Advances in Applied Mathematics 应用数学进展, 2019, 8(10), 1636-1649
Published Online October 2019 in Hans. http://www.hanspub.org/journal/aam
https://doi.org/10.12677/aam.2019.810193

Two Classes of Third Order Boundary Value Problems of Finite Spectrum with Transmission Conditions and Spectral Parameters in the Boundary Conditions

Junwei Zhu

School of Science, Lanzhou University of Technology, Lanzhou Gansu
Email: 1739274449@qq.com

Received: Oct. 2nd, 2019; accepted: Oct. 21st, 2019; published: Oct. 28th, 2019

Abstract

In this paper, we study the following two classes of third order boundary value problems of finite spectrum with transmission conditions and spectral parameters in the boundary conditions

$$\begin{cases}(py'')'+qy=\lambda wy\\ A_\lambda Y(a)+B_\lambda Y(b)=0\\ CY(c-)+DY(c+)=0\end{cases},\begin{cases}(py')''+qy=\lambda wy\\ A_\lambda Y(a)+B_\lambda Y(b)=0\\ CY(c-)+DY(c+)=0\end{cases},$$

for any positive integer *n*,*m*, based on the spectral parameters in the boundary conditions, we construct two classes of third order boundary value problems with transmission conditions, and it is calculated that there are at most $m+n+2$ eigenvalues. The main tool used in this paper is iterative construction of the characteristic function and Rouche's theorem.

Keywords

Transmission Conditions, Spectral Parameters, Characteristic Function, Rouche's Theorem

具有转移条件且边界条件中含有谱参数的两类三阶有限谱边值问题

朱军伟

兰州理工大学理学院，甘肃 兰州
Email: 1739274449@qq.com

文章引用: 朱军伟. 具有转移条件且边界条件中含有谱参数的两类三阶有限谱边值问题[J]. 应用数学进展, 2019, 8(10): 1636-1649. DOI: 10.12677/aam.2019.810193

收稿日期：2019年10月2日；录用日期：2019年10月21日；发布日期：2019年10月28日

摘　要

本文主要研究下述具有转移条件且边界条件中含有谱参数的两类三阶有限谱边值问题

$$\begin{cases}(py'')'+qy=\lambda wy\\ A_\lambda Y(a)+B_\lambda Y(b)=0\\ CY(c-)+DY(c+)=0\end{cases},\begin{cases}(py')''+qy=\lambda wy\\ A_\lambda Y(a)+B_\lambda Y(b)=0\\ CY(c-)+DY(c+)=0\end{cases},$$

对于任意正整数*n,m*，在边界条件含有谱参数的基础上，通过构造带有转移条件的两类三阶边值问题，经计算，至多有 $m+n+2$ 个特征值。所用的工具主要是判断函数的迭代和Rouche定理。

关键词

转移条件，谱参数，判断函数，Rouche定理

1. 引言

近年来，在二阶 Sturm-Liouville 有限谱问题[1]的研究基础上，对于 Sturm-Liouville 问题阶数的研究逐渐推广到四阶[2]，甚至 $2n$ 阶[3]，正因为都是偶数阶，所以可以很自然的推广到高阶偶数阶问题上去。但是对于奇数阶具有有限谱的微分方程边值问题的研究还是比较困难的，随着 Ao 等人对于两类具有有限谱的三阶微分方程边值问题的提出，使得三阶微分方程边值问题成为解决实际问题的关键[4] [5]。理论上三阶微分方程边值问题是否具有有限谱，它的转移条件是否对于其特征值个数会产生影响，以及带谱参数的边界条件是否会对其特征值个数产生影响，对于这些问题的研究都是有必要的。尤其将转移条件与带谱参数边界条件结合在一起研究两类三阶微分方程边值问题是非常有意义的。

目前，学者们对于微分方程边值问题是否具有有限谱的研究做出了大量的杰出工作[6] [7]。2013 年，[8]讨论了一类边界条件带有谱参数且具有转移条件的正则的 Sturm-Liouville 问题的谱个数。值得一提的是，在 2017 年，[9]研究了下述两类具有有限谱的三阶边值问题

$$\begin{cases}(py'')'+qy=\lambda wy\\ AY(a)+BY(b)=0\end{cases},\begin{cases}(py')''+qy=\lambda wy\\ AY(a)+BY(b)=0\end{cases}$$

对于每一个正整数 m，该问题至多有 $2m+1$ 个特征值，他们运用的主要工具有判断函数的迭代、代数学基本定理等。此外，其他关于微分方程边值问题[10] [11] [12]以及 Sturm-Liouville 问题是否具有有限谱的文章可参见文献[13] [14] [15]。

受以上杰出工作的启发，本文考虑下述具有转移条件且边界条件中含有谱参数的两类三阶有限谱边值问题(简记为 BVPS)

$$\begin{cases} (py'')' + qy = \lambda wy \\ A_\lambda Y(a) + B_\lambda Y(b) = 0 \\ CY(c-) + DY(c+) = 0 \end{cases} \tag{1}$$

$$\begin{cases} (py')'' + qy = \lambda wy \\ A_\lambda Y(a) + B_\lambda Y(b) = 0 \\ CY(c-) + DY(c+) = 0 \end{cases} \tag{2}$$

其中， $y = y(t)$， $t \in J = (a,c) \cup (c,b)$， $-\infty < a < b < +\infty$。 $A_\lambda = \begin{pmatrix} \lambda\alpha_1' - \alpha_1 & -\lambda\alpha_2' + \alpha_2 & \lambda\alpha_3' + \alpha_3 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$，

$B_\lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda\beta_1' + \beta_1 & -\lambda\beta_2' - \beta_2 & \lambda\beta_3' - \beta_3 \end{pmatrix}$， $\alpha_i, \alpha_i', \beta_i, \beta_i' \in R, i = 1,2,3$。且满足 $\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1' & \alpha_2' & \alpha_3' \\ 1 & 1 & 1 \end{vmatrix} \neq 0$，

$\begin{vmatrix} \beta_1 & \beta_2 & \beta_3 \\ \beta_1' & \beta_2' & \beta_3' \\ 1 & 1 & 1 \end{vmatrix} \neq 0$， $C, D \in M_3(\mathbb{R})$， $|C| = \rho > 0$， $|D| = \theta > 0$。此处 λ 为谱参数，且系数满足基本条件

$$r = \frac{1}{p}, q, w \in L(J, \mathbb{C}). \tag{3}$$

其中 $L(J, \mathbb{C})$ 表示在 J 上 Lebesgue 可积的复值函数构成的集合。本文基于边界条件中含有谱参数，通过构造带有转移条件的两类三阶边值问题至多有 $m+n+2$ 个特征值，同时还进一步用判断函数验证了特征值个数。

2. 预备知识及说明

令 $u_1 = y, u_2 = y', u_3 = py''$，则与方程 $(py'')' + qy = \lambda wy$ 等价的系统表示为：

$$u_1' = u_2, u_2' = ru_3, u_3' = (\lambda w - q)u_1. \tag{4}$$

可写成如下矩阵形式

$$\begin{pmatrix} y \\ y' \\ py'' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda w - q & 0 & 0 \end{pmatrix} \begin{pmatrix} y \\ y' \\ py'' \end{pmatrix}. \tag{5}$$

同时有

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda w - q & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}. \tag{6}$$

记

$$A(t)=\begin{pmatrix}0&1&0\\0&0&1\\\lambda w-q&0&0\end{pmatrix}$$

容易得出 $A(t)$ 在 J 上是可积的。

定义 1 设 $y=y(t)$ 为问题的解

若 $y\equiv 0,u_2=y'\equiv 0,u_3=py''\equiv 0$，则称 y 为问题的平凡解，反之称为非平凡解；

若对于某个 λ 值，问题存在非平凡解，则 λ 就为该问题的特征值，对应的非平凡解称为对应于特征值 λ 的特征函数。

引理 1 假设(3)成立，$\Phi(x,\lambda)=\left[\phi_{ij}(x,\lambda)\right]$ 为满足 $\Phi(a,\lambda)=I$ 的(4)的基解矩阵，则 $\lambda\in\mathbb{C}$ 是具有转移条件且边界条件中含有谱参数的第一类三阶有限谱边值问题(1)的特征值当且仅当

$$\Delta(\lambda)=det\left[A_\lambda+B_\lambda\Phi(b,\lambda)\right]=0. \tag{7}$$

特别地：

$$\Delta(\lambda)=\sum_{i=1,j=1}^{3}h_{ij}(\lambda)\phi_{ij}(b,\lambda), \tag{8}$$

其中

$$H(\lambda):=\begin{pmatrix}h_{11}(\lambda)&h_{12}(\lambda)&h_{13}(\lambda)\\h_{21}(\lambda)&h_{22}(\lambda)&h_{23}(\lambda)\\h_{31}(\lambda)&h_{32}(\lambda)&h_{33}(\lambda)\end{pmatrix}. \tag{9}$$

$$h_{11}(\lambda)=-(\lambda\beta_1'+\beta_1)(\lambda\alpha_2'-\alpha_2+\lambda\alpha_3'+\alpha_3);\quad h_{12}(\lambda)=(\lambda\beta_1'+\beta_1)(\lambda\alpha_3'+\alpha_3-\lambda\alpha_1'+\alpha_1);$$

$$h_{13}(\lambda)=-(\lambda\beta_1'+\beta_1)(\lambda\alpha_2'+\alpha_2-\lambda\alpha_1'+\alpha_1);\quad h_{21}(\lambda)=(\lambda\beta_2'+\beta_2)(\lambda\alpha_3'+\alpha_3+\lambda\alpha_2'-\alpha_2);$$

$$h_{22}(\lambda)=(\lambda\beta_2'+\beta_2)(\lambda\alpha_1'-\alpha_1-\lambda\alpha_3'-\alpha_3);\quad h_{23}(\lambda)=-(\lambda\beta_2'+\beta_2)(\lambda\alpha_2'-\alpha_2+\lambda\alpha_1'-\alpha_1);$$

$$h_{31}(\lambda)=-(\lambda\beta_3'-\beta_3)(\lambda\alpha_2'-\alpha_2+\lambda\alpha_3'+\alpha_3);\quad h_{32}(\lambda)=(\lambda\beta_3'-\beta_3)(\lambda\alpha_3'+\alpha_3-\lambda\alpha_1'+\alpha_1);$$

$$h_{33}(\lambda)=(\lambda\beta_3'-\beta_3)(\lambda\alpha_1'-\alpha_1-\lambda\alpha_2'-\alpha_2)$$

证明 设 $\Delta(\lambda)=0$，则 $\left[A_\lambda+B_\lambda\Phi(b,\lambda)\right]C=0$ 有一个平凡向量解。解初值问题：

$$Y'=\begin{pmatrix}0&1&0\\0&0&\frac{1}{p}\\\lambda w-q&0&0\end{pmatrix}Y,\quad Y(a)=C,$$

得到 $Y(b)=\Phi(b,\lambda)Y(a)$ 和 $\left[A_\lambda+B_\lambda\Phi(b,\lambda)\right]Y(a)=0$。由此可得 Y 的第一个分量 y 是问题(1)的特征函数，这表明 λ 是这个问题的一个特征值。相反地，如果 λ 是一个特征值，并且 Y 为 λ 所对应的特征函数，则 $Y=\begin{pmatrix}y\\y'\\py''\end{pmatrix}$ 满足 $Y(b)=\Phi(b,\lambda)Y(a)$，因此 $\left[A_\lambda+B_\lambda\Phi(b,\lambda)\right]Y(a)=0$，若 $Y(a)=0$，则 y 是平凡解，这与 y 是特征函数矛盾，故有 $\Delta(\lambda)=det\left[A_\lambda+B_\lambda\Phi(b,\lambda)\right]=0$。式(8)直接计算可得。

$$
\begin{aligned}
\Delta(\lambda) &= det\left[A_\lambda + B_\lambda \Phi(b,\lambda)\right] \\
&= det\left[\begin{pmatrix} \lambda\alpha_1' - \alpha_1 & -\lambda\alpha_2' + \alpha_2 & \lambda\alpha_3' + \alpha_3 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right. \\
&\quad \left. + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda\beta_1' + \beta_1 & -\lambda\beta_2' - \beta_2 & \lambda\beta_3' - \beta_3 \end{pmatrix} \begin{pmatrix} \phi_{11}(b,\lambda) & \phi_{12}(b,\lambda) & \phi_{13}(b,\lambda) \\ \phi_{21}(b,\lambda) & \phi_{22}(b,\lambda) & \phi_{23}(b,\lambda) \\ \phi_{31}(b,\lambda) & \phi_{32}(b,\lambda) & \phi_{33}(b,\lambda) \end{pmatrix}\right] \\
&= h_{11}(\lambda)\phi_{11}(b,\lambda) + h_{12}(\lambda)\phi_{12}(b,\lambda) + h_{13}(\lambda)\lambda\phi_{13}(b,\lambda) + h_{21}(\lambda)\phi_{21}(b,\lambda) + h_{22}(\lambda)\phi_{22}(b,\lambda) \\
&\quad + h_{23}(\lambda)\phi_{23}(b,\lambda) + h_{31}(\lambda)\phi_{31}(b,\lambda) + h_{32}(\lambda)\phi_{32}(b,\lambda) + h_{33}(\lambda)\lambda\phi_{33}(b,\lambda)
\end{aligned}
$$

其中

$$
H(\lambda) := \begin{pmatrix} h_{11}(\lambda) & h_{12}(\lambda) & h_{13}(\lambda) \\ h_{21}(\lambda) & h_{22}(\lambda) & h_{23}(\lambda) \\ h_{31}(\lambda) & h_{32}(\lambda) & h_{33}(\lambda) \end{pmatrix},
$$

综合上面证明，可知(8)明显成立。 □

定义 2 若对所有的 $\lambda \in \mathbb{C}, \Delta\lambda = 0$，或对任意的 $\lambda \in \mathbb{C}\Delta(\lambda) \neq 0$，则称具有转移条件且边界条件中含有谱参数的三阶有限谱边值问题(1) (或等价问题具有转移条件且边界条件中含有谱参数的三阶有限谱问题(4))为退化的。

3. 具有转移条件且边界条件中含有谱参数的三阶有限谱边值问题

在这一部分中我们设(3)成立，并且对于正整数 m, n 对区间 $J = (a,c) \cup (c,b)$ 有分割

$$
\begin{aligned}
&a = a_0 < a_1 < a_2 < \cdots < a_{2m} < a_{2m+1} = c, \\
&c = b_0 < b_1 < b_2 < \cdots < b_{2n} < b_{2n+1} = b,
\end{aligned} \tag{10}
$$

使得满足当 $r(t) = \dfrac{1}{p(t)} = 0$ 时，有

$$
\begin{aligned}
&\int_{a_{2k}}^{a_{2k+1}} w(t)\mathrm{d}t \neq 0, \quad \int_{a_{2k}}^{a_{2k+1}} w(t)t\mathrm{d}t \neq 0, \quad k = 0,1,\cdots,m, \quad t \in (a_{2k}, a_{2k+1}), \\
&\int_{b_{2i}}^{a_{2i+1}} w(t)\mathrm{d}t \neq 0, \quad \int_{a_{2i}}^{a_{2i+1}} w(t)t\mathrm{d}t \neq 0, \quad i = 0,1,\cdots,n, \quad t \in (b_{2i}, a_{2i+1}),
\end{aligned} \tag{11}
$$

而 $q(t) = w(t) = 0$ 时，则有

$$
\begin{aligned}
&\int_{a_{2k+1}}^{a_{2k+2}} r(t)\mathrm{d}t \neq 0, \quad \int_{a_{2k+1}}^{a_{2k+2}} r(t)t\mathrm{d}t \neq 0, \quad k = 0,1,\cdots,m-1, \quad t \in (a_{2k+1}, a_{2k+2}), \\
&\int_{b_{2i+1}}^{b_{2i+2}} r(t)\mathrm{d}t \neq 0, \quad \int_{b_{2i+1}}^{b_{2i+2}} r(t)t\mathrm{d}t \neq 0, \quad i = 0,1,\cdots,n-1, \quad t \in (b_{2i+1}, b_{2i+2}),
\end{aligned} \tag{12}
$$

由(10)~(12)，令

$$
\begin{aligned}
&r_k = \int_{a_{2k+1}}^{a_{2k+2}} r, \quad \hat{r}_k = \int_{a_{2k+1}}^{a_{2k+2}} r(t)t\mathrm{d}t, \quad k = 0,1,\cdots,m-1; \\
&q_k = \int_{a_{2k}}^{a_{2k+1}} q, \quad \hat{q}_k = \int_{a_{2k}}^{a_{2k+1}} q(t)t\mathrm{d}t, \quad k = 0,1,\cdots,m; \\
&w_k = \int_{a_{2k}}^{a_{2k+1}} w, \quad \hat{w}_k = \int_{a_{2k}}^{a_{2k+1}} w(t)t\mathrm{d}t, \quad k = 0,1,\cdots,m; \\
&\tilde{r}_i = \int_{b_{2i+1}}^{b_{2i+2}} r, \quad \breve{r}_i = \int_{b_{2i+1}}^{b_{2i+2}} r(t)t\mathrm{d}t, \quad i = 0,1,\cdots,n-1; \\
&\tilde{q}_i = \int_{b_{2i}}^{b_{2i+1}} q, \quad \breve{q}_i = \int_{b_{2i}}^{b_{2i+1}} q(t)t\mathrm{d}t, \quad k = 0,1,\cdots,n; \\
&w_i = \int_{b_{2i}}^{b_{2i+1}} w, \quad \hat{w}_i = \int_{b_{2i}}^{b_{2i+1}} w(t)t\mathrm{d}t, \quad k = 0,1,\cdots,n.
\end{aligned} \tag{13}
$$

引理 2 设(3)，(10)~(12)成立。令 $\Phi(t,\lambda)=\left[\phi_{ij}(t,\lambda)\right]$ 为系统(4)满足初始条件 $\Phi(a,\lambda)=I$ 的的基解矩阵。则有

$$\Phi(a_1,\lambda)=\begin{pmatrix} 1 & a_1-a_0 & 0 \\ 0 & 1 & 0 \\ \lambda w_0-q_0 & (\lambda w_0-q_0)(a_1-a_0) & 1 \end{pmatrix}, \tag{14}$$

$$\Phi(a_3,\lambda)=\begin{pmatrix} \phi_{11}(a_3,\lambda) & \phi_{12}(a_3,\lambda) & r_0(a_3-a_1) \\ r_0(\lambda w_0-q_0) & 1+r_0(\lambda w_0-q_0)(a_1-a_0) & r_0 \\ \phi_{31}(a_3,\lambda) & \phi_{32}(a_3,\lambda) & (\lambda w_1-q_1)\left[r_0(a_3-a_1)\right]+1 \end{pmatrix}, \tag{15}$$

其中

$$\phi_{11}(a_3,\lambda)=(a_3-a_1)r_0(\lambda w_0-q_0)+1,$$

$$\phi_{12}(a_3,\lambda)=(a_3-a_1)\left[r_0(\lambda w_0-q_0)(a_1-a_0)+1\right]+(a_1-a_0),$$

$$\phi_{31}(a_3,\lambda)=\left[(a_3-a_1)r_0(\lambda w_0-q_0)+1\right](\lambda w_1-q_1)+(\lambda w_0-q_0),$$

$$\phi_{32}(a_3,\lambda)=(\lambda w_1-q_1)(a_3-a_1)r_0+1.$$

更一般地，对 $1\le i\le m$，有

$$\Phi(a_{2i+1},\lambda)=\begin{pmatrix} 1 & 1 & r_{i-1} \\ 0 & 1 & r_{i-1} \\ \lambda w_i-q_i & \lambda w_i-q_i & r_{i-1}(\lambda w_i-q_i) \end{pmatrix}\Phi(a_{2i-1},\lambda). \tag{16}$$

证明 由(4)可知在 r 恒等于零的子区间上 u_2 是常数，而在 q,w 恒等于零的子区间上 u_3 是常数。通过反复应用(4)，即可得出结论。 □

引理 3 设(3)，(10)~(12)成立。令 $\Psi(t,\lambda)=\left[\psi_{ij}(t,\lambda)\right]$ 为系统(4)满足初始条件 $\Psi(c,\lambda)=I$ (此处 $\Psi(c,\lambda)=\Psi(c+,\lambda)$ 表示在 c 处的右极限)的基解矩阵。则有

$$\Psi(b_1,\lambda)=\begin{pmatrix} 1 & b_1-b_0 & 0 \\ 0 & 1 & 0 \\ \lambda\tilde{w}_0-\tilde{q}_0 & (\lambda\tilde{w}_0-\tilde{q}_0)(b_1-b_0) & 1 \end{pmatrix}, \tag{17}$$

$$\Psi(b_3,\lambda)=\begin{pmatrix} \psi_{11}(b_3,\lambda) & \psi_{12}(b_3,\lambda) & \tilde{r}_0(b_3-b_1) \\ \tilde{r}_0(\lambda\tilde{w}_0-\tilde{q}_0) & 1+\tilde{r}_0(\lambda\tilde{w}_0-\tilde{q}_0)(b_1-b_0) & \tilde{r}_0 \\ \psi_{31}(b_3,\lambda) & \psi_{32}(b_3,\lambda) & (\lambda\tilde{w}_1-\tilde{q}_1)\left[\tilde{r}_0(b_3-b_1)\right]+1 \end{pmatrix}, \tag{18}$$

其中

$$\psi_{11}(b_3,\lambda)=(b_3-b_1)\tilde{r}_0(\lambda\tilde{w}_0-\tilde{q}_0)+1,$$

$$\psi_{12}(b_3,\lambda)=(b_3-b_1)\left[\tilde{r}_0(\lambda\tilde{w}_0-\tilde{q}_0)(b_1-b_0)+1\right]+(b_1-b_0),$$

$$\psi_{31}(b_3,\lambda)=\left[(b_3-b_1)\tilde{r}_0(\lambda\tilde{w}_0-\tilde{q}_0)+1\right](\lambda\tilde{w}_1-\tilde{q}_1)+(\lambda\tilde{w}_0-\tilde{q}_0),$$

$$\psi_{32}(b_3,\lambda)=(\lambda\tilde{w}_1-\tilde{q}_1)(b_3-b_1)\tilde{r}_0+1.$$

更一般地，对 $1\le j\le n$，有

$$\Psi(b_{2j+1},\lambda)=\begin{pmatrix}1 & 1 & \tilde{r}_{j-1}\\ 0 & 1 & r_{j-1}\\ \lambda\tilde{w}_j-\tilde{q}_j & \lambda\tilde{w}_j-\tilde{q}_j & \tilde{r}_{j-1}(\lambda\tilde{w}_j-\tilde{q}_j)\end{pmatrix}\Psi(b_{2j-1},\lambda). \tag{19}$$

证明 证明与引理 2 相同。 □

引理 4 设(3)，(10)~(12)成立。令 $\Phi(t,\lambda)=[\phi_{ij}(t,\lambda)]$ 为系统(4)满足初始条件 $\Phi(a,\lambda)=I$ 的基解矩阵，且 $\Psi(t,\lambda)=[\psi_{ij}(t,\lambda)]$ 如引理 3 中给出。则有

$$\Phi(b,\lambda)=\Psi(b,\lambda)G\Phi(c,\lambda), \tag{20}$$

其中 $G=[g_{ij}]_{3\times3}=-D^{-1}C$，而 $\Psi(c,\lambda)=\Psi(c-,\lambda)$ 表示在 c 点处的左极限。

证明 由转移条件 $CY(c-)+DY(c+)=0$ 可知

$$C\Phi(c-,\lambda)+D\Phi(c+,\lambda)=0,$$

从而有

$$\Phi(c+,\lambda)=-D^{-1}C\Phi(c-,\lambda)=G\Phi(c-,\lambda),$$

注意到 $\Phi(c,\lambda)=\Phi(c-,\lambda)=\Phi(a_{2m+1},\lambda),\Psi(b,\lambda)=\Phi(b_{2n+1},\lambda)$ 及 $\Phi(c-\lambda)=I$，故由引理(2)，(3)得

$$\Phi(b,\lambda)=\Psi(b,\lambda)G\Phi(c,\lambda),$$

其中 $\Psi(b,\lambda)=\Phi(b_{2n+1},\lambda)$。证毕。 □

引理 5 设(3)，(10)~(12)成立。令 $\Phi(t,\lambda)=[\phi_{ij}(t,\lambda)]$ 为系统(4)满足初始条件 $\Phi(a,\lambda)=I$ 的的基解矩阵，对于每一个 $\lambda\in C$，$\Phi(b,\lambda)$ 则有如下结果

$$\phi_{11}(b,\lambda)=RR^*G^*\cdot\prod_{k=0}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{11}(b,\lambda),$$

$$\phi_{12}(b,\lambda)=RR^*G^*\cdot\prod_{k=0}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{12}(b,\lambda),$$

$$\phi_{13}(b,\lambda)=RR^*G^*\cdot\prod_{k=1}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{13}(b,\lambda),$$

$$\phi_{21}(b,\lambda)=RR^*G^*\cdot\prod_{k=0}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{21}(b,\lambda),$$

$$\phi_{22}(b,\lambda)=RR^*G^*\cdot\prod_{k=0}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{22}(b,\lambda),$$

$$\phi_{23}(b,\lambda)=RR^*G^*\cdot\prod_{k=1}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{23}(b,\lambda),$$

$$\phi_{31}(b,\lambda)=RR^*G^*\cdot\prod_{k=0}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{31}(b,\lambda),$$

$$\phi_{32}(b,\lambda)=RR^*G^*\cdot\prod_{k=0}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{32}(b,\lambda),$$

$$\phi_{33}(b,\lambda)=RR^*G^*\cdot\prod_{k=1}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{33}(b,\lambda),$$

其中

$$G^{*}=g_{11}\left(\lambda w_0-q_0\right)+g_{21}\left(\lambda w_0-q_0\right)+g_{31}+g_{12}\left(\lambda w_0-q_0\right)+g_{22}\left(\lambda w_0-q_0\right)$$
$$+g_{32}+g_{13}\left(\lambda w_0-q_0\right)+g_{23}\left(\lambda w_0-q_0\right)+g_{33},$$

$$R=\prod_{k=0}^{m-1}r_k,\ R^{*}=\prod_{i=0}^{n-1}r_i,\ \phi_{ij}'\left(b,\lambda\right)=o\left(RR^{*}\right),\ i,j=1,2,3.$$

证明 由引理 1 可知

$$\begin{aligned}\Phi(c,\lambda)&=\Phi\left(a_{2m+1},\lambda\right)\\
&=\begin{pmatrix}1&0&r_{m-1}\\0&1&r_{m-1}\\\lambda w_m-q_m&\lambda w_m-q_m&\left(\lambda w_m-q_m\right)r_{m-1}\end{pmatrix}\Phi\left(a_{2m-1},\lambda\right)\\
&=\begin{pmatrix}1&1&r_{m-1}\\0&1&r_{m-1}\\\lambda w_m-q_m&\lambda w_m-q_m&\left(\lambda w_m-q_m\right)r_{m-1}\end{pmatrix}\\
&\quad\times\begin{pmatrix}1&1&r_{m-2}\\0&1&r_{m-2}\\\lambda w_{m-1}-q_{m-1}&\lambda w_{m-1}-q_{m-1}&\left(\lambda w_{m-1}-q_{m-1}\right)r_{m-2}\end{pmatrix}\Phi\left(a_{2m-3},\lambda\right)\\
&=\begin{pmatrix}\theta_{11}&\theta_{12}&\theta_{13}\\\theta_{21}&\theta_{22}&\theta_{13}\\\theta_{21}&\theta_{22}&\theta_{13}\end{pmatrix}\Phi\left(a_{2m-3},\lambda\right),\end{aligned}$$

其中

$$\begin{aligned}
\theta_{11}&=1+r_{m-1}\left(\lambda w_{m-1}-q_{m-1}\right)=r_{m-1}\left(\lambda w_{m-1}-q_{m-1}\right)+o\left(\lambda w_{m-1}-q_{m-1}\right),\\
\theta_{12}&=2+r_{m-1}\left(\lambda w_{m-1}-q_{m-1}\right)=r_{m-1}\left(\lambda w_{m-1}-q_{m-1}\right)+o\left(\lambda w_{m-1}-q_{m-1}\right),\\
\theta_{13}&=2r_{m-2}+r_{m-1}r_{m-2}\left(\lambda w_{m-1}-q_{m-1}\right)=r_{m-1}r_{m-2}\left(\lambda w_{m-1}-q_{m-1}\right)+o\left(r_{m-1}r_{m-2}\left(\lambda w_{m-1}-q_{m-1}\right)\right),\\
\theta_{21}&=r_{m-1}\left(\lambda w_{m-1}-q_{m-1}\right)=r_{m-1}\left(\lambda w_{m-1}-q_{m-1}\right)+o\left(\lambda w_{m-1}-q_{m-1}\right),\\
\theta_{22}&=1+r_{m-1}\left(\lambda w_{m-1}-q_{m-1}\right)=r_{m-1}\left(\lambda w_{m-1}-q_{m-1}\right)+o\left(r_{m-1}\left(\lambda w_{m-1}-q_{m-1}\right)\right),\\
\theta_{23}&=r_{m-2}+r_{m-1}r_{m-2}\left(\lambda w_{m-1}-q_{m-1}\right)=r_{m-1}r_{m-2}\left(\lambda w_{m-1}-q_{m-1}\right)+o\left(r_{m-1}r_{m-2}\left(\lambda w_{m-1}-q_{m-1}\right)\right),\\
\theta_{31}&=\lambda w_m-q_m+r_{m-1}\left(\lambda w_m-q_m\right)\left(\lambda w_{m-1}-q_{m-1}\right)\\
&=r_{m-1}\left(\lambda w_m-q_m\right)\left(\lambda w_{m-1}-q_{m-1}\right)+o\left(r_{m-1}\left(\lambda w_m-q_m\right)\left(\lambda w_{m-1}-q_{m-1}\right)\right),\\
\theta_{32}&=2\left(\lambda w_m-q_m\right)+r_{m-1}\left(\lambda w_m-q_m\right)\left(\lambda w_{m-1}-q_{m-1}\right)\\
&=r_{m-1}\left(\lambda w_m-q_m\right)\left(\lambda w_{m-1}-q_{m-1}\right)+o\left(r_{m-1}\left(\lambda w_m-q_m\right)\left(\lambda w_{m-1}-q_{m-1}\right)\right),\\
\theta_{33}&=2r_{m-2}\left(\lambda w_m-q_m\right)+r_{m-1}r_{m-2}\left(\lambda w_m-q_m\right)\left(\lambda w_{m-1}-q_{m-1}\right)\\
&=r_{m-1}r_{m-2}\left(\lambda w_m-q_m\right)\left(\lambda w_{m-1}-q_{m-1}\right)+o\left(r_{m-1}r_{m-2}\left(\lambda w_m-q_m\right)\left(\lambda w_{m-1}-q_{m-1}\right)\right),
\end{aligned}$$

又因为

$$\Phi\left(a_{2m-3},\lambda\right)=\begin{pmatrix}1&1&r_{m-3}\\0&1&r_{m-3}\\\lambda w_{m-2}-q_{m-2}&\lambda w_{m-2}-q_{m-2}&\left(\lambda w_{m-2}-q_{m-2}\right)r_{m-3}\end{pmatrix}\Phi\left(a_{2m-5},\lambda\right),$$

故

$$\Phi(c,\lambda)=\begin{pmatrix}\theta_{11}&\theta_{12}&\theta_{13}\\\theta_{21}&\theta_{22}&\theta_{23}\\\theta_{31}&\theta_{32}&\theta_{33}\end{pmatrix}\Phi(a_{2m-3},\lambda)$$
$$=\begin{pmatrix}\theta_{11}&\theta_{12}&\theta_{13}\\\theta_{11}&\theta_{12}&\theta_{13}\\\theta_{11}&\theta_{12}&\theta_{13}\end{pmatrix}\begin{pmatrix}1&1&r_{m-3}\\0&1&r_{m-3}\\\lambda w_{m-2}-q_{m-2}&\lambda w_{m-2}-q_{m-2}&(\lambda w_{m-2}-q_{m-2})r_{m-3}\end{pmatrix}\Phi(a_{2m-5},\lambda)$$
$$=\begin{pmatrix}\eta_{11}&\eta_{12}&\eta_{13}\\\eta_{21}&\eta_{22}&\eta_{23}\\\eta_{31}&\eta_{32}&\eta_{33}\end{pmatrix}\Phi(a_{2m-5},\lambda)$$

其中

$$\begin{aligned}\eta_{11}&=\theta_{11}+(\lambda w_{m-2}-q_{m-2})\theta_{13}\\&=r_{m-1}r_{m-2}(\lambda w_{m-1}-q_{m-1})(\lambda w_{m-2}-q_{m-2})\\&\quad+o(r_{m-1}r_{m-2}(\lambda w_{m-1}-q_{m-1})(\lambda w_{m-2}-q_{m-2})),\end{aligned}$$

$$\begin{aligned}\eta_{12}&=\theta_{11}+\theta_{12}+(\lambda w_{m-2}-q_{m-2})\theta_{13}\\&=r_{m-1}r_{m-2}(\lambda w_{m-1}-q_{m-1})(\lambda w_{m-2}-q_{m-2})\\&\quad+o(r_{m-1}r_{m-2}(\lambda w_{m-1}-q_{m-1})(\lambda w_{m-2}-q_{m-2})),\end{aligned}$$

$$\begin{aligned}\eta_{13}&=r_{m-3}\theta_{11}+r_{m-3}\theta_{12}+r_{m-3}(\lambda w_{m-2}-q_{m-2})\theta_{13}\\&=r_{m-1}r_{m-2}r_{m-3}(\lambda w_{m-1}-q_{m-1})(\lambda w_{m-2}-q_{m-2})\\&\quad+o(r_{m-1}r_{m-2}r_{m-3}(\lambda w_{m-1}-q_{m-1})(\lambda w_{m-2}-q_{m-2})),\end{aligned}$$

$$\begin{aligned}\eta_{21}&=\theta_{21}+(\lambda w_{m-2}-q_{m-2})\theta_{23}\\&=r_{m-1}r_{m-2}(\lambda w_{m-1}-q_{m-1})(\lambda w_{m-2}-q_{m-2})\\&\quad+o(r_{m-1}r_{m-2}(\lambda w_{m-1}-q_{m-1})(\lambda w_{m-2}-q_{m-2})),\end{aligned}$$

$$\begin{aligned}\eta_{22}&=\theta_{21}+\theta_{22}+(\lambda w_{m-2}-q_{m-2})\theta_{23}\\&=r_{m-1}r_{m-2}(\lambda w_{m-1}-q_{m-1})(\lambda w_{m-2}-q_{m-2})\\&\quad+o(r_{m-1}r_{m-2}(\lambda w_{m-1}-q_{m-1}))((\lambda w_{m-2}-q_{m-2})),\end{aligned}$$

$$\begin{aligned}\eta_{23}&=r_{m-3}\theta_{21}+r_{m-3}\theta_{22}+r_{m-3}(\lambda w_{m-2}-q_{m-2})\theta_{23}\\&=r_{m-1}r_{m-2}r_{m-3}(\lambda w_{m-1}-q_{m-1})(\lambda w_{m-2}-q_{m-2})\\&\quad+o(r_{m-1}r_{m-2}r_{m-3}(\lambda w_{m-1}-q_{m-1})(\lambda w_{m-2}-q_{m-2})),\end{aligned}$$

$$\begin{aligned}\eta_{31}&=\theta_{31}+(\lambda w_{m-2}-q_{m-2})\theta_{33}\\&=r_{m-1}r_{m-2}(\lambda w_{m}-q_{m})(\lambda w_{m-1}-q_{m-1})(\lambda w_{m-2}-q_{m-2})\\&\quad+o(r_{m-1}r_{m-2}(\lambda w_{m}-q_{m})(\lambda w_{m-1}-q_{m-1})(\lambda w_{m-2}-q_{m-2})),\end{aligned}$$

$$\begin{aligned}\eta_{32}&=\theta_{31}+\theta_{32}+(\lambda w_{m-2}-q_{m-2})\theta_{33}\\&=r_{m-1}r_{m-2}(\lambda w_{m}-q_{m})(\lambda w_{m-1}-q_{m-1})(\lambda w_{m-2}-q_{m-2})\\&\quad+o(r_{m-1}r_{m-2}(\lambda w_{m}-q_{m})(\lambda w_{m-1}-q_{m-1})(\lambda w_{m-2}-q_{m-2})),\end{aligned}$$

$$\begin{aligned}\eta_{33} &= r_{m-3}\theta_{31} + r_{m-3}\theta_{32} + r_{m-3}\left(\lambda w_{m-2} - q_{m-2}\right)\theta_{33} \\ &= r_{m-1}r_{m-2}r_{m-3}\left(\lambda w_m - q_m\right)\left(\lambda w_{m-1} - q_{m-1}\right)\left(\lambda w_{m-2} - q_{m-2}\right) \\ &\quad + o\left(r_{m-1}r_{m-2}r_{m-3}\left(\lambda w_m - q_m\right)\left(\lambda w_{m-1} - q_{m-1}\right)\left(\lambda w_{m-2} - q_{m-2}\right)\right), \\ &\cdots\end{aligned}$$

重复上述方法，最终可以得到

$$\Phi(c,\lambda) = \begin{pmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ \xi_{31} & \xi_{32} & \xi_{33} \end{pmatrix} \Phi(a_1,\lambda),$$

其中

$$\xi_{11} = \prod_{k=1}^{m-1} r_k \prod_{k=1}^{m-1}\left(\lambda w_k - q_k\right) + o\left(\prod_{k=1}^{m-1} r_k \prod_{k=1}^{m-1}\left(\lambda w_k - q_k\right)\right),$$

$$\xi_{12} = \prod_{k=1}^{m-1} r_k \prod_{k=1}^{m-1}\left(\lambda w_k - q_k\right) + o\left(\prod_{k=1}^{m-1} r_k \prod_{k=1}^{m-1}\left(\lambda w_k - q_k\right)\right),$$

$$\xi_{13} = \prod_{k=0}^{m-1} r_k \prod_{k=1}^{m-1}\left(\lambda w_k - q_k\right) + o\left(\prod_{k=1}^{m-1} r_k \prod_{k=1}^{m-1}\left(\lambda w_k - q_k\right)\right),$$

$$\xi_{21} = \prod_{k=1}^{m-1} r_k \prod_{k=1}^{m-1}\left(\lambda w_k - q_k\right) + o\left(\prod_{k=1}^{m-1} r_k \prod_{k=1}^{m-1}\left(\lambda w_k - q_k\right)\right),$$

$$\xi_{22} = \prod_{k=1}^{m-1} r_k \prod_{k=1}^{m-1}\left(\lambda w_k - q_k\right) + o\left(\prod_{k=1}^{m-1} r_k \prod_{k=1}^{m-1}\left(\lambda w_k - q_k\right)\right),$$

$$\xi_{23} = \prod_{k=0}^{m-1} r_k \prod_{k=1}^{m-1}\left(\lambda w_k - q_k\right) + o\left(\prod_{k=1}^{m-1} r_k \prod_{k=1}^{m-1}\left(\lambda w_k - q_k\right)\right),$$

$$\xi_{31} = \prod_{k=1}^{m-1} r_k \prod_{k=1}^{m-1}\left(\lambda w_k - q_k\right) + o\left(\prod_{k=1}^{m-1} r_k \prod_{k=1}^{m-1}\left(\lambda w_k - q_k\right)\right),$$

$$\xi_{32} = \prod_{k=1}^{m-1} r_k \prod_{k=1}^{m-1}\left(\lambda w_k - q_k\right) + o\left(\prod_{k=1}^{m-1} r_k \prod_{k=1}^{m-1}\left(\lambda w_k - q_k\right)\right),$$

$$\xi_{33} = \prod_{k=0}^{m-1} r_k \prod_{k=1}^{m-1}\left(\lambda w_k - q_k\right) + o\left(\prod_{k=1}^{m-1} r_k \prod_{k=1}^{m-1}\left(\lambda w_k - q_k\right)\right),$$

又

$$\Phi(a_1,\lambda) = \begin{pmatrix} 1 & a_1 - a_0 & 0 \\ 0 & 1 & 0 \\ \lambda w_0 - q_0 & \left(\lambda w_0 - q_0\right)\left(a_1 - a_0\right) & 1 \end{pmatrix},$$

所以有

$$\begin{aligned}\Phi(c,\lambda) &= \begin{pmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ \xi_{31} & \xi_{32} & \xi_{33} \end{pmatrix} \Phi(a_1,\lambda) \\ &= \begin{pmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ \xi_{31} & \xi_{32} & \xi_{33} \end{pmatrix} \begin{pmatrix} 1 & a_1 - a_0 & 0 \\ 0 & 1 & 0 \\ \lambda w_0 - q_0 & \left(\lambda w_0 - q_0\right)\left(a_1 - a_0\right) & 1 \end{pmatrix} \\ &= \begin{pmatrix} \phi_{11}(c,\lambda) & \phi_{12}(c,\lambda) & \phi_{13}(c,\lambda) \\ \phi_{21}(c,\lambda) & \phi_{22}(c,\lambda) & \phi_{23}(c,\lambda) \\ \phi_{31}(c,\lambda) & \phi_{32}(c,\lambda) & \phi_{33}(c,\lambda) \end{pmatrix}.\end{aligned}$$

故

$$\begin{aligned}
\phi_{11}(c,\lambda) &= \prod_{k=0}^{m-1} r_k \prod_{k=0}^{m-1}(\lambda w_k - q_k) + o\left(\prod_{k=0}^{m_0-1} r_k \prod_{k=0}^{m_0-1}(q_k - \lambda w_k)\right),\\
\phi_{12}(c,\lambda) &= \prod_{k=0}^{m-1} r_k \prod_{k=0}^{m-1}(\lambda w_k - q_k) + o\left(\prod_{k=1}^{m_0-1} r_k \prod_{k=1}^{m_0-1}(q_k - \lambda w_k)\right),\\
\phi_{13}(c,\lambda) &= \prod_{k=0}^{m-1} r_k \prod_{k=1}^{m-1}(\lambda w_k - q_k) + o\left(\prod_{k=0}^{m_0-1} r_k \prod_{k=0}^{m_0}(q_k - \lambda w_k)\right),\\
\phi_{21}(c,\lambda) &= \prod_{k=0}^{m-1} r_k \prod_{k=0}^{m-1}(\lambda w_k - q_k) + o\left(\prod_{k=1}^{m_0-1} r_k \prod_{k=1}^{m_0-1}(q_k - \lambda w_k)\right),\\
\phi_{22}(c,\lambda) &= \prod_{k=0}^{m-1} r_k \prod_{k=0}^{m-1}(\lambda w_k - q_k) + o\left(\prod_{k=0}^{m_0-1} r_k \prod_{k=0}^{m_0-1}(q_k - \lambda w_k)\right),\\
\phi_{23}(c,\lambda) &= \prod_{k=0}^{m-1} r_k \prod_{k=1}^{m-1}(\lambda w_k - q_k) + o\left(\prod_{k=1}^{m_0-1} r_k \prod_{k=0}^{m_0-1}(q_k - \lambda w_k)\right),\\
\phi_{31}(c,\lambda) &= \prod_{k=0}^{m-1} r_k \prod_{k=0}^{m-1}(\lambda w_k - q_k) + o\left(\prod_{k=0}^{m_0-1} r_k \prod_{k=0}^{m_0-1}(q_k - \lambda w_k)\right),\\
\phi_{32}(c,\lambda) &= \prod_{k=0}^{m-1} r_k \prod_{k=0}^{m-1}(\lambda w_k - q_k) + o\left(\prod_{k=1}^{m_0-1} r_k \prod_{k=0}^{m_0-1}(q_k - \lambda w_k)\right),\\
\phi_{33}(c,\lambda) &= \prod_{k=0}^{m-1} r_k \prod_{k=1}^{m-1}(\lambda w_k - q_k) + o\left(\prod_{k=0}^{m_0-1} r_k \prod_{k=0}^{m_0-1}(q_k - \lambda w_k)\right),
\end{aligned} \tag{21}$$

沿用上面的方法，则有

$$\begin{aligned}
\psi_{11}(b,\lambda) &= \prod_{i=0}^{n-1} r_i \prod_{i=0}^{n-1}(\lambda w_i - q_i) + o\left(\prod_{i=0}^{n-1} r_i \prod_{i=0}^{n-1}(\lambda w_i - q_i)\right),\\
\psi_{12}(b,\lambda) &= \prod_{i=0}^{n-1} r_i \prod_{i=0}^{n-1}(\lambda w_i - q_i) + o\left(\prod_{i=0}^{n-1} r_i \prod_{i=0}^{n-1}(\lambda w_i - q_i)\right),\\
\psi_{13}(b,\lambda) &= \prod_{i=0}^{n-1} r_i \prod_{i=1}^{n-1}(\lambda w_i - q_i) + o\left(\prod_{i=0}^{n-1} r_i \prod_{i=1}^{n-1}(\lambda w_i - q_i)\right),\\
\psi_{21}(b,\lambda) &= \prod_{i=0}^{n-1} r_i \prod_{i=0}^{n-1}(\lambda w_i - q_i) + o\left(\prod_{i=0}^{n-1} r_i \prod_{i=0}^{n-1}(\lambda w_i - q_i)\right),\\
\psi_{22}(b,\lambda) &= \prod_{i=0}^{n-1} r_i \prod_{i=0}^{n-1}(\lambda w_i - q_i) + o\left(\prod_{i=0}^{n-1} r_i \prod_{i=0}^{n-1}(\lambda w_i - q_i)\right),\\
\psi_{23}(b,\lambda) &= \prod_{i=0}^{n-1} r_i \prod_{i=1}^{n-1}(\lambda w_i - q_i) + o\left(\prod_{i=0}^{n-1} r_i \prod_{i=1}^{n-1}(\lambda w_i - q_i)\right),\\
\psi_{31}(b,\lambda) &= \prod_{i=0}^{n-1} r_i \prod_{i=0}^{n-1}(\lambda w_i - q_i) + o\left(\prod_{i=0}^{n-1} r_i \prod_{i=0}^{n-1}(\lambda w_i - q_i)\right),\\
\psi_{32}(b,\lambda) &= \prod_{i=0}^{n-1} r_i \prod_{i=0}^{n-1}(\lambda w_i - q_i) + o\left(\prod_{i=0}^{n-1} r_i \prod_{i=0}^{n-1}(\lambda w_i - q_i)\right),\\
\psi_{33}(b,\lambda) &= \prod_{i=0}^{n-1} r_i \prod_{i=1}^{n-1}(\lambda w_i - q_i) + o\left(\prod_{i=0}^{n-1} r_i \prod_{i=1}^{n-1}(\lambda w_i - q_i)\right).
\end{aligned} \tag{22}$$

又因为

$$G=\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}, \tag{23}$$

由引理 4 可知

$$\Phi(b,\lambda)=\Psi(b,\lambda)G\Phi(c,\lambda),$$

结合(21)~(23)，故有

$$\phi_{11}(b,\lambda)=RR^*G^*\cdot\prod_{k=0}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{11}(b,\lambda),$$

$$\phi_{12}(b,\lambda)=RR^*G^*\cdot\prod_{k=0}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{12}(b,\lambda),$$

$$\phi_{13}(b,\lambda)=RR^*G^*\cdot\prod_{k=1}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{13}(b,\lambda),$$

$$\phi_{21}(b,\lambda)=RR^*G^*\cdot\prod_{k=0}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{21}(b,\lambda),$$

$$\phi_{22}(b,\lambda)=RR^*G^*\cdot\prod_{k=0}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{22}(b,\lambda),$$

$$\phi_{23}(b,\lambda)=RR^*G^*\cdot\prod_{k=1}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{23}(b,\lambda),$$

$$\phi_{31}(b,\lambda)=RR^*G^*\cdot\prod_{k=0}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{31}(b,\lambda),$$

$$\phi_{32}(b,\lambda)=RR^*G^*\cdot\prod_{k=0}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{32}(b,\lambda),$$

$$\phi_{33}(b,\lambda)=RR^*G^*\cdot\prod_{k=1}^{m-1}(\lambda w_k-q_k)\prod_{i=1}^{n-1}(\lambda w_i-q_i)+\phi'_{33}(b,\lambda),$$

其中

$$G^*=g_{11}(\lambda w_0-q_0)+g_{21}(\lambda w_0-q_0)+g_{31}+g_{12}(\lambda w_0-q_0)+g_{22}(\lambda w_0-q_0)$$
$$+g_{32}+g_{13}(\lambda w_0-q_0)+g_{23}(\lambda w_0-q_0)+g_{33},$$

$$R=\prod_{k=0}^{m-1}r_k,\ R^*=\prod_{i=0}^{n-1}r_i,\ \phi'_{ij}(b,\lambda)=o(RR^*),\ i,j=1,2,3.$$

结论得证。 □

定理 3 设 $m,n\in N$，$g_{12}\neq 0$，且(3)，(10)~(12)成立。$H(\lambda)=(h_{ij}(\lambda))_{3\times 3}$ 引理 1 中定义，则具有转移条件且边界条件中含有谱参数的第一类三阶有限谱边值问题(1)至多只有 $m+n+2$ 个特征值。

证明 由引理 1 可知

$$\Delta(\lambda)=\sum_{i=1,j=1}^{3}h_{ij}(\lambda)\phi_{ij}(b,\lambda),$$

再由引理 5 可知，$\phi_{11}(b,\lambda),\phi_{12}(b,\lambda),\phi_{13}(b,\lambda),\phi_{21}(b,\lambda),\phi_{22}(b,\lambda),\phi_{23}(b,\lambda),\phi_{31}(b,\lambda),\phi_{32}(b,\lambda),\phi_{33}(b,\lambda)$ 关于 λ 的次数分别为 $m+n,m+n,m+n-1,m+n,m+n,m+n-1,m+n,m+n,m+n-1$。所以当 $h_{11}\neq 0$ 时，$\Delta(\lambda)$ 的

次数为 $m+n+2$，由代数学基本定理可知，$\Delta(\lambda)=0$ 时，至多有 $m+n+2$ 个特征值。其它情况判断函数 $\Delta(\lambda)$ 关于 λ 的次数必定小于或等于 $m+n+2$。定理得证。 □

4. 第二类三阶有限谱边值问题

在这部分内容中，我们讨论第二类具有转移条件且边界条件中含有谱参数的三阶有限谱边值问题

$$\begin{cases}(py')''+qy=\lambda wy,\\ A_\lambda Y(a)+B_\lambda Y(b)=0\\ CY(c-)+DY(c+)=0\end{cases}$$

两类问题的方程形式虽有不同，但结果是一致的，下面我们只阐述在证明过程中的不同部分。令 $u_1=y,u_2=y',u_3=py''$，则与方程 $(py')''+qy=\lambda wy$ 等价的系统表示为：

$$u_1'=ru_2,u_2'=u_3,u_3'=(\lambda w-q)u_1. \tag{24}$$

与引理 2 的结论类似，但由(24)可知，在 r 恒等于零的子区间上 u_1 是常数，而在 q,w 恒等于零的子区间上 u_3 是常数。通过反复应用(24)，即可得出新的引理。经反复利用引理，可得出 $\Delta(\lambda)$ 的次数仍为 $m+n+2$，结论一致。

基金项目

兰州理工大学研究生科研创新基金(No. 56256016)资助项目。

参考文献

[1] Kong, Q., Wu, H. and Zettl, A. (2001) Sturm-Liouville Problems with Finite Spectrum. *Mathematical Analysis and Applications*, **263**, 748-762. https://doi.org/10.1006/jmaa.2001.7661

[2] Ao, J.J., Bo, F.Z. and Sun, J. (2014) Fourth Order Boundary Value Problems with Finite Spectrum. *Applied Mathematics and Computation*, **244**, 952-958. https://doi.org/10.1016/j.amc.2014.07.054

[3] Ao, J.J., Sun, J. and Zettl, A. (2015) Finite Spectrum of 2*n*th Order Boundary Value Problems. *Applied Mathematics Letters*, **42**, 1-8. https://doi.org/10.1016/j.aml.2014.10.003

[4] Wu, Y.Y. and Zhao, Z.Q. (2011) Positive Solutions for Third-Order Boundary Value Problems with Change of Signs. *Applied Mathematics and Computation*, **218**, 2744-2749. https://doi.org/10.1016/j.amc.2011.08.015

[5] Greenberg, M. (1987) Third Order Linear Differential Equations. REIDEL, Dordrecht.

[6] Kong, Q., Wu, H. and Zettl, A. (1999) Dependence of the *n*th Sturm-Liouville Eigenvalue on the Problem. *Journal of Differential Equations*, **156**, 328-354. https://doi.org/10.1006/jdeq.1998.3613

[7] Xu, M.Z., Wang, W.Y. and Ao, J.J. (2018) Finite Spectrum of Sturm-Liouville Problems with *n* Transmission Conditions. *Iranian Mathematical Society*, **42**, 811-817. https://doi.org/10.1007/s40995-016-0072-1

[8] Ao, J.J., Sun, J. and Zhang, M.Z. (2013) The Finite Spectrum of Sturm-Liouville Problems with Transmission Conditions and Eigenparameter-Dependent Boundary Conditions. *Results in Mathematics*, **63**, 1057-1070. https://doi.org/10.1007/s00025-012-0252-z

[9] Ao, J.J. (2017) On Two Classes of Third Order Boundary Value Problems with Finite Spectrum. *Iranian Mathematical Society*, **43**, 1089-1099.

[10] Yang, C. and Sun, J. (2012) Existence of Multiple Positive Solutions for Fourth-Order Boundary Value Problems in Banach Spaces. *Boundary Value Problems*, **107**, 1-13.

[11] Graef, J.R., Kong, L., Kong, Q. and Yang, B. (2011) Positive Solutions to a Fourth Order Boundary Value Problem. *Results in Mathematics*, **59**, 141-155. https://doi.org/10.1007/s00025-010-0068-7

[12] Atkinson, F.V. (1964) Discrete and Continuous Boundary Problems. Academic Press, New York, London. https://doi.org/10.1063/1.3051875

[13] Chanane, B. (2010) Accurate Solutions of Fourth Order Sturm-Liouville Problems. *Applied Mathematics and Computation*, **234**, 3064-3071. https://doi.org/10.1016/j.cam.2010.04.023

[14] Zettl, A. (2005) Sturm-Liouvile Theory. In: *Mathematical Surveys and Monographs*, Volume 121, American Mathematical Society, Washington DC.

[15] Everitt, W.N. and Race, D. (1977) On Necessary and Sufficient Conditions for the Existence of Caratheodory Solutions of Ordinary Differential Equations. *Quaestiones Mathematicae*, **78**, 507-512. https://doi.org/10.1080/16073606.1978.9631549