

# Conservation Laws of the Defocusing Hirota Equation under Non-Zero Background

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## Abstract

The conservation laws are important indexes for the integrability of nonlinear equations. It plays an important role in the research of partial differential equation systems. Based on Lax pair, this paper studies infinite conservation laws and higher-order conserved quantities of the one-dimensional defocusing Hirota equation under non-zero backgrounds.

## Keywords

Hirota Equation, Lax Pair, Conservation Laws

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# 散焦HIROTA方程在非零背景下的无穷守恒律

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## 摘要

守恒律是非线性方程可积的一个重要指标, 在偏微分方程系统的研究中扮演着重要的角色。本文基于Lax对, 研究了一维散焦Hirota方程在非零背景下的无穷守恒律及其高阶守恒量。

## 关键词

Hirota方程, Lax对, 守恒律

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## 1. 问题背景及主要结论

随着现代科学的发展, 线性微分方程已经无法确切的描述现实社会中各种现象出现的个体内在关系, 随之而来的, 非线性科学得以快速的发展, 并在数学、物理、流体等领域中广泛应用。也正源于此, 越来越多科学家投身到非线性偏微分方程的研究之中, 并取得巨大的成就。而在这之间, 数学家发现许多具有广泛应用背景的方程如: KdV 方程、Schrödinger 方程、sine-Gordon 方程等都存在孤立子解, 于此, 孤立子理论得以产生并迅速发展。同时, 因为光孤子在光纤通信中的重要作用, 也使得众多的数学家开始致力于描述光孤子运动的非线性发展方程的研究, 而这其中最具有代表性的方程就是 Hirota 方程。

广义的 Hirota 方程如下:

$$i(\psi_t + \mu_1 \psi + (\nu_1 + \mu_1 x) \psi_x) + (\nu_2 + \mu_2 x)(\psi_{xx} + 2|\psi|^2 \psi) + 2\mu_2 \left( \psi_x + \psi \int_{-\infty}^x |\psi|^2 dx \right) + i\nu(\psi_{xxx} + 6|\psi|^2 \psi_x) = 0 \quad (1.1)$$

针对广义的 Hirota 方程, Porsezian [1]、田播[2]等人分别利用 Bäcklund、Hirota 双线性方法得出了方程的孤子解, 并分析了该孤子解的性质。而当方程(1.1)中系数  $\mu_1 = \nu = \nu_1 = 0$  时候, 方程就退化为广义的非线性 Schrödinger 方程:

$$i\psi_t + (\nu_2 + \mu_2 x)(\psi_{xx} + 2|\psi|^2 \psi) + 2\mu_2 \left( \psi_x + \psi \int_{-\infty}^x |\psi|^2 dx \right) = 0 \quad (1.2)$$

该方程对于描述连续极限条件下不同磁性相互作用的动力学特性具有明显的效果, 该 Schrödinger 方程的 Lax 对在[3]文献中也已经给出。当方程(1.1)中系数  $\mu_1 = \mu_2 = \nu_1 = 0, \nu_2 = 1, \nu = \epsilon$ , 方程(1.1)就变成了变系数 Hirota 方程:

$$i\psi_x + \psi_{xx} + 2|\psi|^2 \psi + i\epsilon(\psi_{xxx} + 6|\psi|^2 \psi_x) = 0 \quad (1.3)$$

方程(1.3)的可积性在文献[4]中给出, 同时, Radhakrishnan 等人[5]利用 Hirota 双线性方法得出了该方程的孤子解。当方程(1.1)中系数  $\mu_1 = \mu_2 = \nu_1 = 0, \nu_2 = \frac{1}{2}, \nu = \frac{\sqrt{2}}{4}$ , 得到方程:

$$i\psi_x + \frac{1}{2}(\psi_{xx} + 2|\psi|^2 \psi) + \frac{\sqrt{2}}{4}i(\psi_{xxx} + 6|\psi|^2 \psi_x) = 0 \quad (1.4)$$

对式子(1.4)作坐标变换:  $x = \frac{1}{\sqrt{2}}x$ , 即可得到方程:

$$i\psi_x + \psi_{xx} + |\psi|^2 \psi + i(\psi_{xxx} + 3|\psi|^2 \psi_x) = 0 \quad (1.5)$$

相似于方程(1.5), 本文深入研究了下列一维 Hirota 方程:

$$i\psi_x + \psi_{xx} - |\psi|^2 \psi + i(\psi_{xxx} - 3|\psi|^2 \psi_x) = 0 \quad (1.6)$$

其中:  $\psi = \psi(x, t) \in \mathbb{C}, (x, t) \in \mathbb{R} \times \mathbb{R}$ 。这是一个散焦立方非线性方程, 包括三阶色散项和非线性色散项。守恒律反映了某物理量不随时间改变的一种现象。而在孤子理论中, 人们发现这样一个事实, 如果非线

性发展方程具有孤立子解, 那么几乎也具有无穷守恒律。因此, 就一个孤子系统而言, 寻找其无穷守恒律, 对于证明此系统的可积性, 甚至在证明其适定性方面有重大意义。下面我们将基于 Lax 对[6], 来研究散焦 Hirota 方程(1.6)在非零背景下的无穷守恒律及其高阶守恒量。

定义 1.1 [7]给定一般的非线性偏微分方程:

$$F(x, t, u) = 0, \quad (1.7)$$

其中:  $u = u(x, t)$  是关于  $x$  和  $t$  的函数, 而  $F(x, t, u)$  是  $x, t, u$  及其导数的函数。若存在一对连续可微函数  $\omega(x, t, u)$  和  $J(x, t, u)$ , 使得  $u$  按照方程(1.7)发展时候满足关系

$$\partial_t \omega(x, t, u) = \partial_x J(x, t, u), \quad (1.8)$$

则称此关系式为式(1.7)的守恒律, 而  $\omega(x, t, u)$  和  $J(x, t, u)$  分别称为守恒密度与连带流。

如果当  $|x|$  趋于无穷时, 密度与流充分地趋于零, 则将守恒律在整个数轴上对  $x$  积分得:

$$\frac{d}{dt} \int_{-\infty}^{\infty} \omega(x, t, u) dx = 0, \quad (1.9)$$

则积分  $\int_{-\infty}^{\infty} \omega(x, t, u) dx$  与时间  $t$  无关, 是方程(1.7)的守恒量。

注: 设二阶矩阵

$$M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix},$$

我们定义如下符号意义:

$$M^{off} = \begin{pmatrix} 0 & m_2 \\ m_3 & 0 \end{pmatrix}, \quad M^{diag} = \begin{pmatrix} m_1 & 0 \\ 0 & m_4 \end{pmatrix}.$$

**定理 1.1** HIROTA 方程(1.6)具有无穷守恒律。

**推论 1.1** 已知方程(1.6), 则有

$$S(\psi) = \int_{\mathbb{R}} \left[ |\psi_{xx}|^2 + 3|\psi|^2 |\psi_x|^2 + \frac{1}{2}(\bar{\psi}\psi_x + \psi\bar{\psi}_x)^2 + (1 - |\psi|^2)^2 \left(1 + \frac{1}{2}|\psi|^2\right) \right] dx. \quad (1.10)$$

是其高阶守恒量。

## 2. 定理的证明

为了证明定理 1.1, 参考[7]中第三章内容和文献[8] [9]中孤立子理论, 首先需要推导出方程(1.6)所对应的 Lax 对, 而方程(1.6)可以由方程

$$i\psi_x + \frac{1}{2}(\psi_{xx} - 2|\psi|^2 \psi) + \frac{\sqrt{2}}{4}i(\psi_{xxx} - 6|\psi|^2 \psi_x) = 0. \quad (2.1)$$

作坐标变换得到, 则首先可以研究方程(2.1)的 Lax 对。

**命题 2.1** 方程(2.1)对应的 Lax 对为:

$$\Phi_x = (i\lambda\sigma_3 + Q)\Phi, \quad (2.2)$$

$$\begin{aligned} \Phi_t = & \left[ \sqrt{2} \left[ i\sigma_3 \lambda^3 + Q\lambda^2 + \left( -\frac{i}{2}\sigma_3 Q_x + \frac{i}{2}\sigma_3 Q^2 \right) \lambda \right. \right. \\ & \left. \left. + \left( -\frac{1}{4}Q_{xx} + \frac{1}{2}Q^3 + \frac{1}{4}(Q_x Q - Q Q_x) \right) \right] \right] \Phi \\ & - \left[ i\sigma_3 \lambda^2 + Q\lambda + \left( -\frac{i}{2}\sigma_3 Q_x + \frac{i}{2}\sigma_3 Q^2 \right) \right] \Phi \end{aligned} \quad (2.3)$$

其中:

$$Q = \begin{pmatrix} 0 & \psi \\ \bar{\psi} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

**证明** 我们假设方程(2.1)对应的 Lax 对具有如下形式:

$$\begin{cases} \Phi_x = (i\lambda\sigma_3 + Q)\Phi \\ \Phi_t = \left( \sum_{k=0}^3 v_k \lambda^k \right) \Phi \end{cases}$$

$\Phi(\lambda; x, t)$  为复平面上关于  $\lambda$  的亚纯函数。因为  $\Phi_{xt} = \Phi_{tx}$ , 则有

$$\begin{cases} \Phi_{xt} = \left[ (i\lambda\sigma_3 + Q)_t + (i\lambda\sigma_3 + Q) \left( \sum_{k=0}^3 v_k \lambda^k \right) \right] \Phi \\ \Phi_{tx} = \left[ \left( \sum_{k=0}^3 v_k \lambda^k \right)_x + \left( \sum_{k=0}^3 v_k \lambda^k \right) (i\lambda\sigma_3 + Q) \right] \Phi \end{cases}$$

定义运算符:  $[U, V] = UV - VU$ , 则

$$(i\lambda\sigma_3 + Q)_t - \left( \sum_{k=0}^3 v_k \lambda^k \right)_x + \left[ i\lambda\sigma_3 + Q, \sum_{k=0}^3 v_k \lambda^k \right] = 0.$$

下面, 我们对比  $\lambda^k$  的系数:

$$\lambda^4 : [i\sigma_3, v_3] = 0 \quad (2.4)$$

$$\lambda^3 : -v_{3,x} + [i\sigma_3, v_2] + [Q, v_3] = 0 \quad (2.5)$$

$$\lambda^2 : -v_{2,x} + [i\sigma_3, v_1] + [Q, v_2] = 0 \quad (2.6)$$

$$\lambda^1 : -v_{1,x} + [i\sigma_3, v_0] + [Q, v_1] = 0 \quad (2.7)$$

$$\lambda^0 : Q_t - v_{0,x} + [Q, v_0] = 0. \quad (2.8)$$

根据式(2.4), 并选择  $-v_{3,x}^{diag} = 0$ ,  $v_3^{diag} = i\sigma_3 \alpha$  其中  $\alpha$  是常数。再根据式(2.5), 则得到:  $v_2^{off} = \alpha Q$ 。

根据式(2.6), 并选择  $-v_{2,x}^{diag} + [Q, v_2^{off}] = 0$ ,  $v_2^{diag} = i\sigma_3 \beta$  其中  $\beta$  是常数。则有  $2i\sigma_3 v_1^{off} = \alpha Q_x + 2i\sigma_3 \beta Q$ ,

$$v_1^{off} = -\frac{i}{2}\sigma_3 \alpha Q_x + \beta Q.$$

根据式(2.7), 并选择  $-v_{1,x}^{diag} + [Q, v_1^{off}] = 0$ ,  $v_1^{diag} = \frac{i}{2}\sigma_3 \alpha Q^2$  则

$$\begin{aligned}
 2i\sigma_3 v_0^{off} &= v_{1,x}^{off} + [v_1^{diag}, Q] \\
 &= -\frac{i}{2}\sigma_3\alpha Q_{xx} + \beta Q_x + \frac{i}{2}(\sigma_3\alpha Q^3 - Q\sigma_3\alpha Q^2) \\
 &= -\frac{i}{2}\sigma_3\alpha Q_{xx} + \beta Q_x + i\sigma_3\alpha Q^3
 \end{aligned} \tag{2.9}$$

求解式(2.9), 可得

$$v_0^{off} = -\frac{1}{4}\alpha Q_{xx} - \frac{i}{2}\sigma_3\beta Q_x + \frac{1}{2}Q^3\alpha。$$

根据式子(2.8), 并令  $-v_{0,x}^{diag} + [Q, v_0^{off}] = 0$ , 则

$$\begin{aligned}
 v_{0,x}^{diag} &= \frac{1}{4}\alpha Q_{xx}Q - Q\frac{1}{4}\alpha Q_{xx} + \frac{i}{2}\sigma_3\beta Q_xQ - \frac{i}{2}Q\sigma_3\beta Q_x \\
 &= \frac{1}{4}\alpha(Q_{xx}Q - QQ_{xx}) + \frac{i}{2}\beta(\sigma_3\beta Q_xQ + \sigma_3QQ_x)
 \end{aligned} \tag{2.10}$$

求解式(2.10), 可得

$$v_0^{diag} = \frac{1}{4}\alpha(Q_xQ - QQ_x) + \frac{i}{2}\sigma_3\beta Q^2。$$

又根据

$$\begin{aligned}
 Q_t - v_{0,x}^{off} + [Q, v_0^{diag}] &= 0, \\
 -v_{0,x}^{off} &= \frac{1}{4}\alpha Q_{xxx} + \frac{i}{2}\sigma_3\beta Q_{xx} - \frac{1}{2}\alpha(Q_xQ^2 + QQ_xQ + Q^2Q_x), \\
 [Q, v_0^{diag}] &= \frac{\alpha}{4}(2QQ_xQ - Q^2Q_x - Q_xQ^2) - i\sigma_3\beta Q_{xx}
 \end{aligned}$$

得到

$$Q_t + \frac{i}{2}\beta(\sigma_3Q_{xx} - 2\sigma_3Q^3) + \frac{\alpha}{4}(Q_{xxx} - 3(Q_xQ^2 + Q^2Q_x)) = 0。 \tag{2.11}$$

将  $Q = \begin{pmatrix} 0 & \psi \\ \bar{\psi} & 0 \end{pmatrix}$  代入式(2.11), 得到

$$\psi_t + \frac{i}{2}\beta(\psi_{xx} - 2|\psi|^2\psi) + \frac{\alpha}{4}(\psi_{xxx} - 6|\psi|^2\psi_x) = 0。 \tag{2.12}$$

令  $\beta = -1$ ,  $\alpha = \sqrt{2}$ , 则可以得到:

$$i\psi_t + \frac{1}{2}(\psi_{xx} - 2|\psi|^2\psi) + i\frac{\sqrt{2}}{4}(\psi_{xxx} - 6|\psi|^2\psi_x) = 0。 \tag{2.13}$$

综合以上, 根据已得:  $v_0, v_1, v_2, v_3, \beta, \alpha$  的值, 可得式子(2.1)的 Lax 对。

**命题 2.2** 方程(1.6)对应的 Lax 对为:

$$\Phi_x = \frac{1}{\sqrt{2}} \begin{pmatrix} i\lambda & \psi \\ \bar{\psi} & -i\lambda \end{pmatrix} \Phi, \tag{2.14}$$

$$\Phi_t = \begin{pmatrix} p_1 & p_2 \\ p_3 & -p_1 \end{pmatrix} \Phi. \tag{2.15}$$

其中

$$\begin{aligned} p_1 &= -i\lambda^2 - \frac{i}{2}|\psi|^2 + \sqrt{2}i\lambda^3 + \frac{\sqrt{2}i\lambda}{2}|\psi|^2 + \frac{1}{2}(\psi_x\bar{\psi} - \psi\bar{\psi}_x), \\ p_2 &= -\lambda\psi + \frac{\sqrt{2}}{2}i\psi_x + \sqrt{2}\lambda^2\psi - i\lambda\psi_x + \frac{\sqrt{2}}{2}(|\psi|^2\psi - \psi_{xx}), \\ p_3 &= -\lambda\bar{\psi} - \frac{\sqrt{2}}{2}i\bar{\psi}_x + \sqrt{2}\lambda^2\bar{\psi} + i\lambda\bar{\psi}_x + \frac{\sqrt{2}}{2}(|\psi|^2\bar{\psi} - \bar{\psi}_{xx}). \end{aligned}$$

**证明** 根据式子(1.6)和(2.1)之间的坐标变换, 易证。

**命题 2.3** 已知方程(1.6)的 Lax 对, 在特殊解为  $\psi(x,t) = e^{-it}$  的情况下, 方程有如下形式的基本解

$$\Phi_0 = D^{-1}C e^{\left[\frac{1}{\sqrt{2}}i\xi x + \left[\sqrt{2}\left(\lambda^2 + \frac{1}{2}\right) - \lambda\right]i\xi t\right]\sigma_3},$$

其中:  $\lambda^2 + \xi^2 = 1$ ,  $D = e^{\frac{i}{2}t\sigma_3}$ ,  $C = \begin{pmatrix} 1 & 1 \\ -i & -i \\ \lambda + \xi & \lambda - \xi \end{pmatrix}$ 。

**证明** 利用命题 2.2 的 Lax 对, 有

$$\Phi_{0,x} = \frac{1}{\sqrt{2}} \begin{pmatrix} i\lambda & e^{-it} \\ e^{it} & -i\lambda \end{pmatrix} \Phi_0,$$

引入算子  $D = e^{\frac{i}{2}t\sigma_3}$ , 作变换:  $\tilde{\Phi}_0 = D\Phi_0$ 。则

$$(\tilde{\Phi}_0)_x = \frac{1}{\sqrt{2}} D \begin{pmatrix} i\lambda & e^{-it} \\ e^{it} & -i\lambda \end{pmatrix} D^{-1} \tilde{\Phi}_0 = \begin{pmatrix} \frac{1}{\sqrt{2}}i\lambda & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}i\lambda \end{pmatrix} \tilde{\Phi}_0.$$

利用线性代数可得到矩阵的对角化形式

$$\begin{pmatrix} \frac{1}{\sqrt{2}}i(\lambda - \xi) & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}i(\lambda + \xi) \end{pmatrix} = C \begin{pmatrix} \frac{1}{\sqrt{2}}i\xi & 0 \\ 0 & -\frac{1}{\sqrt{2}}i\xi \end{pmatrix} C^{-1}.$$

引入  $\hat{\Phi}_0 = C^{-1}\tilde{\Phi}_0$ , 则

$$(\hat{\Phi}_0)_x = (C^{-1}\tilde{\Phi}_0)_x = \begin{pmatrix} \frac{1}{\sqrt{2}}i\xi & 0 \\ 0 & -\frac{1}{\sqrt{2}}i\xi \end{pmatrix} \hat{\Phi}_0. \tag{2.16}$$

同理, 由于方程满足定命题 2.2 中 lax 对, 关于参数  $t$ , 亦可以得到:

$$(\hat{\Phi}_0)_t = \left[ \sqrt{2} \left( \lambda^2 + \frac{1}{2} \right) - \lambda \right] \begin{pmatrix} i\xi & 0 \\ 0 & -i\xi \end{pmatrix} \hat{\Phi}_0. \quad (2.17)$$

结合式(2.16)和式(2.17)可以得到

$$\hat{\Phi}_0 = C^{-1} \tilde{\Phi}_0 = e^{\left[ \frac{1}{\sqrt{2}} i\xi x + \left[ \sqrt{2} \left( \lambda^2 + \frac{1}{2} \right) - \lambda \right] i\xi t \right] \sigma_3}.$$

继而

$$\begin{aligned} \tilde{\Phi}_0 &= C e^{\left[ \frac{1}{\sqrt{2}} i\xi x + \left[ \sqrt{2} \left( \lambda^2 + \frac{1}{2} \right) - \lambda \right] i\xi t \right] \sigma_3} \\ \Phi_0 &= D^{-1} C e^{\left[ \frac{1}{\sqrt{2}} i\xi x + \left[ \sqrt{2} \left( \lambda^2 + \frac{1}{2} \right) - \lambda \right] i\xi t \right] \sigma_3}. \end{aligned}$$

**定理 1.1 的证明** 根据命题 2.2, 我们知道方程(1.6)的 Lax 对。设  $\Phi = (\varphi_1, \varphi_2)^T$ 。根据式(2.14), 得到

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}_x = \begin{pmatrix} \frac{1}{\sqrt{2}} i\lambda & \frac{1}{\sqrt{2}} \psi \\ \frac{1}{\sqrt{2}} \bar{\psi} & -\frac{1}{\sqrt{2}} i\lambda \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \quad (2.18)$$

继而利用式(2.18), 得到

$$\frac{\varphi_{1,x}}{\varphi_1} = (\ln \varphi_1)_x = \frac{1}{\sqrt{2}} i\lambda + \frac{1}{\sqrt{2}} \frac{\varphi_2}{\varphi_1}.$$

和

$$\begin{aligned} \left( \frac{\varphi_2}{\varphi_1} \right)_x &= \frac{\varphi_{2,x}}{\varphi_1} - \frac{\varphi_2 \varphi_{1,x}}{\varphi_1^2} \\ &= \frac{\frac{1}{\sqrt{2}} \bar{\psi} \varphi_1 - \frac{1}{\sqrt{2}} i\lambda \varphi_2}{\varphi_1} - \frac{\frac{1}{\sqrt{2}} i\lambda \varphi_1 \varphi_2 + \frac{1}{\sqrt{2}} \psi \varphi_2^2}{\varphi_1^2}. \\ &= \frac{1}{\sqrt{2}} \bar{\psi} - \sqrt{2} i\lambda \frac{\varphi_2}{\varphi_1} - \frac{1}{\sqrt{2}} \psi \left( \frac{\varphi_2}{\varphi_1} \right)^2 \end{aligned} \quad (2.19)$$

假设  $\phi = \frac{\varphi_2}{\varphi_1}$ ,  $\phi = \sum_{k=1}^{+\infty} \omega_k \lambda^{-k}$ , 将其带入式(2.19), 得到

$$\phi_x = \frac{1}{\sqrt{2}} \bar{\psi} - \sqrt{2} i\lambda \phi - \frac{1}{\sqrt{2}} \psi \phi^2.$$

和

$$\begin{aligned} \sum_{k=1}^{+\infty} \omega_{k,x} \lambda^{-k} &= \frac{1}{\sqrt{2}} \bar{\psi} - \sqrt{2} i\lambda \sum_{k=1}^{+\infty} \omega_k \lambda^{-k} - \frac{1}{\sqrt{2}} \psi \left( \sum_{k=1}^{+\infty} \omega_k \lambda^{-k} \right)^2 \\ &= \frac{1}{\sqrt{2}} \bar{\psi} - \sqrt{2} i \sum_{k=0}^{+\infty} \omega_{k+1} \lambda^{-k} - \frac{1}{\sqrt{2}} \psi \sum_{k=2}^{+\infty} \left( \sum_{j=1}^k \omega_j \omega_{k-j} \right) \lambda^{-k}. \end{aligned}$$

下面，我们比较  $\lambda^k$  同次幂的系数，则得到：

$$\begin{aligned} \lambda^0 : i\sqrt{2}\omega_1 - \frac{1}{\sqrt{2}}\bar{\psi} &= 0 \\ \lambda^{-1} : \omega_{1,x} + \sqrt{2}i\omega_2 &= 0 \\ \lambda^{-2} : \omega_{2,x} + \sqrt{2}i\omega_3 + \frac{1}{\sqrt{2}}\psi\omega_1^2 &= 0 \\ \lambda^{-3} : \omega_{3,x} + \sqrt{2}i\omega_4 + \frac{1}{\sqrt{2}}\psi 2\omega_1\omega_2 &= 0 \\ \lambda^{-k} : \omega_{k+1} &= \frac{1}{\sqrt{2}}i \left[ \omega_{k,x} + \frac{1}{\sqrt{2}}\psi \sum_{j=1}^k \omega_j \omega_{k-j} \right], (k \geq 4) \end{aligned}$$

根据同阶系数相等，可以算出  $\omega_k$ ，下面列出前五个系数

$$\begin{aligned} \omega_1 &= -\frac{i}{2}\bar{\psi} \\ \omega_2 &= \frac{\sqrt{2}}{4}\bar{\psi}_x \\ \omega_3 &= \frac{i}{8}(2\bar{\psi}_{xx} - \bar{\psi}^2\psi) \\ \omega_4 &= \frac{1}{\sqrt{2}} \left( \frac{1}{4}\psi\bar{\psi}\bar{\psi}_x + \frac{1}{8}(\psi\bar{\psi}^2 - 2\bar{\psi}_{xx})_x \right) \end{aligned}$$

和

$$\begin{aligned} \omega_5 &= i \frac{1}{\sqrt{2}} \left[ \omega_{4,x} + \frac{1}{\sqrt{2}}\psi(\omega_1\omega_3 + \omega_2\omega_2 + \omega_3\omega_1) \right] \\ &= \frac{i}{2} \left[ \frac{1}{4}(\psi\bar{\psi}\bar{\psi}_x)_x + \frac{1}{8}[(\psi\bar{\psi}^2)_{xx} - 2\bar{\psi}_{xxxx}] + \psi \left[ \frac{1}{8}\bar{\psi}(2\bar{\psi}_{xx} - \psi\bar{\psi}^2) + \frac{1}{8}\bar{\psi}_x^2 \right] \right] \\ &= \frac{i}{8} \left[ (\bar{\psi}\psi\bar{\psi}_x)_x + \frac{1}{2}(\psi\bar{\psi}^2)_{xx} - \bar{\psi}_{xxxx} + \psi \left[ \bar{\psi}(\bar{\psi}_{xx} - \frac{1}{2}\psi\bar{\psi}^2) + \frac{1}{2}\bar{\psi}_x^2 \right] \right] \end{aligned}$$

根据命题 2.3 中得到的基本解的形式，假设  $\varphi_1 = \Theta(\lambda; x, t) e^{\left[ \frac{1}{\sqrt{2}}i\xi x + \left[ \sqrt{2}(\lambda^2 + \frac{1}{2}) - \lambda \right] i\xi t \right]}$ ，其中  $\Theta$  为关于  $\lambda$  的解析函数。则

$$\frac{\varphi_{1,x}}{\varphi_1} = \frac{\Theta_x + \frac{1}{\sqrt{2}}i\xi\Theta}{\Theta} = \frac{1}{\sqrt{2}} \sum_{k=1}^{+\infty} (\psi\omega_k)\lambda^{-k} + \frac{i\lambda}{\sqrt{2}}$$

则有

$$\frac{\Theta_x}{\Theta} = \frac{1}{\sqrt{2}} \sum_{k=1}^{+\infty} (\psi\omega_k)\lambda^{-k} + \frac{i\lambda}{\sqrt{2}} - \frac{i\xi}{\sqrt{2}} \tag{2.20}$$

已知  $\Theta$  为关于  $\lambda$  解析函数，且  $\xi = \sqrt{\lambda^2 - 1}$ ，下面分析公式(2.20)右边部分，对  $\xi$  关于  $\lambda$  展开，并合并  $\lambda$  的同阶系数，则



$$\begin{aligned}
 \frac{\Theta_x}{\Theta} &= \frac{1}{\sqrt{2}} \sum_{k=1}^{+\infty} (\psi \omega_k) \lambda^{-k} + \frac{i\lambda}{\sqrt{2}} - \frac{1}{\sqrt{2}} i\xi \\
 &= \frac{1}{\sqrt{2}} \sum_{k=1}^{+\infty} (\psi \omega_k) \lambda^{-k} + \frac{i\lambda}{\sqrt{2}} - \frac{1}{\sqrt{2}} i\lambda \left( \sum_{k=0}^{+\infty} \frac{\prod_{j=0}^{k-1} \left( j - \frac{1}{2} \right)}{k!} \lambda^{-2k} \right) \\
 &= \frac{1}{\sqrt{2}} \sum_{k=1}^{+\infty} (\psi \omega_k) \lambda^{-k} - \frac{1}{\sqrt{2}} i \left( \sum_{k=1}^{+\infty} \frac{\prod_{j=0}^{k-1} \left( j - \frac{1}{2} \right)}{k!} \lambda^{-2k+1} \right)
 \end{aligned} \tag{2.21}$$

同样，利用式(2.15)有

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}_t = \begin{pmatrix} p_1 & p_2 \\ p_3 & -p_1 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \tag{2.22}$$

根据式(2.22)，有

$$\frac{\varphi_{1,t}}{\varphi_1} = (\ln \varphi_1)_t = p_1 + p_2 \phi = \sum_{k=-3}^{+\infty} T_k \lambda^{-k}.$$

其中：

$$\begin{aligned}
 T_{-3} &= \sqrt{2}i, \\
 T_{-2} &= -i, \\
 T_{-1} &= \frac{\sqrt{2}i}{2} |\psi|^2 + \sqrt{2} \psi \omega_1, \\
 T_0 &= -\frac{i}{2} |\psi|^2 + \frac{1}{2} (\psi_x \bar{\psi} - \psi_x \bar{\psi}_x) - \psi \omega_1 - i \psi_x \omega_1 + \sqrt{2} \psi \omega_2, \\
 T_k &= \frac{\sqrt{2}i}{2} \psi_x \omega_k + \frac{\sqrt{2}}{2} (|\psi|^2 \psi - \psi_{xx}) \omega_1 - (\psi + i \psi_x) \omega_{k+1} + \sqrt{2} \psi \omega_{k+2}, (k \geq 1)
 \end{aligned}$$

综合以上，我们得到  $\varphi_1$  的关于  $x, t$  的二阶微分

$$\begin{aligned}
 \left( \frac{\varphi_{1,x}}{\varphi_1} \right)_t &= (\ln \varphi_1)_{xt} = \frac{1}{\sqrt{2}} \sum_{k=1}^{+\infty} (\psi \omega_k)_t \lambda^{-k} + \left( \frac{i\lambda}{\sqrt{2}} \right)_t \\
 \left( \frac{\varphi_{1,t}}{\varphi_1} \right)_x &= (\ln \varphi_1)_{tx} = \sum_{k=-3}^{+\infty} (T_k)_x \lambda^{-k}
 \end{aligned}$$

由  $\varphi_1 = \Theta(\lambda; x, t) e^{\left[ \frac{1}{\sqrt{2}} i \xi x + \left[ \sqrt{2} \left( \lambda^2 + \frac{1}{2} \right) - \lambda \right] i \xi t \right]}$ ，则有

$$\left( \frac{\Theta_x}{\Theta} \right)_t = \left( \frac{\Theta_t}{\Theta} \right)_x = \left( \frac{1}{\sqrt{2}} \sum_{k=1}^{+\infty} (\psi \omega_k) \lambda^{-k} - \frac{1}{\sqrt{2}} i \left( \sum_{k=1}^{+\infty} \frac{\prod_{j=0}^{k-1} \left( j - \frac{1}{2} \right)}{k!} \lambda^{-2k+1} \right) \right)_t = \sum_{k=-3}^{+\infty} (T_k)_t \lambda^{-k}.$$

根据相容性条件  $(\ln \Theta)_{xt} = (\ln \Theta)_x$ ，可得到 AKNS 方程族的守恒律。

当  $k = 2n + 1, n \in N$ ，即  $k$  为奇数时，

$$\left( \frac{1}{\sqrt{2}} \psi \omega_k - \frac{\prod_{j=0}^{k-1} \left( j - \frac{1}{2} \right)}{k!} \right)_t = (T_k)_x \quad (2.23)$$

当  $k = 2n, n \in N$ ，即  $k$  为偶数时，

$$\left( \frac{1}{\sqrt{2}} \psi \omega_k \right)_t = (T_k)_x \quad (2.24)$$

结合公式(2.23)和公式(2.24)，下面列出其中两个守恒律

$$\left( \frac{1}{\sqrt{2}} \psi \omega_1 + \frac{\sqrt{2}}{4} i \right)_t = \left( \frac{1}{\sqrt{2}} (-|\psi|^2 + 1) \right)_t = (T_1)_x \quad (2.25)$$

$$\begin{aligned} & \left( \frac{1}{\sqrt{2}} \psi \omega_5 + \frac{\sqrt{2}}{32} i \right)_t \\ &= \left\{ \frac{1}{\sqrt{2}} \psi \left[ \frac{i}{8} \left[ (\bar{\psi} \psi \bar{\psi}_x)_x + \frac{1}{2} (\psi \bar{\psi}^2)_{xx} - \bar{\psi}_{xxx} + \psi \left[ \bar{\psi} \left( \bar{\psi}_{xx} - \frac{1}{2} \psi \bar{\psi}^2 \right) + \frac{1}{2} \bar{\psi}^2 \right] \right] \right] \right\} + \frac{\sqrt{2}}{32} i \right\}_t \\ &= (T_5)_x \end{aligned} \quad (2.26)$$

**推论 1.1 的证明** 根据定理 1.1，我们知道方程(1.6)的无穷守恒律如公式(2.23)和(2.24)所示。根据式(2.26)得出守恒量

$$\int_{\mathbb{R}} |\psi_{xx}|^2 + \frac{1}{2} (\psi_x^2 \bar{\psi}^2 + 2 \bar{\psi} \bar{\psi}_x \psi \psi_x + \psi^2 \bar{\psi}^2) + 3 |\psi|^2 |\psi_x|^2 + \frac{1}{2} |\psi|^6 - \frac{1}{2} dx \quad (2.27)$$

根据式(2.25)得出守恒量

$$\int_{\mathbb{R}} \frac{1}{\sqrt{2}} (-|\psi|^2 + 1) dx \quad (2.28)$$

则根据上面两个守恒量，我们得出

$$\begin{aligned} & \int_{\mathbb{R}} |\psi_{xx}|^2 + \frac{1}{2} (\psi_x^2 \bar{\psi}^2 + 2 \bar{\psi} \bar{\psi}_x \psi \psi_x + \psi^2 \bar{\psi}^2) + 3 |\psi|^2 |\psi_x|^2 + \frac{1}{2} |\psi|^6 - \frac{1}{2} dx + \frac{3\sqrt{2}}{2} \int_{\mathbb{R}} \frac{1}{\sqrt{2}} (-|\psi|^2 + 1) dx \\ &= \int_{\mathbb{R}} \left[ |\psi_{xx}|^2 + 3 |\psi|^2 |\psi_x|^2 + \frac{1}{2} (\bar{\psi} \psi_x + \psi \bar{\psi}_x)^2 + (1 - |\psi|^2)^2 \left( 1 + \frac{1}{2} |\psi|^2 \right) \right] dx \\ &= S(\psi) \end{aligned} \quad (2.29)$$

**意义：**文献[10] [11]在证明解的轨道稳定性时，利用了高阶守恒量，但未给出证明，本文给出了基于 Lax 对证明方程无穷守恒律和高阶守恒量的方法。此外，该部分内容为我们进一步利用无约束变分的方法研究方程：

$$i(\psi_x + \psi_t) + \psi_{xx} - |\psi|^2 \psi + i(\psi_{xxx} - 3|\psi|^2 \psi_x) = 0。$$

的暗孤子解:  $\psi = e^{-it} u_0(x)$ ,  $u_0(x) = \tanh\left(\frac{x}{\sqrt{2}}\right)$ ,  $x \in \mathbb{R}$  的轨道稳定性奠定了基础。

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