\\ \bar{u}_{2i}''=\dfrac{\partial a_2}{\partial y_1}\left(x,\varphi_1,\varphi_2\right)\cdot\bar{u}_{1(i+2)}\left(x\right)+\dfrac{\partial a_2}{\partial y_2}\left(x,\varphi_1,\varphi_2\right)\cdot\bar{u}_{2(i+2)}\left(x\right)+\bar{h}_{i+2}\left(x\right),\end{cases}\tag{2.9}$$

其中 $\bar{g}_{i+2}\left(x\right),\bar{h}_{i+2}\left(x\right)$ 为 $x,\varphi_1\left(x\right),\cdots,\bar{u}_{1(i+1)}\left(x\right),\varphi_2,\cdots,\bar{u}_{2(i+1)}\left(x\right)$ 的函数，由假设(H_1)和(H_2)可知(2.8)式有唯一解，又由假设(H_3)知，从式(2.9)可依次求出 $u_{1i}\left(x\right)$ 和 $u_{2i}\left(x\right)$。

将(2.3b)代入(2.5)中，得

$$\varepsilon^0:\begin{cases}\bar{v}_{10}''=a_1\left(0,\varphi_1\left(0\right)+\bar{v}_{10},\varphi_2\left(0\right)+\bar{v}_{20}\right)-a_1\left(0,\varphi_1\left(0\right),\varphi_2\left(0\right)\right),\\ \bar{v}_{20}''=a_2\left(0,\varphi_1\left(0\right)+\bar{v}_{10},\varphi_2\left(0\right)+\bar{v}_{20}\right)-a_2\left(0,\varphi_1\left(0\right),\varphi_2\left(0\right)\right),\\ \bar{v}_{10}\left(0\right)=p_1-\varphi_1\left(0\right),\\ \bar{v}_{20}\left(0\right)=p_2-\varphi_2\left(0\right),\end{cases}\tag{2.10}$$

$$\varepsilon^i:\begin{cases}\bar{v}_{1i}''=\dfrac{\partial a_1}{\partial y_1}\left(0,\varphi_1\left(0\right)+\bar{v}_{10},\varphi_2\left(0\right)+\bar{v}_{20}\right)\cdot\bar{v}_{1i}\left(\tau\right)+\dfrac{\partial a_1}{\partial y_2}\left(0,\varphi_1\left(0\right)+\bar{v}_{10},\varphi_2\left(0\right)+\bar{v}_{20}\right)\cdot\bar{v}_{2i}\left(\tau\right)+\bar{F}_i\left(\tau\right),\\ \bar{v}_{2i}''=\dfrac{\partial a_2}{\partial y_1}\left(0,\varphi_1\left(0\right)+\bar{v}_{10},\varphi_2\left(0\right)+\bar{v}_{20}\right)\cdot\bar{v}_{1i}\left(\tau\right)+\dfrac{\partial a_2}{\partial y_2}\left(0,\varphi_1\left(0\right)+\bar{v}_{10},\varphi_2\left(0\right)+\bar{v}_{20}\right)\cdot\bar{v}_{2i}\left(\tau\right)+\bar{G}_i\left(\tau\right),\end{cases}\tag{2.11}$$

其中，$\bar{F}_i\left(\tau\right),\bar{G}_i\left(\tau\right)$ 为 $\tau,\bar{v}_{10}\left(\tau\right),\cdots,\bar{v}_{1(i-1)}\left(\tau\right),\bar{v}_{20}\left(\tau\right),\cdots,\bar{v}_{2(i-1)}\left(\tau\right)$ 的函数。

由边界层函数的性质可知：$\lim\limits_{\tau\to+\infty}\bar{v}_{10}\left(\tau\right)=\lim\limits_{\tau\to+\infty}\bar{v}_{10}'\left(\tau\right)=\lim\limits_{\tau\to+\infty}\bar{v}_{20}\left(\tau\right)=\lim\limits_{\tau\to+\infty}\bar{v}_{20}'\left(\tau\right)=0$。

利用文[4]中引理 3.5.6，由假设(H_3)和(H_4)可知，存在 $\delta_1,\delta_2>0$ 使得

$$\bar{v}_{10}\left(\tau\right)=O\left(\mathrm{e}^{-\sqrt{\delta_1}\tau}\right),\ \bar{v}_{20}\left(\tau\right)=O\left(\mathrm{e}^{-\sqrt{\delta_2}\tau}\right).$$

同理，由式(2.11)知：存在 $\bar{k}_{1i},\bar{k}_{2i}>0$，且 $\sqrt{\delta_1}>\bar{k}_{1i},\sqrt{\delta_2}>\bar{k}_{2i}$，使得

$$\bar{v}_{1i}\left(\tau\right)=O\left(\mathrm{e}^{-\left(\sqrt{\delta_1}-\bar{k}_{1i}\right)\tau}\right),\ \bar{v}_{2i}\left(\tau\right)=O\left(\mathrm{e}^{-\left(\sqrt{\delta_2}-\bar{k}_{2i}\right)\tau}\right).$$

将(2.3c)代入(2.6)中，可得

$$\varepsilon^0:\begin{cases}\bar{w}_{10}''=a_1\left(c,\varphi_1\left(c\right)+\bar{w}_{10},\varphi_2\left(c\right)+\bar{w}_{20}\right)-a_1\left(c,\varphi_1\left(c\right),\varphi_2\left(c\right)\right),\\ \bar{w}_{20}''=a_2\left(c,\varphi_1\left(c\right)+\bar{w}_{10},\varphi_2\left(c\right)+\bar{w}_{20}\right)-a_2\left(c,\varphi_1\left(c\right),\varphi_2\left(c\right)\right),\\ \bar{w}_{10}\left(0\right)=\gamma_{10}-\varphi_1\left(c\right),\\ \bar{w}_{20}\left(0\right)=\gamma_{20}-\varphi_2\left(c\right),\end{cases}\tag{2.12}$$

$$\varepsilon^i:\begin{cases}\bar{w}_{1i}''=\dfrac{\partial a_1}{\partial y_1}\left(c,\varphi_1\left(c\right)+\bar{w}_{10},\varphi_2\left(c\right)+\bar{w}_{20}\right)\cdot\bar{w}_{1i}\left(\eta\right)+\dfrac{\partial a_1}{\partial y_2}\left(c,\varphi_1\left(c\right)+\bar{w}_{10},\varphi_2\left(c\right)+\bar{w}_{20}\right)\cdot\bar{w}_{2i}\left(\eta\right)+\bar{H}_i\left(\eta\right),\\ \bar{w}_{2i}''=\dfrac{\partial a_2}{\partial y_1}\left(c,\varphi_1\left(c\right)+\bar{w}_{10},\varphi_2\left(c\right)+\bar{w}_{20}\right)\cdot\bar{w}_{1i}\left(\eta\right)+\dfrac{\partial a_2}{\partial y_2}\left(c,\varphi_1\left(c\right)+\bar{w}_{10},\varphi_2\left(c\right)+\bar{w}_{20}\right)\cdot\bar{w}_{2i}\left(\eta\right)+\bar{W}_i\left(\eta\right),\end{cases}$$

其中，$\bar{H}_i\left(\eta\right),\bar{W}_i\left(\eta\right)$ 为 $\eta,\bar{w}_{10}\left(\eta\right),\cdots,\bar{w}_{1(i-1)}\left(\eta\right),\bar{w}_{20}\left(\eta\right),\cdots,\bar{w}_{2(i-1)}\left(\eta\right)$ 的函数。

由边界层的性质可知： $\lim\limits_{\eta\to-\infty}\overline{w}_{10}(\eta)=\lim\limits_{\eta\to\infty}\overline{w}'_{10}(\eta)=\lim\limits_{\eta\to\infty}\overline{w}_{20}(\eta)=\lim\limits_{\eta\to\infty}\overline{w}'_{20}(\eta)=0$ 。

利用文[4]中引理 3.5.6，由假设(H_3)和(H_4)可知：

$$\overline{w}_{10}(\eta)=O\left(\mathrm{e}^{\sqrt{\delta_1}\eta}\right),\ \overline{w}_{20}(\eta)=O\left(\mathrm{e}^{\sqrt{\delta_2}\eta}\right),$$

$$\overline{w}_{1i}(\eta)=O\left(\mathrm{e}^{\left(\sqrt{\delta_1}-\overline{l}_{1i}\right)\eta}\right),\ \overline{w}_{2i}(\eta)=O\left(\mathrm{e}^{\left(\sqrt{\delta_2}-\overline{l}_{2i}\right)\eta}\right),$$

其中 $0<\overline{l}_{1i}<\sqrt{\delta_1},0<\overline{l}_{2i}<\sqrt{\delta_2}$ 。

类似地，可求出右问题的形式渐近解

$$\begin{cases}\tilde{y}_1=\tilde{u}_1(x)+\tilde{v}_1(\eta)+\tilde{w}_1(\xi),\\ \tilde{y}_2=\tilde{u}_2(x)+\tilde{v}_2(\eta)+\tilde{w}_2(\xi),\end{cases}\tag{2.13}$$

其中 $x\in(c,1]$， $\tilde{v}_j(\eta),\tilde{w}_j(\xi),\xi=\dfrac{x-1}{\varepsilon},j=1,2$ 分别是左、右边界层校正项，且满足

$$\tilde{v}_{10}(\eta)=O\left(\mathrm{e}^{-\sqrt{\delta_1}\eta}\right),\ \tilde{v}_{20}(\eta)=O\left(\mathrm{e}^{-\sqrt{\delta_2}\eta}\right),$$

$$\tilde{w}_{10}(\xi)=O\left(\mathrm{e}^{\sqrt{\delta_1}\xi}\right),\ \tilde{w}_{20}(\xi)=O\left(\mathrm{e}^{\sqrt{\delta_2}\xi}\right),$$

$$\tilde{v}_{1i}(\eta)=O\left(\mathrm{e}^{-\left(\sqrt{\delta_1}-\tilde{k}_{1i}\right)\eta}\right),\ \tilde{v}_{2i}(\eta)=O\left(\mathrm{e}^{-\left(\sqrt{\delta_2}-\tilde{k}_{2i}\right)\eta}\right),$$

$$\tilde{w}_{1i}(\xi)=O\left(\mathrm{e}^{\left(\sqrt{\delta_1}-\tilde{l}_{1i}\right)\xi}\right),\ \tilde{w}_{2i}(\xi)=O\left(\mathrm{e}^{\left(\sqrt{\delta_2}-\tilde{l}_{2i}\right)\xi}\right),$$

其中， $0<\tilde{k}_{1i}<\sqrt{\delta_1},0<\tilde{k}_{2i}<\sqrt{\delta_2},0<\tilde{l}_{1i}<\sqrt{\delta_1},0<\tilde{l}_{2i}<\sqrt{\delta_2}$ 。

为左、右问题的解在 $x=c$ 处光滑连接，须有

$$\begin{cases}\overline{y}'_1(c)=\tilde{y}'_1(c),\\ \overline{y}'_2(c)=\tilde{y}'_2(c).\end{cases}\tag{2.14}$$

将式(2.3)，(2.13)代入(2.14)，考虑到 $\overline{v}_1(\tau),\tilde{w}_1(\eta),\overline{v}'_2(\tau),\tilde{w}'_2(\eta)$ 在 $x=c$ 处的指数衰减性，从而有

$$\begin{cases}\left.\dfrac{\mathrm{d}\overline{w}_{10}}{\mathrm{d}\eta}\right|_{\eta=0}=\left.\dfrac{\mathrm{d}\tilde{v}_{10}}{\mathrm{d}\eta}\right|_{\eta=0},\\ \left.\dfrac{\mathrm{d}\overline{w}_{20}}{\mathrm{d}\eta}\right|_{\eta=0}=\left.\dfrac{\mathrm{d}\tilde{v}_{20}}{\mathrm{d}\eta}\right|_{\eta=0}.\end{cases}\tag{2.15}$$

令 $\dfrac{\mathrm{d}\overline{w}_{10}}{\mathrm{d}\eta}=V$ ，对(2.12)变形为：

$$2V\frac{\mathrm{d}V}{\mathrm{d}\eta}=2\left[a_1(\cdot)-f_{11}(c)\right]\cdot V.$$

两边对 η 在 $(-\infty,0)$ 上积分得：

$$\int_{-\infty}^{0}2V\frac{\mathrm{d}V}{\mathrm{d}\eta}\mathrm{d}\eta=\int_{-\infty}^{0}2a_1(\cdot)\frac{\mathrm{d}\overline{w}_{10}}{\mathrm{d}\eta}\mathrm{d}\eta-\int_{-\infty}^{0}2f_{11}(c)\frac{\mathrm{d}\overline{w}_{10}}{\mathrm{d}\eta}\mathrm{d}\eta,$$

$$\begin{aligned}V^2(0)&=2a_1(\cdot)\cdot\overline{w}_{10}(0)-2f_{11}(c)\cdot\overline{w}_{10}(0)\\&\quad-2\int_{-\infty}^{0}\left[\frac{\partial a_1}{\partial\overline{y}_1}(\cdot)\cdot\frac{\mathrm{d}\overline{w}_{10}(\eta)}{\mathrm{d}\eta}+\frac{\partial a_1}{\partial\overline{y}_2}(\cdot)\cdot\frac{\mathrm{d}\overline{w}_{20}(\eta)}{\mathrm{d}\eta}\right]\cdot\overline{w}_{10}(\eta)\cdot\mathrm{d}\eta.\end{aligned}$$

即

$$\left(\frac{\mathrm{d}\bar{w}_{10}(\eta)}{\mathrm{d}\eta}\right)^2\Bigg|_{\eta=0}=2a_1(\cdot)\cdot\bar{w}_{10}(0)-2f_{11}(c)\cdot\bar{w}_{10}(0)-2\int_{-\infty}^{0}\left[\frac{\partial a_1}{\partial\bar{y}_1}(\cdot)\cdot\frac{\mathrm{d}\bar{w}_{10}(\eta)}{\mathrm{d}\eta}+\frac{\partial a_1}{\partial\bar{y}_2}(\cdot)\cdot\frac{\mathrm{d}\bar{w}_{20}(\eta)}{\mathrm{d}\eta}\right]\cdot\bar{w}_{10}(\eta)\cdot\mathrm{d}\eta.$$

类似地，有

$$\left(\frac{\mathrm{d}\tilde{v}_{10}}{\mathrm{d}\eta}\right)^2\Bigg|_{\eta=0}=2a_1(*)\cdot\tilde{v}_{10}(0)-2f_{12}(c)\cdot\tilde{v}_{10}(0)+2\int_{0}^{+\infty}\left[\frac{\partial a_1}{\partial\tilde{y}_1}(*)\cdot\frac{\mathrm{d}\tilde{v}_{10}(\eta)}{\mathrm{d}\eta}+\frac{\partial a_1}{\partial\tilde{y}_2}(*)\cdot\frac{\mathrm{d}\tilde{v}_{20}(\eta)}{\mathrm{d}\eta}\right]\cdot\tilde{v}_{10}(\eta)\cdot\mathrm{d}\eta.$$

由连接条件(2.15)知，

$$\begin{aligned}&2a_1(\cdot)\cdot\bar{w}_{10}(0)-2f_{11}(c)\cdot\bar{w}_{10}(0)-2\int_{-\infty}^{0}\left[\frac{\partial a_1}{\partial\bar{y}_1}(\cdot)\cdot\frac{\mathrm{d}\bar{w}_{10}(\eta)}{\mathrm{d}\eta}+\frac{\partial a_1}{\partial\bar{y}_2}(\cdot)\cdot\frac{\mathrm{d}\bar{w}_{20}(\eta)}{\mathrm{d}\eta}\right]\cdot\bar{w}_{10}(\eta)\cdot\mathrm{d}\eta\\&=2a_1(*)\cdot\tilde{v}_{10}(0)-2f_{12}(c)\cdot\tilde{v}_{10}(0)+2\int_{0}^{+\infty}\left[\frac{\partial a_1}{\partial\tilde{y}_1}(*)\cdot\frac{\mathrm{d}\tilde{v}_{10}(\eta)}{\mathrm{d}\eta}+\frac{\partial a_1}{\partial\tilde{y}_2}(*)\cdot\frac{\mathrm{d}\tilde{v}_{20}(\eta)}{\mathrm{d}\eta}\right]\cdot\tilde{v}_{10}(\eta)\cdot\mathrm{d}\eta.\end{aligned}$$

即

$$\begin{aligned}&a_1(c,\gamma_{10},\gamma_{20})\cdot(\gamma_{10}-\varphi_1(c))-f_{11}(c)\cdot(\gamma_{10}-\varphi_1(c))\\&+a_1(c,\gamma_{10},\gamma_{20})\cdot(\psi_1(c)-\gamma_{10})+f_{12}(c)\cdot(\psi_1(c)-\gamma_{10})\\&=\int_{0}^{+\infty}\left[\frac{\partial a_1}{\partial\tilde{y}_1}(*)\cdot\frac{\mathrm{d}\tilde{v}_{10}(\eta)}{\mathrm{d}\eta}+\frac{\partial a_1}{\partial\tilde{y}_2}(*)\cdot\frac{\mathrm{d}\tilde{v}_{20}(\eta)}{\mathrm{d}\eta}\right]\cdot\tilde{v}_{10}(\eta)\cdot\mathrm{d}\eta\\&\quad+\int_{-\infty}^{0}\left[\frac{\partial a_1}{\partial\bar{y}_1}(\cdot)\cdot\frac{\mathrm{d}\bar{w}_{10}(\eta)}{\mathrm{d}\eta}+\frac{\partial a_1}{\partial\bar{y}_2}(\cdot)\cdot\frac{\mathrm{d}\bar{w}_{20}(\eta)}{\mathrm{d}\eta}\right]\cdot\bar{w}_{10}(\eta)\cdot\mathrm{d}\eta\\&=P_1(\gamma_{10},\gamma_{20}).\end{aligned}$$

考虑第二个分量，类似可得

$$\begin{aligned}&a_2(c,\gamma_{10},\gamma_{20})\cdot(\gamma_{20}-\varphi_2(c))-f_{21}(c)\cdot(\gamma_{20}-\varphi_2(c))\\&+a_2(c,\gamma_{10},\gamma_{20})\cdot(\psi_2(c)-\gamma_{20})+f_{22}(c)\cdot(\psi_2(c)-\gamma_{20})\\&=\int_{0}^{+\infty}\left[\frac{\partial a_2}{\partial\tilde{y}_1}(*)\cdot\frac{\mathrm{d}\tilde{v}_{10}(\eta)}{\mathrm{d}\eta}+\frac{\partial a_2}{\partial\tilde{y}_2}(*)\cdot\frac{\mathrm{d}\tilde{v}_{20}(\eta)}{\mathrm{d}\eta}\right]\cdot\tilde{v}_{20}(\eta)\cdot\mathrm{d}\eta\\&\quad+\int_{-\infty}^{0}\left[\frac{\partial a_2}{\partial\bar{y}_1}(\cdot)\cdot\frac{\mathrm{d}\bar{w}_{10}(\eta)}{\mathrm{d}\eta}+\frac{\partial a_2}{\partial\bar{y}_2}(\cdot)\cdot\frac{\mathrm{d}\bar{w}_{20}(\eta)}{\mathrm{d}\eta}\right]\cdot\bar{w}_{20}(\eta)\cdot\mathrm{d}\eta\\&=P_2(\gamma_{10},\gamma_{20}).\end{aligned}$$

其中，以上诸式中 $(\cdot)=(c,\varphi_1(c)+\bar{w}_{10}(0),\varphi_2(c)+\bar{w}_{20}(0))$，$(*)=(c,\psi_1(c)+\tilde{v}_{10}(0),\psi_2(c)+\tilde{v}_{20}(0))$

(H_5)假设方程关于 γ_{10},γ_{20} 的方程组

$$\begin{cases}\left(a_1\left(c,\gamma_{10},\gamma_{20}\right)-f_{11}\left(c\right)\right)\cdot\left(\gamma_{10}-\varphi_1\left(c\right)\right)+\left(a_1\left(c,\gamma_{10},\gamma_{20}\right)+f_{12}\left(c\right)\right)\cdot\left(\psi_1\left(c\right)-\gamma_{10}\right)=P_1\left(\gamma_{10},\gamma_{20}\right)\\ \left(a_2\left(c,\gamma_{10},\gamma_{20}\right)-f_{21}\left(c\right)\right)\cdot\left(\gamma_{20}-\varphi_2\left(c\right)\right)+\left(a_2\left(c,\gamma_{10},\gamma_{20}\right)+f_{22}\left(c\right)\right)\cdot\left(\psi_1\left(c\right)-\gamma_{20}\right)=P_2\left(\gamma_{10},\gamma_{20}\right)\end{cases}$$

有唯一解 γ_{10},γ_{20}。

至此，我们构造出原问题(1.1)的形式渐近解

$$\hat{y}_i\left(x\right)=\begin{cases}\overline{y}_i\left(x\right), & x\in\left[0,c\right),\\ \gamma_{10}, & x=c,\\ \tilde{y}_i\left(x\right), & x\in\left(c,1\right], \ i=1,2.\end{cases}\tag{2.16}$$

3. 主要结论及证明

定理 1 假设(H_1)-(H_5)成立，对于充分小的 $\varepsilon>0$，问题(1.1)存在解 $\left(y_1\left(x\right),y_2\left(x\right)\right)$，满足

$$\begin{aligned}&y_i\left(x\right)\in C\left[0,1\right]\cap C^2\left(\left[0,c\right)\cup\left(c,1\right]\right),\\ &y_i\left(x\right)=\hat{y}_i\left(x\right)+O\left(\varepsilon^{m+1}\right), \ i=1,2,\end{aligned}\tag{3.1}$$

其中 $\hat{y}_1\left(x\right),\hat{y}_2\left(x\right)$ 为式(2.16)所定义。

证明：构造辅助函数

$$\alpha_i\left(x\right)=\begin{cases}\overline{z}_i\left(x\right)-\gamma\varepsilon^{m+1}, & x\in\left[0,c\right),\\ \gamma_i-\gamma\varepsilon^{m+1}, & x=c, \ i=1,2,\\ \tilde{z}_i\left(x\right)-\gamma\varepsilon^{m+1}, & x\in\left(c,1\right],\end{cases}$$

$$\beta_i\left(x\right)=\begin{cases}\overline{z}_i\left(x\right)+\gamma\varepsilon^{m+1}, & x\in\left[0,c\right),\\ \gamma_i+\gamma\varepsilon^{m+1}, & x=c, \ i=1,2,\\ \tilde{z}_i\left(x\right)+\gamma\varepsilon^{m+1}, & x\in\left(c,1\right],\end{cases}$$

其中，

$$\begin{cases}\overline{z}_i=\sum_{j=0}^{m}\left(\overline{u}_{ij}\left(x\right)+\overline{v}_{ij}\left(\tau\right)+\overline{w}_{ij}\left(\eta\right)\right)\cdot\varepsilon^j,\\ \tilde{z}_i=\sum_{j=0}^{m}\left(\tilde{u}_{ij}\left(x\right)+\tilde{v}_{ij}\left(\eta\right)+\tilde{w}_{ij}\left(\xi\right)\right)\cdot\varepsilon^j,\end{cases} \quad i=1,2.$$

显然，有 $\alpha_1\left(x\right)\le\beta_1\left(x\right),\alpha_2\left(x\right)\le\beta_2\left(x\right)$。由边界条件易得，

$$\begin{cases}\alpha_1\left(0\right)\le p_1\le\beta_1\left(0\right), \ \alpha_2\left(0\right)\le p_2\le\beta_2\left(0\right),\\ \alpha_1\left(1\right)\le q_1\le\beta_1\left(1\right), \ \alpha_2\left(1\right)\le q_2\le\beta_2\left(1\right).\end{cases}$$

对 $x\in\left[0,c\right)$，由条件(H_3)及中值定理得

$$\begin{aligned}&a_1\left(x,\alpha_1\left(x\right),\overline{y}_2\right)-a_1\left(x,\overline{z}_1\left(x\right),\overline{z}_2\right)\\ &=\frac{\partial a_1}{\partial y_1}\cdot\left(\alpha_1-\overline{z}_1\right)+\frac{\partial a_1}{\partial y_2}\cdot\left(\overline{y}_2-\overline{z}_2\right)\le-\sigma_1\gamma\varepsilon^{m+1}+\frac{\sigma_1}{2}\gamma\varepsilon^{m+1}=-\frac{\sigma_1}{2}\gamma\varepsilon^{m+1}.\end{aligned}$$

由形式解的构造过程知

$$\left|\varepsilon^2\cdot\frac{\mathrm{d}^2\overline{z}_1}{\mathrm{d}x^2}-\left(a_1\left(x,\overline{z}_1,\overline{z}_2\right)-f_{11}\left(x\right)\right)\right|=O\left(\varepsilon^{m+1}\right).$$

因此，存在 $\varepsilon_1, \mu_1 > 0$，当 $0 \le x < c, 0 < \varepsilon < \varepsilon_1$ 时，有

$$\left|\varepsilon^2 \cdot \frac{d^2 \bar{z}_1}{dx^2} - \left(a_1\left(x, \bar{z}_1, \bar{z}_2\right) - f_{11}\left(x\right)\right)\right| \le \mu_1 \cdot \varepsilon^{m+1}.$$

那么，取 $\gamma \ge \dfrac{2\mu_1}{\sigma_1}$ 时，有

$$\begin{aligned}&\varepsilon^2 \alpha_1'' - a_1\left(x, \alpha_1, \bar{y}_2\right) + f_{11}\left(x\right)\\&= \varepsilon^2 \frac{d\bar{z}_1^2}{dx^2} - a_1\left(x, \bar{z}_1, \bar{z}_2\right) + f_{11}\left(x\right) + a_1\left(x, \bar{z}_1, \bar{z}_2\right) - a_1\left(x, \alpha_1, \bar{y}_2\right)\\&\ge \frac{\sigma_1}{2}\gamma\varepsilon^{m+1} - \mu_1\varepsilon^{m+1} = \left(\frac{\sigma_1}{2}\gamma - \mu_1\right) \cdot \varepsilon^{m+1} \ge 0.\end{aligned}$$

对 $x \in [0, c)$，存在 $\varepsilon_2, \mu_2 > 0$，$\gamma \ge \dfrac{2\mu_1}{\sigma_1}$，当 $0 < \varepsilon < \varepsilon_2$ 时，

$$\begin{aligned}&\varepsilon^2 \beta_1'' - a_1\left(x, \beta_1, \bar{y}_2\right) + f_{11}\left(x\right)\\&= \varepsilon^2 \frac{d\bar{z}_1^2}{dx^2} - a_1\left(x, \bar{z}_1, \bar{z}_2\right) + f_{11}\left(x\right) + a_1\left(x, \bar{z}_1, \bar{z}_2\right) - a_1\left(x, \beta_1, \bar{y}_2\right)\\&\le -\frac{\sigma_1}{2}\gamma\varepsilon^{m+1} + \mu_2\varepsilon^{m+1} \le 0.\end{aligned}$$

同理可证，在 $x \in (c, 1]$ 上有

$$\varepsilon^2 \alpha_1'' - a_1\left(x, \alpha_1, \tilde{y}_2\right) + f_{12}\left(x\right) \ge 0,$$

$$\varepsilon^2 \beta_1'' - a_1\left(x, \beta_1, \tilde{y}_2\right) + f_{12}\left(x\right) \le 0.$$

类似可证，在 $x \in [0, c) \cup (c, 1]$ 上有

$$\varepsilon^2 \alpha_2'' - a_2\left(x, \bar{y}_1, \alpha_2\right) + f_{21}\left(x\right) \ge 0, \varepsilon^2 \alpha_2'' - a_2\left(x, \bar{y}_1, \beta_2\right) + f_{22}\left(x\right) \ge 0$$

$$\varepsilon^2 \beta_2'' - a_2\left(x, \bar{y}_1, \beta_2\right) + f_{21}\left(x\right) \le 0, \varepsilon^2 \beta_2'' - a_2\left(x, \bar{y}_1, \beta_2\right) + f_{22}\left(x\right) \le 0.$$

因此，(α_1, α_2) 和 (β_1, β_2) 分别是问题(1.1)的一个下解和上解。由微分不等式定理知[2]，问题(1.1)存在解 $\left(y_1(x), y_2(x)\right)$ 且满足(3.1)。证毕。

4. 例子

考虑如下的边值问题：

$$\begin{cases}-\varepsilon^2 y_1'' + 5y_1 - 2y_2 = f_1(x), \ x \in \left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right],\\-\varepsilon^2 y_2'' - 2y_1 + 8y_2 = f_2(x), \ x \in \left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right],\\y_1(0) = 1, \ y_2(0) = 2,\\y_1(1) = 0, \ y_2(1) = 1,\end{cases} \tag{4.1}$$

其中，

$$f_1(x)=\begin{cases}x+2, & x\in\left[0,\frac{1}{2}\right),\\ 4, & x\in\left(\frac{1}{2},1\right],\end{cases}\qquad f_2(x)=\begin{cases}x-2, & x\in\left[0,\frac{1}{2}\right),\\ 2, & x\in\left(\frac{1}{2},1\right].\end{cases}$$

显然，由 $f_{11}(x)=x+2, f_{21}(x)=x-2$ 可知，$\varphi_1(x)=\frac{5x+6}{18},\varphi_2(x)=\frac{7x-6}{36}$，且由 $f_{12}=4, f_{22}=2$ 知 $\psi_1(x)=1,\psi_2(x)=\frac{1}{2}$。

可将问题(4.1)看成如下左、右问题的耦合。

左问题(L)：

$$\begin{cases}-\varepsilon^2 y_1''+5y_1-2y_2=x+2, & x\in\left[0,\frac{1}{2}\right),\\ -\varepsilon^2 y_2''-2y_1+8y_2=x-2, & x\in\left[0,\frac{1}{2}\right),\\ y_1(0)=1,\ y_2(0)=2,\\ y_1\left(\frac{1}{2}\right)=\gamma_1(\varepsilon),\ y_2\left(\frac{1}{2}\right)=\gamma_2(\varepsilon).\end{cases}\tag{4.2}$$

右问题(R)：

$$\begin{cases}-\varepsilon^2 y_1''+5y_1-2y_2=4, & x\in\left(\frac{1}{2},1\right],\\ -\varepsilon^2 y_2''-2y_1+8y_2=2, & x\in\left(\frac{1}{2},1\right],\\ y_1(1)=0,\ y_2(1)=1,\\ y_1\left(\frac{1}{2}\right)=\gamma_1(\varepsilon),\ y_2\left(\frac{1}{2}\right)=\gamma_2(\varepsilon).\end{cases}\tag{4.3}$$

其中 $\gamma_1(\varepsilon)=\gamma_{10}+\varepsilon\gamma_{11}+\cdots$，$\gamma_2(\varepsilon)=\gamma_{20}+\varepsilon\gamma_{21}+\cdots$ 为 ε 有关的待定常数。

由渐近解的构造方法可求出左、右问题的零阶近似：

左问题(4.2)的边界层函数 $\bar{V}_{10}(\tau),\bar{V}_{20}(\tau),\bar{W}_{10}(\eta),\bar{W}_{20}(\eta)$：

$$\bar{V}_{10}(\tau)=-\frac{11}{15}e^{-3\tau}+\frac{7}{5}e^{-2\tau},\ \bar{V}_{20}(\tau)=\frac{22}{15}e^{-3\tau}+\frac{7}{10}e^{-2\tau},$$

$$\bar{W}_{10}(\eta)=\left(2\gamma_{10}-\frac{7}{20}\right)e^{2\eta}+\left(\gamma_{20}-\frac{11}{90}\right)e^{3\eta},$$

$$\bar{W}_{20}(\eta)=\left(\gamma_{10}-\frac{7}{40}\right)e^{2\eta}-\left(2\gamma_{20}-\frac{22}{90}\right)e^{3\eta}.$$

右问题(4.3)的边界层函数 $\tilde{V}_{10}(\eta),\tilde{V}_{20}(\eta),\tilde{W}_{10}(\xi),\tilde{W}_{20}(\xi)$：

$$\tilde{V}_{10}(\eta)=(2\gamma_{10}-1)e^{-2\eta}+\gamma_{20}e^{-3\eta},$$

$$\tilde{V}_{20}(\eta)=-2\gamma_{20}e^{-3\eta}+\left(\gamma_{10}-\frac{1}{2}\right)e^{-2\eta},$$

$$\tilde{W}_{10}(\xi)=-\frac{3}{5}\mathrm{e}^{-2\xi}-\frac{2}{5}\mathrm{e}^{-3\xi},\ \tilde{W}_{20}(\xi)=-\frac{3}{10}\mathrm{e}^{-2\xi}+\frac{4}{5}\mathrm{e}^{-3\xi}.$$

其中，$\tau=x\cdot\varepsilon^{-1},\eta=\left(x-\frac{1}{2}\right)\cdot\varepsilon^{-1},\xi=(x-1)\cdot\varepsilon^{-1}$。

由式子(2.24)，可得$\gamma_{10}=\frac{27}{80},\gamma_{20}=\frac{11}{180}$，因此边值问题(4.1)的零阶近似解为：

$$\hat{y}_1(x)=\begin{cases}\frac{5x+6}{18}-\frac{11}{15}\mathrm{e}^{-\frac{3x}{\varepsilon}}+\frac{7}{5}\mathrm{e}^{-\frac{2x}{\varepsilon}}+\frac{11}{720}\mathrm{e}^{\frac{6x-3}{2\varepsilon}}+\frac{13}{40}\mathrm{e}^{\frac{2x-1}{\varepsilon}},\ x\in\left[0,\frac{1}{2}\right),\\ \frac{27}{80},\ x=\frac{1}{2},\\ 1-\frac{13}{40}\mathrm{e}^{\frac{1-2x}{\varepsilon}}-\frac{3}{5}\mathrm{e}^{\frac{2x-2}{\varepsilon}}+\frac{11}{180}\mathrm{e}^{\frac{3-6x}{2\varepsilon}}-\frac{2}{5}\mathrm{e}^{\frac{3x-3}{\varepsilon}},\ x\in\left(\frac{1}{2},1\right].\end{cases}$$

$$\hat{y}_2(x)=\begin{cases}\frac{7x-6}{36}+\frac{22}{15}\mathrm{e}^{-\frac{3x}{\varepsilon}}+\frac{7}{10}\mathrm{e}^{-\frac{2x}{\varepsilon}}-\frac{11}{360}\mathrm{e}^{\frac{6x-3}{2\varepsilon}}+\frac{13}{180}\mathrm{e}^{\frac{2x-1}{\varepsilon}},\ x\in\left[0,\frac{1}{2}\right),\\ \frac{11}{180},\ x=\frac{1}{2},\\ \frac{1}{2}-\frac{13}{180}\mathrm{e}^{\frac{1-2x}{\varepsilon}}-\frac{3}{10}\mathrm{e}^{\frac{2x-2}{\varepsilon}}-\frac{11}{90}\mathrm{e}^{\frac{3-6x}{2\varepsilon}}+\frac{4}{5}\mathrm{e}^{\frac{3x-3}{\varepsilon}},\ x\in\left(\frac{1}{2},1\right].\end{cases}$$

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