Advances in Applied Mathematics 应用数学进展, 2021, 10(4), 1255-1269
Published Online April 2021 in Hans. http://www.hanspub.org/journal/aam
https://doi.org/10.12677/aam.2021.104136

Hans汉斯

具有左右分数阶导数和时滞的非瞬时脉冲微分方程非线性边值问题

张雨馨

上海理工大学理学院，上海
Email: yuxinzhang9@163.com

收稿日期：2021年3月27日；录用日期：2021年4月15日；发布日期：2021年4月29日

摘　要

本文研究了一类特殊的具有左右分数阶导数和时滞的非瞬时脉冲微分方程，该方程具有交叉时滞，且带有非线性边界条件。并基于上下解方法得到多个正解存在性定理。

关键词

左右分数阶导数，时滞，非瞬时脉冲微分方程，非线性边界条件，上下解方法

Nonlinear Boundary Value Problems for Non Instantaneous Pulse Differential Equations with Left-Right Fractional Derivatives and Delays

Yuxin Zhang

College of Science, China University of Shanghai for Science and Technology, Shanghai
Email: yuxinzhang9@163.com

Received: Mar. 27th, 2021; accepted: Apr. 15th, 2021; published: Apr. 29th, 2021

Abstract

In this paper, we study a class of special non-instantaneous impulsive differential equations with left and right fractional derivatives and delays. The equations have cross delays and nonlinear

文章引用: 张雨馨. 具有左右分数阶导数和时滞的非瞬时脉冲微分方程非线性边值问题[J]. 应用数学进展, 2021, 10(4): 1255-1269. DOI: 10.12677/aam.2021.104136

boundary conditions. Based on the upper and lower solution method, we obtain the existence theorems of multiple positive solutions.

Keywords

Left-Right Fractional Derivatives, Time Delay, Non-Instantaneous Pulse, Nonlinear Boundary Conditions, Upper and Lower Solution Method

1. 引言

分数阶微分方程非常适合刻画具有记忆和遗传性质的材料及过程，其对复杂系统的描述具有建模简单、描述准确、参数物理意义清楚等优势，因此也是复杂力学、物理过程数学建模的重要工具，如利用分数阶微积分在不同粘弹性流体的本构关系，在非牛顿流体中进行应用；分数阶 SEIR 传染病模型可以准确研究传染病、社交网络信息传播等方面的问题；在含未知参数的情况下，利用非线性分数阶系统状态估计分别含有分数阶有色过程噪声和有色测量噪声的连续时间问题。

在分数阶微分方程边值问题的研究[1] [2] [3] [4] [5]中，有时需要考虑左侧和右侧不同的定义，而同时带有左侧和右侧分数阶导数的微分方程相对于只含有分数阶右导数或左导数的分数阶微分方程，它的应用范围[6] [7] [8] [9]更加广泛，其在机械力学、生物工程、物理学、经济学等自然科学领域建立的数学模型中经常出现，并且具有很重要的作用，如用来分析空气中充满粒状材料时的室内外的温度数据等。

文献[10]研究了带有左右分数阶导数的微分方程边值问题：

$$\begin{cases} {}^{c}D_{b^-}^{\alpha}\, {}^{c}D_{a^+}^{\alpha} T(t) + \lambda T(t) = 0, \\ T(a) = T_0, T(b) = T_1, \end{cases}$$

其中，$T \in C[0,1]$，${}^{c}D_{b^-}^{\alpha}$ 为 Caputo 分数阶右导数，${}^{c}D_{a^+}^{\alpha}$ 为 Caputo 分数阶左导数。作者利用分数阶微分方程的数值解针对实际问题进行分析。

文献[3]研究了带有左右阶导数的耦合微分方程边值问题：

$$\begin{cases} -\left({}_{0^+}D_t^{\alpha} u(t)\right) = f(t, v(t)), \quad t \in (0,T), \\ -\left({}_{t}D_{T^-}^{\beta} v(t)\right) = g(t, u(t)), \quad t \in (0,T), \\ u(0) = 0, \quad {}_{0^+}D_t^{\alpha-1} u(T) = r_1\, {}_{t}D_{T^-}^{\beta-1} v(\xi), \\ v(T) = 0, \quad {}_{t}D_{T^-}^{\beta-1} v(0) = r_2\, {}_{0^+}D_t^{\alpha-1} u(\xi), \end{cases}$$

其中，${}_{0^+}D_t^{\alpha}, {}_{t}D_{T^-}^{\beta}$ 分别是 α 阶 R-L 分数阶左导数和 β 阶 R-L 分数阶右导数，$0 < \alpha, \beta \le 2$，$\xi \in [0,T]$。作者运用上下解方法获得了边值问题解的存在性定理。

基于以上启发，本文研究含有左右分数阶导数和时滞的非瞬时脉冲微分方程非线性边值问题：

$$\begin{cases} {}_{t}^{c}D_{\xi^-}^{\alpha}u(t)=f_1\left(t,u(t),u(t+\tau_1)\right), & t\in[0,\xi], \\ {}_{\xi^+}^{\ c}D_{t}^{\beta}u(t)=f_2\left(t,u(t),u(t-\tau_2)\right), & t\in(\xi,1], \\ \Delta u(\xi)=I\left(\xi,u(\xi)\right),\ \Delta u'(\xi)=Q\left(\xi,u(\xi)\right), \\ h_0\left(u(0),u(1)\right)=0, \quad h_1\left({}_{t}^{c}D_{\xi^-}^{\alpha-1}u(0),{}_{\xi^+}^{\ c}D_{t}^{\beta-1}u(1)\right)=0 \end{cases} \tag{1}$$

解的存在性与多解性。其中，${}_{t}^{c}D_{\xi^-}^{\alpha}$ 是右侧 Caputo 分数阶导数，${}_{\xi^+}^{\ c}D_{t}^{\beta}$ 是左侧 Caputo 分数阶导数，$1<\alpha,\beta\le 2$，$\xi\in(0,1)$，$u(\xi^+)=\lim_{\varepsilon\to 0^+}u(\xi+\varepsilon)$，$u(\xi^-)=\lim_{\varepsilon\to 0^-}u(\xi+\varepsilon)$，$\tau_1\in(0,1-\xi)$，$\tau_2\in(0,\xi)$，$f_1\in C\left([0,\xi]\times\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+\right)$，$f_2\in C\left((\xi,1]\times\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+\right)$，$I,Q\in C\left(\mathbb{R},\mathbb{R}^+\right)$，$h_0,h_1\in C\left(\mathbb{R}^2,\mathbb{R}\right)$ 为给定的非线性函数。

2. 线性边值问题

定义 1 [11]：若 $\alpha>0$，$a<b\in\mathbb{R}$ 则

$$ {}_{a^+}I_t^{\alpha}\ {}_{a^+}^{\ c}D_t^{\alpha}u(t)=u(t)+c_1+c_2(t-a)+c_3(t-a)^2+\cdots+c_n(t-a)^{n-1}, $$

$$ {}_{t}I_{b^-}^{\alpha}\ {}_{t}^{c}D_{b^-}^{\alpha}u(t)=u(t)+d_1+d_2(b-t)+d_3(b-t)^2+\cdots+d_n(b-t)^{n-1}, $$

其中 $c_i,d_i\in\mathbb{R},i=1,2,\cdots,n,n\in\mathbb{N}$。

引理 1 [12]：令 E 为 Banach 空间，且 $P\subset E$ 是一个正规体锥。如果存在 α_1，β_1，α_2，$\beta_2\in P$ 使

$$\alpha_1\prec\beta_1\prec\alpha_2\prec\beta_2,$$

且 $A:[\alpha_1,\beta_2]\to E$ 是全连续算子，且为强增算子，使

$$\alpha_1\prec A\alpha_1,\quad A\beta_1\prec\beta_1,\quad \alpha_2\prec A\alpha_2,\quad A\beta_2\preceq\beta_2.$$

则算子 A 至少有三个不动点 x_1,x_2,x_3 使得

$$\alpha_1\preceq x_1\prec\prec\beta_1,\quad \alpha_2\prec\prec x_2\preceq\beta_2,\quad \alpha_2\npreceq x_2\npreceq\beta_2.$$

令 $J=[0,1]$，$J_0=J\setminus\xi$，

$E=PC[J,\mathbb{R}]=\left\{u:J\to\mathbb{R}:u\text{在}J_0\text{上是连续的},u(\xi^+)\text{与}u(\xi^-)\text{存在且}u(\xi^-)=u(\xi)\right\}$。显然 E 是 Banach 空间且定义其范数为

$$\|u\|=\sup_{t\in[0,1]}|u(t)|.$$

令 $\|u\|_{[0,\xi]}=\sup_{t\in[0,\xi]}|u(t)|$，$\|u\|_{(\xi,1]}=\sup_{t\in(\xi,1]}|u(t)|$，则 $\|u\|=\max\left\{\|u\|_{[0,\xi]},\|u\|_{(\xi,1]}\right\}$。

引理 2：令 $h\in C\left([0,\xi],\mathbb{R}^+\right),y\in C\left((\xi,1],\mathbb{R}^+\right)$，对任意 $m_i,n_i\in\mathbb{R},i=1,2$，且 $\Delta_1\neq 0$。则边值问题

$$\begin{cases} {}_{t}^{c}D_{\xi^-}^{\alpha}u(t)=h(t), \ t\in(0,\xi), \\ {}_{\xi^+}^{\ c}D_{t}^{\beta}u(t)=y(t), t\in(\xi,1), \\ \Delta u(\xi)=I,\ \Delta u'(\xi)=Q, \\ m_1u(0)+n_1u(1)=\gamma_0,\ m_2\,{}_{t}^{c}D_{\xi^-}^{\alpha-1}u(0)+n_2\,{}_{\xi^+}^{\ c}D_{t}^{\beta-1}u(1)=\gamma_1 \end{cases} \tag{2}$$

在 E 中存在唯一解

$$u(t)=\begin{cases}\int_0^{\xi}G_1(t,s)h(s)\mathrm{d}s+\int_{\xi}^{1}g_2(t,s)y(s)\mathrm{d}s+\Delta_2+\dfrac{t}{\Delta_1}\left(-\dfrac{Qn_2(1-\xi)^{2-\beta}}{\Gamma(3-\beta)}-\gamma_1\right),t\in[0,\xi],\\ \int_{\xi}^{1}G_2(t,s)y(s)\mathrm{d}s+\int_0^{\xi}g_1(t,s)h(s)\mathrm{d}s+\Delta_3+\dfrac{t}{\Delta_1}\left(\dfrac{Qm_2}{\Gamma(3-\beta)}\xi^{2-\alpha}-\gamma_1\right),t\in(\xi,1].\end{cases} \tag{3}$$

其中

$$G_1(t,s)=\begin{cases}g_1(t,s),\ 0\le s\le t\le\xi,\\ g_1(t,s)+\dfrac{1}{\Gamma(\alpha)}(s-t)^{\alpha-1},\ 0\le t\le s\le\xi;\end{cases} \tag{4}$$

$$G_2(t,s)=\begin{cases}g_2(t,s)+\dfrac{1}{\Gamma(\beta)}(t-s)^{\beta-1},\ \xi\le s\le t\le 1,\\ g_2(t,s),\ \xi\le t\le s\le 1;\end{cases} \tag{5}$$

记

$$\Delta_1=\frac{m_2\xi^{2-\alpha}}{\Gamma(3-\alpha)}-\frac{n_2(1-\xi)^{2-\beta}}{\Gamma(3-\beta)};\Delta_2=-\frac{1}{m_1+n_1}\left(n_1I+n_1\left(1-\xi+\frac{n_2(1-\xi)^{2-\beta}}{\Delta_1\Gamma(3-\beta)}\right)Q-\frac{n_1}{\Delta_1}\gamma_1-\gamma_0\right);$$

$$\Delta_3=-\frac{1}{m_1+n_1}\left(-m_1I+\left(m_1\xi+n_1+\frac{n_1n_2(1-\xi)^{2-\beta}}{\Delta_1\Gamma(3-\beta)}\right)Q-\frac{n_1}{\Delta_1}\gamma_1-\gamma_0\right);$$

$$g_1(t,s)=-\frac{1}{m_1+n_1}\left(\frac{m_1}{\Gamma(\alpha)}s^{\alpha-1}+\frac{n_1m_2}{\Delta_1}\right)+\frac{tm_2}{\Delta_1};g_2(t,s)=-\frac{n_1}{m_1+n_1}\left(\frac{1}{\Gamma(\beta)}(1-s)^{\beta-1}+\frac{n_2}{\Delta_1}\right)-\frac{tn_2}{\Delta_1}.$$

证明：设$u\in E$是边值问题(2)的解，则由定义 1 可知存在常数$c_i\in\mathbb{R}$，$i=0,1,2,3$使${}_t^CD_{\xi^-}^{\alpha}u(t)=h(t)$的解为：

$$u(t)={}_tI_{\xi^-}^{\alpha}h(t)+c_0+c_1t=\frac{1}{\Gamma(\alpha)}\int_t^{\xi}(s-t)^{\alpha-1}h(s)\mathrm{d}s+c_0+c_1t,$$

$$u'(t)=-\frac{1}{\Gamma(\alpha-1)}\int_t^{\xi}(s-t)^{\alpha-2}h(s)\mathrm{d}s+c_1,$$

$${}_t^cD_{\xi^-}^{\alpha-1}u(t)=\int_t^{\xi}h(s)\mathrm{d}s-\frac{c_1}{\Gamma(3-\alpha)}(\xi-t)^{2-\alpha}$$

${}_{\xi^+}^{\ c}D_t^{\beta}u(t)=y(t)$的解为：

$$u(t)={}_{\xi^+}I_t^{\beta}y(t)+c_2+c_3t=\frac{1}{\Gamma(\beta)}\int_{\xi}^{t}(t-s)^{\beta-1}y(s)\mathrm{d}s+c_2+c_3t,$$

$$u'(t)=\frac{1}{\Gamma(\beta-1)}\int_{\xi}^{t}(t-s)^{\beta-2}y(s)\mathrm{d}s+c_3,$$

$${}_{\xi^+}^{\ c}D_t^{\beta-1}u(t)=\int_{\xi}^{t}y(s)\mathrm{d}s+\frac{c_3}{\Gamma(3-\beta)}(t-\xi)^{2-\beta}$$

由边值条件$\Delta u(\xi)=I,\Delta u'(\xi)=Q$得

$$\begin{cases} c_2 - c_0 = I - Q\xi, \\ c_3 - c_1 = Q. \end{cases}$$

再由边值条件 $m_1u(0)+n_1u(1)=\gamma_0$， $m_2\,{}_t^cD_{\xi^-}^{\alpha-1}u(0)+n_2\,{}_{\xi^+}^{\;c}D_t^{\beta-1}u(1)=\gamma_1$，可得

$$\begin{cases} c_0 = -\dfrac{1}{m_1+n_1}\left(\displaystyle\int_0^\xi\left(\frac{m_1}{\Gamma(\alpha)}s^{\alpha-1}+\frac{n_1m_2}{\Delta_1}\right)h(s)\mathrm{d}s+n_1\int_\xi^1\left(\frac{1}{\Gamma(\beta)}(1-s)^{\beta-1}-\frac{n_2}{\Delta_1}\right)y(s)\mathrm{d}s\right. \\ \qquad\left.+n_1I+n_1\left(1-\xi+\dfrac{n_2(1-\xi)^{2-\beta}}{\Delta_1\Gamma(3-\beta)}\right)Q-\dfrac{n_1}{\Delta_1}\gamma_1-\gamma_0\right), \\ c_1 = \dfrac{1}{\Delta_1}\left(m_2\displaystyle\int_0^\xi h(s)\mathrm{d}s+n_2\int_\xi^1 y(s)\mathrm{d}s+\frac{Qn_2(1-\xi)^{2-\beta}}{\Gamma(3-\beta)}-\gamma_1\right), \\ c_2 = -\dfrac{1}{m_1+n_1}\left(\displaystyle\int_0^\xi\left(\frac{m_1s^{\alpha-1}}{\Gamma(\alpha)}+\frac{n_1m_2}{\Delta_1}\right)h(s)\mathrm{d}s+n_1\int_\xi^1\left(\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}-\frac{n_2}{\Delta_1}\right)y(s)\mathrm{d}s\right. \\ \qquad\left.-m_1I+\left(m_1\xi+n_1+\dfrac{n_1n_2(1-\xi)^{2-\beta}}{\Delta_1\Gamma(3-\beta)}\right)Q-\dfrac{n_1}{\Delta_1}\gamma_1-\gamma_0\right), \\ c_3 = \dfrac{1}{\Delta_1}\left(m_2\displaystyle\int_0^\xi h(s)\mathrm{d}s+n_2\int_\xi^1 y(s)\mathrm{d}s+\frac{Qm_2}{\Gamma(3-\beta)}\xi^{2-\alpha}-\gamma_1\right). \end{cases}$$

因此，当 $t\in[0,\xi]$ 时，

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)}\int_t^\xi(s-t)^{\alpha-1}h(s)\mathrm{d}s-\frac{1}{m_1+n_1}\left(\int_0^\xi\left(\frac{m_1}{\Gamma(\alpha)}s^{\alpha-1}+\frac{n_1m_2}{\Delta_1}\right)h(s)\mathrm{d}s\right. \\ &\quad\left.+n_1\int_\xi^1\left(\frac{1}{\Gamma(\beta)}(1-s)^{\beta-1}-\frac{n_2}{\Delta_1}\right)y(s)\mathrm{d}s+n_1I+n_1\left(1-\xi+\frac{n_2(1-\xi)^{2-\beta}}{\Delta_1\Gamma(3-\beta)}\right)Q-\frac{n_1}{\Delta_1}\gamma_1-\gamma_0\right) \\ &\quad+\frac{t}{\Delta_1}\left(m_2\int_0^\xi h(s)\mathrm{d}s+n_2\int_\xi^1 y(s)\mathrm{d}s+\frac{Qn_2(1-\xi)^{2-\beta}}{\Gamma(3-\beta)}-\gamma_1\right) \\ &= \int_0^\xi G_1(t,s)h(s)\mathrm{d}s+\int_\xi^1 g_2(t,s)y(s)\mathrm{d}s+\Delta_2+\frac{t}{\Delta_1}\left(\frac{Qn_2(1-\xi)^{2-\beta}}{\Gamma(3-\beta)}-\gamma_1\right). \end{aligned}$$

当 $t\in(\xi,1]$ 时，

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\beta)}\int_\xi^t(t-s)^{\beta-1}y(s)\mathrm{d}s-\frac{1}{m_1+n_1}\left(\int_0^\xi\left(\frac{m_1}{\Gamma(\alpha)}s^{\alpha-1}+\frac{n_1m_2}{\Delta_1}\right)h(s)\mathrm{d}s\right. \\ &\quad\left.+n_1\int_\xi^1\left(\frac{1}{\Gamma(\beta)}(1-s)^{\beta-1}+\frac{n_2}{\Delta_1}\right)y(s)\mathrm{d}s-m_1I+\left(m_1\xi+n_1+\frac{n_1n_2(1-\xi)^{2-\beta}}{\Delta_1\Gamma(3-\beta)}\right)Q-\frac{n_1}{\Delta_1}\gamma_1-\gamma_0\right) \\ &\quad+\frac{t}{\Delta_1}\left(m_2\int_0^\xi h(s)\mathrm{d}s+n_2\int_\xi^1 y(s)\mathrm{d}s+\frac{Qm_2}{\Gamma(3-\beta)}\xi^{2-\alpha}-\gamma_1\right) \\ &= \int_\xi^1 G_2(t,s)y(s)\mathrm{d}s+\int_0^\xi g_1(t,s)h(s)\mathrm{d}s+\Delta_3+\frac{t}{\Delta_1}\left(\frac{Qm_2}{\Gamma(3-\beta)}\xi^{2-\alpha}-\gamma_1\right). \end{aligned}$$

易证(3)是方程(2)的解，反之亦然。

证毕。

为了以后证明，我们给出如下假设：

(H1) $m_i,n_i\in\mathbb{R}(i=1,2)$，$n_1>0,n_2>0,-n_1>m_1>-\frac{n_1}{\xi},\gamma_1\le 0,\gamma_0\le 0$，

$$m_2>\max\left\{\frac{\Gamma(3-\alpha)\xi^{\alpha-1}n_2}{\Gamma(3-\beta)(\Gamma(3-\alpha)\Gamma(\alpha)-\xi)},-\frac{\Gamma(3-\alpha)n_2(1-\xi)^{2-\beta}}{\Gamma(3-\beta)\xi^{2-\alpha}}\right\}.$$

引理 3： 假设(H1)成立，则由式(4)、(5)定义的函数 $G_i(t,s),i=1,2$ 满足以下性质：

1) $0<G_1(0,s)\le G_1(t,s)\le G_1(\xi,s)$，对任意 $(t,s)\in[0,\xi]\times[0,\xi]$；

2) $0<G_2(\xi,s)\le G_2(t,s)\le G_2(1,s)$，对任意 $(t,s)\in[\xi,1]\times[\xi,1]$。

证明：1) 显然 $G_i(t,s),i=1,2$ 为连续函数。 由(H1)知，$\Delta_1>0$，$m_2>\frac{\xi^{\alpha-1}}{\Gamma(\alpha)}\Delta_1$，则对于 $t\in[0,\xi]$，当 $0\le s\le t\le\xi$ 时，

$$G_1(t,s)=g_1(t,s)=-\frac{1}{m_1+n_1}\left(\frac{m_1}{\Gamma(\alpha)}s^{\alpha-1}+\frac{n_1m_2}{\Delta_1}\right)+\frac{tm_2}{\Delta_1},\frac{\partial g_1(t,s)}{\partial t}=\frac{m_2}{\Delta_1}>0;$$

当 $0\le s\le t\le\xi$ 时，由于 $G_1(t,s)=g_1(t,s)+\frac{1}{\Gamma(\alpha)}(s-t)^{\alpha-1}$，

$$\frac{\partial G_1(t,s)}{\partial t}=-\frac{1}{\Gamma(\alpha-1)}(s-t)^{\alpha-2}+\frac{m_2}{\Delta_1},\frac{\partial^2 G_1(t,s)}{\partial t^2}=\frac{\alpha-2}{\Gamma(\alpha-1)}(s-t)^{\alpha-3}<0,$$

则 $\frac{\partial G_1(t,s)}{\partial t}\ge\frac{\partial G_1(s,s)}{\partial t}=\frac{m_2}{\Delta_1}>0$。因此，$G_1(t,s)$ 是关于 t 的单调递增函数，且

$$G_1(0,s)\le G_1(t,s)\le G_1(\xi,s).$$

又

$$G_1(0,s)=g_1(0,s)+\frac{1}{\Gamma(\alpha)}s^{\alpha-1}=-\frac{1}{m_1+n_1}\left(\frac{m_1}{\Gamma(\alpha)}s^{\alpha-1}+\frac{n_1m_2}{\Delta_1}\right)+\frac{1}{\Gamma(\alpha)}s^{\alpha-1}$$
$$=-\frac{1}{m_1+n_1}\left(\frac{-n_1}{\Gamma(\alpha)}s^{\alpha-1}+\frac{n_1m_2}{\Delta_1}\right)>-\frac{n_1}{m_1+n_1}\left(\frac{-1}{\Gamma(\alpha)}\xi^{\alpha-1}+\frac{m_2}{\Delta_1}\right)>0,$$

则 $0<G_1(0,s)\le G_1(t,s)\le G_1(\xi,s)$ 成立。

2) 对于 $t\in[\xi,1]$，当 $\xi\le t\le s\le 1$ 时，

$$G_2(t,s)=g_2(t,s)=-\frac{n_1}{m_1+n_1}\left(\frac{1}{\Gamma(\beta)}(1-s)^{\beta-1}+\frac{n_2}{\Delta_1}\right)+\frac{tn_2}{\Delta_1},\frac{\partial g_2(t,s)}{\partial t}=\frac{n_2}{\Delta_1}>0;$$

当 $\xi\le s\le t\le 1$ 时，

$$G_2(t,s)=g_2(t,s)+\frac{1}{\Gamma(\beta)}(t-s)^{\beta-1},\frac{\partial G_2(t,s)}{\partial t}=\frac{n_2}{\Delta_1}+\frac{1}{\Gamma(\beta-1)}(t-s)^{\beta-2}>0,$$

则 $G_2(t,s)$ 是关于 t 的单调递增函数，那么

$$G_2(\xi,s)\le G_2(t,s)\le G_2(1,s).$$

又

$$G_2(\xi,s)=-\frac{n_1}{m_1+n_1}\left(\frac{1}{\Gamma(\beta)}(1-s)^{\beta-1}+\frac{n_2}{\Delta_1}\right)+\frac{n_2}{\Delta_1}=\frac{1}{m_1+n_1}\left(\frac{n_1}{\Gamma(\beta)}(1-s)^{\beta-1}+\frac{n_2\left(1-(m_1+n_1)\xi\right)}{\Delta_1}\right)>0,$$

因此，$0<G_2(\xi,s)\le G_2(t,s)\le G_2(1,s)$ 成立。

证毕。

引理 4：若(H1)成立，$I,Q\in\mathbb{R}^+$，$\Delta_1\ne 0$。若 u 满足

$$\begin{cases} {}_t^cD_{\xi^-}^{\alpha}u(t)\ge 0,\ t\in(0,\xi),\\ {}_{\xi^+}^{\ c}D_t^{\beta}u(t)\ge 0,\ t\in(\xi,1),\\ \Delta u(\xi)=I,\ \Delta u'(\xi)=Q,\\ m_1u(0)+n_1u(1)\le 0,\ m_2\,{}_t^cD_{\xi^-}^{\alpha-1}u(0)+n_2\,{}_{\xi^+}^{\ c}D_t^{\beta-1}u(1)\ge 0, \end{cases}\tag{4}$$

则 $u(t)\ge 0$，$t\in[0,1]$。

证明：对任意 $h\in C([0,\xi],\mathbb{R}^+)$，$y\in C((\xi,1],\mathbb{R}^+)$，由于 $\gamma_1\le 0$，$\gamma_0\le 0$ 为常数。考虑以下边值问题:

$$\begin{cases} {}_t^cD_{\xi^-}^{\alpha}u(t)=h(t),\ t\in[0,\xi],\\ {}_{\xi^+}^{\ c}D_t^{\beta}u(t)=y(t),\ t\in(\xi,1],\\ \Delta u(\xi)=I,\ \Delta u'(\xi)=Q,\\ m_1u(0)+n_1u(1)=\gamma_0,\ m_2\,{}_t^cD_{\xi^-}^{\alpha-1}u(0)+n_2\,{}_{\xi^+}^{\ c}D_t^{\beta-1}u(1)=\gamma_1. \end{cases}$$

由引理 2 可得

$$u(t)=\begin{cases} \int_0^{\xi}G_1(t,s)h(s)\mathrm{d}s+\int_{\xi}^{1}g_2(t,s)y(s)\mathrm{d}s+\Delta_2+\dfrac{t}{\Delta_1}\left(\dfrac{Qn_2(1-\xi)^{2-\beta}}{\Gamma(3-\beta)}-\gamma_1\right),t\in[0,\xi],\\ \int_{\xi}^{1}G_2(t,s)y(s)\mathrm{d}s+\int_0^{\xi}g_1(t,s)h(s)\mathrm{d}s+\Delta_2+\dfrac{t}{\Delta_1}\left(\dfrac{Qm_2}{\Gamma(3-\beta)}\xi^{2-\alpha}-\gamma_1\right),t\in(\xi,1]. \end{cases}$$

由(H1)可得，$\Delta_1,\Delta_2,\Delta_3>0$，$\dfrac{Qn_2(1-\xi)^{2-\beta}}{\Gamma(3-\beta)}-\gamma_1>0$，$\dfrac{Qm_2\xi^{2-\alpha}}{\Gamma(3-\beta)}-\gamma_1>0$，$t\in[0,1]$。再由引理 3 可得 $u(t)\ge 0$，$t\in[0,1]$ 显然成立。

证毕。

3. 分数阶微分方程的上下解方法

为方便叙述，我们假设下文满足以下假设:

(H2) 对任意 $u_1\le u_2,v_1\le v_2$，$f(t,u_1,v_1)\le f(t,u_2,v_2),t\in[0,\xi]$，

$$g(t,u_1,v_1)\le g(t,u_2,v_2),t\in(\xi,1],\ \ I(u_1)\le I(u_2),Q(u_1)\le Q(u_2),t\in[0,1].$$

对任意 $u_1<u_2,v_1<v_2$，$f(t,u_1,v_1)<f(t,u_2,v_2),t\in[0,\xi]$，

$$g(t,u_1,v_1)<g(t,u_2,v_2),t\in(\xi,1],\ \ I(u_1)<I(u_2),Q(u_1)<Q(u_2),t\in[0,1].$$

(H3) 若(H1)成立，且 $x_1,x_2,y_1,y_2\in\mathbb{R}$，当 $x_1\le x_2,y_1\le y_2$ 时，

$$h_0(x_2,y_2)-h_0(x_1,y_1)\le -m_1(x_2-x_1)-n_1(y_2-y_1),$$

$$h_1\left(x_2,y_2\right)-h_0\left(x_1,y_1\right)\geq -m_2\left(x_2-x_1\right)-n_2\left(y_2-y_1\right).$$

记 $P=\left\{u\in E:u(t)\geq 0,t\in[0,1]\right\}$，显然 P 为 E 中的正规体锥。且若 $u(t)\leq v(t)$， $t\in[0,1]$，则 $u\preceq v\in P$。对任意 $u\in P$，考虑如下边值问题：

$$\begin{cases} {}_t^cD_{\xi^-}^{\alpha}u(t)=f_1\left(t,x(t),x(t+\tau_1)\right), \quad t\in(0,\xi), \\ {}_{\xi^+}^{\;c}D_t^{\beta}u(t)=f_2\left(t,x(t),x(t-\tau_2)\right), \quad t\in(\xi,1), \\ \Delta u(\xi)=I\left(\xi,x(\xi)\right), \quad \Delta u'(\xi)=Q\left(\xi,x(\xi)\right), \\ m_1u(0)+n_1u(1)=h_0\left(x(0),x(1)\right)+m_1x(0)+n_1x(1):=\gamma_0(x), \\ m_2\,{}_t^cD_{\xi^-}^{\alpha-1}u(0)+n_2\,{}_{\xi^+}^{\;c}D_t^{\beta-1}u(1) \\ =h_1\left({}_t^cD_{\xi^-}^{\alpha-1}x(0),{}_{\xi^+}^{\;c}D_t^{\beta-1}x(1)\right)+m_2\,{}_t^cD_{\xi^-}^{\alpha-1}x(0)+n_2\,{}_{\xi^+}^{\;c}D_t^{\beta-1}x(1):=\gamma_1(x). \end{cases} \tag{5}$$

由引理 2 知，边值问题(5)有唯一解

$$u(t)=\begin{cases} \int_0^{\xi}G_1(t,s)f_1\left(s,x(s),x(s+\tau_1)\right)\mathrm{d}s+\int_{\xi}^1 g_2(t,s)f_2\left(s,x(s),x(s-\tau_2)\right)\mathrm{d}s \\ \quad +\Delta_2^x+\dfrac{t}{\Delta_1}\left(\dfrac{Q\left(\xi,x(\xi)\right)n_2(1-\xi)^{2-\beta}}{\Gamma(3-\beta)}-\gamma_1(x)\right), \ t\in[0,\xi], \\ \int_{\xi}^1 G_2(t,s)f_2\left(s,x(s),x(s-\tau_2)\right)\mathrm{d}s+\int_0^{\xi}g_1(t,s)f_1\left(s,x(s),x(s+\tau_1)\right)\mathrm{d}s \\ \quad +\Delta_3^x+\dfrac{t}{\Delta_1}\left(\dfrac{Q\left(\xi,x(\xi)\right)m_2}{\Gamma(3-\beta)}\xi^{2-\alpha}-\gamma_1(x)\right), \ t\in(\xi,1]. \end{cases}$$

其中

$$\Delta_2^x=-\frac{1}{m_1+n_1}\left(n_1I\left(\xi,x(\xi)\right)+n_1\left(1-\xi+\frac{n_2(1-\xi)^{2-\beta}}{\Delta_1\Gamma(3-\beta)}\right)Q\left(\xi,x(\xi)\right)-\frac{n_1}{\Delta_1}\gamma_1(x)-\gamma_0(x)\right);$$

$$\Delta_3^x=-\frac{1}{m_1+n_1}\left(-m_1I\left(\xi,x(\xi)\right)+\left(m_1\xi+n_1+\frac{n_1n_2}{\Delta_1\Gamma(3-\beta)}\right)Q\left(\xi,x(\xi)\right)-\frac{n_1}{\Delta_1}\gamma_1(x)-\gamma_0(x)\right).$$

定义算子

$$Tx(t)=\begin{cases} \int_0^{\xi}G_1(t,s)f_1\left(s,x(s),x(s+\tau_1)\right)\mathrm{d}s+\int_{\xi}^1 g_2(t,s)f_2\left(s,x(s),x(s-\tau_2)\right)\mathrm{d}s \\ \quad +\Delta_2^x+\dfrac{t}{\Delta_1}\left(\dfrac{Q\left(\xi,x(\xi)\right)n_2(1-\xi)^{2-\beta}}{\Gamma(3-\beta)}-\gamma_1(x)\right), \ t\in[0,\xi], \\ \int_{\xi}^1 G_2(t,s)f_2\left(s,x(s),x(s-\tau_2)\right)\mathrm{d}s+\int_0^{\xi}g_1(t,s)f_1\left(s,x(s),x(s+\tau_1)\right)\mathrm{d}s \\ \quad +\Delta_3^x+\dfrac{t}{\Delta_1}\left(\dfrac{Q\left(\xi,x(\xi)\right)m_2\xi^{2-\alpha}}{\Gamma(3-\beta)}-\gamma_1(x)\right), \ t\in(\xi,1]. \end{cases}$$

引理 5： 若(H1)成立，则 T 为全连续算子。

证明：由引理 3，引理 4 知，对任意 $x\in P$，当 $t\in[0,1]$ 时，$Tx\geq 0$ 显然成立。

因此，$T:P\to P$ 是有意义的。

接下来，我们分两步证明：

第一步： T 是连续算子。

设对任意 $x_n\in P$，$n=1,2,\cdots$ 存在 $x\in P$ 使得当 $n\to\infty$ 时，$\|x_n-x\|\to 0$。则存在 $\bar{M}_0>0$，使得 $\|x_n\|\leq\bar{M}_0$，$\|x\|\leq\bar{M}_0$。又由于 f_1,f_2 连续，$I,Q\in C(\mathbb{R},\mathbb{R})$，且 $\gamma_0(x),\gamma_1(x)\in\mathbb{R}$，则

$$\lim_{n\to\infty}\left(f_1\left(t,x_n(t),x_n(t+\tau_1)\right)-f_1\left(t,x(t),x(t+\tau_1)\right)\right)=0,$$

$$\lim_{n\to\infty}\left(f_2\left(t,x_n(t),x_n(t-\tau_2)\right)-f_2\left(t,x(t),x(t-\tau_2)\right)\right)=0,$$

$$\lim_{n\to\infty}\left|I\left(x_n(t)\right)-I\left(x(t)\right)\right|=0,\ \lim_{n\to\infty}\left|Q\left(x_n(t)\right)-Q\left(x(t)\right)\right|=0,$$

$$\lim_{n\to\infty}\left(\gamma_0(x_n)-\gamma_0(x)\right)=0,\ \lim_{n\to\infty}\left(\gamma_1(x_n)-\gamma_1(x)\right)=0,$$

且存在常数 $\bar{M}_1>0$，使得 $\sup\limits_{(t,u,v)\in A}\left|f_1(t,u,v)\right|\leq\bar{M}_1$，$\sup\limits_{(t,u,v)\in B}\left|f_2(t,u,v)\right|\leq\bar{M}_1$，其中

$$A=[0,\xi]\times\left[-\bar{M}_0,\bar{M}_0\right]\times\left[-\bar{M}_0,\bar{M}_0\right],\ B=[\xi,1]\times\left[-\bar{M}_0,\bar{M}_0\right]\times\left[-\bar{M}_0,\bar{M}_0\right].$$

再由引理 3 可得，当 $t\in[0,\xi]$ 时，

$$\begin{aligned}&\left|T(x_n)-T(x)\right|\\&=\Bigg|\int_0^{\xi}G_1(t,s)\left(f_1\left(s,x_n(s),x_n(s+\tau_1)\right)-f_1\left(s,x(t),x(s+\tau_1)\right)\right)\mathrm{d}s\\&\quad+\int_{\xi}^{1}g_2(t,s)\left(f_2\left(s,x_n(s),x_n(s-\tau_2)\right)-f_2\left(s,x(s),x(s-\tau_2)\right)\right)\mathrm{d}s\\&\quad+\left(\Delta_2^{x_n}-\Delta_2^{x}\right)+\frac{t}{\Delta_1}\left(\frac{\left(Q\left(\xi,x_n(\xi)\right)-Q\left(\xi,x(\xi)\right)\right)n_2(1-\xi)^{2-\beta}}{\Gamma(3-\beta)}-\left(\gamma_1(x_n)-\gamma_1(x)\right)\right)\Bigg|\end{aligned}$$

$$\begin{aligned}&\left|T(x_n)-T(x)\right|\\&=\Bigg|\int_0^{\xi}G_1(t,s)\left(f_1\left(s,x_n(s),x_n(s+\tau_1)\right)-f_1\left(s,x(t),x(s+\tau_1)\right)\right)\mathrm{d}s\\&\quad+\int_{\xi}^{1}g_2(t,s)\left(f_2\left(s,x_n(s),x_n(s-\tau_2)\right)-f_2\left(s,x(s),x(s-\tau_2)\right)\right)\mathrm{d}s\\&\quad+\left(\Delta_2^{x_n}-\Delta_2^{x}\right)+\frac{t}{\Delta_1}\left(\frac{\left(Q\left(\xi,x_n(\xi)\right)-Q\left(\xi,x(\xi)\right)\right)n_2(1-\xi)^{2-\beta}}{\Gamma(3-\beta)}-\left(\gamma_1(x_n)-\gamma_1(x)\right)\right)\Bigg|\end{aligned}$$

则由 Lebesgue 控制收敛定理可知，$\lim\limits_{n\to\infty}\|Tu_n-Tu\|_{[0,\xi]}=0$。同理可得，$\lim\limits_{n\to\infty}\|Tu_n-Tu\|_{(\xi,1]}=0$。

因此，对任意 $t\in[0,1]$，有 $\lim\limits_{n\to\infty}\|Tu_n-Tu\|=0$，则算子 T 是连续算子。

第二步：T 是紧的。

令 $\Omega \subset P$ 为有界集，由 f_1，f_2，I，Q 的连续性得，存在 $\bar{M}_2 > 0$，使得对任意 $t \in [0,\xi]$，$u,v \in \Omega$，有 $|f_1(t,u,v)| \le \bar{M}_2$；对任意 $t \in (\xi,1]$，$u,v \in \Omega$ 有 $|f_2(t,u,v)| \le \bar{M}_2$，$|I| \le \bar{M}_2$，$|Q| \le \bar{M}_2$，$|\gamma_0(x)| \le \bar{M}_2$，$|\gamma_1(x)| \le \bar{M}_2$。

则

$$|\Delta_2^x| = -\frac{1}{m_1+n_1}\left(n_1\left(2-\xi+\frac{n_2(1-\xi)^{2-\beta}+\Gamma(3-\beta)}{\Delta_1\Gamma(3-\beta)}\right)-1\right)\bar{M}_2;$$

$$|\Delta_3^x| = -\frac{1}{m_1+n_1}\left(m_1(\xi-1)+n_1\left(1+\frac{n_2-\Gamma(3-\beta)}{\Delta_1\Gamma(3-\beta)}\right)-1\right)\bar{M}_2$$

$$\begin{aligned}\sup_{t\in[0,\xi]}|Tu(t)| &\le \left|\int_0^{\xi} G_1(\xi,s) f_1(s,x(s),x(s+\tau_1))\mathrm{d}s + \int_{\xi}^{1} g_2(1,s) f_2(s,x(s),x(s-\tau_2))\mathrm{d}s\right.\\ &\quad \left. + \Delta_2^x + \frac{\xi}{\Delta_1}\left(\frac{Q(\xi,x(\xi))n_2(1-\xi)^{2-\beta}}{\Gamma(3-\beta)} - \gamma_1(x)\right)\right|\\ &\le \left(\int_0^{\xi} G_1(\xi,s)\mathrm{d}s + \int_{\xi}^{1} g_2(1,s)\mathrm{d}s - \frac{1}{m_1+n_1}\left(n_1\left(2-\xi+\frac{n_2(1-\xi)^{2-\beta}-\Gamma(3-\beta)}{\Delta_1\Gamma(3-\beta)}\right)-1\right)\right.\\ &\quad \left. + \frac{\xi}{\Delta_1}\left(\frac{n_2(1-\xi)^{2-\beta}}{\Gamma(3-\beta)}-1\right)\right)\bar{M}_2,\end{aligned}$$

$$\begin{aligned}\sup_{t\in(\xi,1]}|Tu(t)| &\le \left|\int_{\xi}^{1} G_2(1,s) f_2(s,x(s),x(s-\tau_2))\mathrm{d}s + \int_0^{\xi} g_1(\xi,s) f_1(s,x(s),x(s+\tau_1))\mathrm{d}s\right.\\ &\quad \left. + \Delta_3^x + \frac{\xi}{\Delta_1}\left(\frac{Q(\xi,x(\xi))m_2}{\Gamma(3-\beta)}\xi^{2-\alpha} - \gamma_1(x)\right)\right|\\ &\le \left(\int_{\xi}^{1} G_2(1,s)\mathrm{d}s + \int_0^{\xi} g_1(\xi,s)\mathrm{d}s - \frac{1}{m_1+n_1}\left(m_1(\xi-1)+n_1\left(1+\frac{n_2-\Gamma(3-\beta)}{\Delta_1\Gamma(3-\beta)}\right)-1\right)\right.\\ &\quad \left. + \frac{1}{\Delta_1}\left(\frac{m_2\xi^{2-\alpha}}{\Gamma(3-\beta)}-1\right)\right)\bar{M}_2.\end{aligned}$$

因此，算子 $T(\Omega)$ 一致有界。

由于 $G_1(t,s)$，$g_2(t,s)$ 在 $[0,\xi]\times[0,\xi]$ 上连续，所以 $G_1(t,s)$，$g_2(t,s)$ 在 $[0,\xi]\times[0,\xi]$ 上一致连续。因此对任意 $\varepsilon > 0$，存在 $0 < \delta_1 < \dfrac{\varepsilon\Delta_1\Gamma(3-\beta)}{2\left|n_2(1-\xi)^{2-\beta}-\Gamma(3-\beta)(1-n_2)\right|\bar{M}_2}$，当 $|t_1-t_2| < \delta_1$ 时，有 $|G_1(t_1,s)-G_1(t_2,s)| < \dfrac{\varepsilon}{4\bar{M}_2}$，$|g_2(t_1,s)-g_2(t_2,s)| < \dfrac{\varepsilon}{4\bar{M}_2}$。因此，对任意的 $t_1,t_2 \in [0,\xi]$，$|t_1-t_2| < \delta_1$，$u \in \Omega$ 有

$$
\begin{aligned}
\left|Tu(t_2)-Tu(t_1)\right| = & \left|\int_0^{\xi}\left(G_1(t_1,s)-G_1(t_2,s)\right) f_1\left(s,u(s),u(s+\tau_1)\right)\mathrm{d}s\right. \\
& +\int_{\xi}^{1}\left(g_2(t_1,s)-g_2(t_2,s)\right) f_2\left(s,u(s),u(s-\tau_2)\right)\mathrm{d}s \\
& \left.+\frac{t_1-t_2}{\Delta_1}\left(\frac{Q\left(\xi,x(\xi)\right)n_2(1-\xi)^{2-\beta}}{\Gamma(3-\beta)}-\gamma_1(x)\right)\right| \\
\le & \bar{M}_2\left(\int_0^{\xi}\left|G_1(t_1,s)-G_1(t_2,s)\right|\mathrm{d}s+\int_{\xi}^{1}\left|g_2(t_1,s)-g_2(t_2,s)\right|\mathrm{d}s\right) \\
& +\left|t_1-t_2\right|\frac{|\,n_2(1-\xi)^{2-\beta}-\Gamma(3-\beta)(1-n_2)\bar{M}_2}{\Delta_1\Gamma(3-\beta)} \\
< & \varepsilon.
\end{aligned}
$$

又由于 $G_2(t,s)$， $g_1(t,s)$ 在 $[\xi,1]\times[\xi,1]$ 上连续，所以 $G_2(t,s)$， $g_1(t,s)$ 在 $[\xi,1]\times[\xi,1]$ 上一致连续。

因此对上述 $\varepsilon>0$，存在 $0<\delta_2<\dfrac{\varepsilon\Delta_1\Gamma(3-\beta)}{2\left|m_2\xi^{2-\alpha}-(m_2+1)\Gamma(3-\beta)\right|}$，当 $\left|t_3-t_4\right|<\delta_2$ 时，有 $\left|G_2(t_3,s)-G_2(t_4,s)\right|<\dfrac{\varepsilon}{4\bar{M}_2}$， $\left|g_1(t_3,s)-g_1(t_4,s)\right|<\dfrac{\varepsilon}{4\bar{M}_2}$。

因此，对任意的 $t_3,t_4\in(\xi,1]$， $\left|t_3-t_4\right|<\delta_2$， $u\in\Omega$，有

$$
\begin{aligned}
\left|Tu(t_3)-Tu(t_4)\right| = & \left|\int_{\xi}^{1}\left(G_2(t_3,s)-G_1(t_4,s)\right) f_2\left(s,u(s),u(s-\tau_2)\right)\mathrm{d}s\right. \\
& +\int_0^{\xi}\left(g_1(t_3,s)-g_1(t_4,s)\right) f_1\left(s,u(s),u(s+\tau_1)\right)\mathrm{d}s \\
& \left.+\frac{t_3-t_4}{\Delta_1}\left(\frac{Q\left(\xi,x(\xi)\right)m_2}{\Gamma(3-\beta)}\xi^{2-\alpha}-\gamma_1(x)\right)\right| \\
\le & \bar{M}_2\left(\int_{\xi}^{1}\left|G_2(t_3,s)-G_1(t_4,s)\right|\mathrm{d}s+\int_0^{\xi}\left|g_1(t_3,s)-g_1(t_4,s)\right|\mathrm{d}s\right) \\
& +\left|t_3-t_4\right|\frac{\left|m_2\xi^{2-\alpha}-(m_2+1)\Gamma(3-\beta)\right|}{\Delta_1\Gamma(3-\beta)}\bar{M}_2 \\
< & \varepsilon.
\end{aligned}
$$

因此，$T(\Omega)$ 在 $[0,\xi],(\xi,1]$ 上等度连续，易知，当 $t\in[0,1]$ 时，对任意 $\varepsilon>0$，存在 $\delta_3>0$，当 $\left|t_5-t_6\right|<\delta_3$ 时 $\left|Tu(t_5)-Tu(t_6)\right|<\varepsilon$， 因此， $T(\Omega)$ 是等度连续的。

由 Arzela-Ascoli 定理知 $T(\Omega)$ 相对列紧。又因为算子 T 是连续算子，所以算子 T 是全连续的。

证毕。

引理 6： T 为强增算子。

证明：对任意 $x_1,x_2\in E, x_1\prec x_2$， 即 $x_1(t)\le x_2(t)$ 且 $x_1(t)\not\equiv x_2(t)$， 由(H2)可得，

$$f_1\left(t,x_2(t),x_2(t+\tau_1)\right)-f_1\left(t,x_1(t),x_1(t+\tau_1)\right)\ge 0,\ t\in[0,\xi],$$

$$f_2\left(t,x_2(t),x_2(t-\tau_2)\right)-f_2\left(t,x_1(t),x_1(t-\tau_2)\right)\ge 0,\ t\in(\xi,1],$$

$$\left(I\left(\xi,x_2(\xi)\right)-I\left(\xi,x_1(\xi)\right)\right)\ge 0,\left(Q\left(\xi,x_2(\xi)\right)-Q\left(\xi,x_1(\xi)\right)\right)\ge 0,\ \ t\in[0,1].$$

由于 $x_1(t)\not\equiv x_2(t)$，则存在区间 $[a,b]\subset[0,\xi]$ 或 $[a,b]\subset(\xi,1]$ 使得当 $t\in[a,b]$ 时，$x_1(t)<x_2(t)$。

因此，当 $[a,b]\subset[0,\xi]$ 时，

$$f_1(t,x_2(t),x_2(t+\tau_1))-f_1(t,x_1(t),x_1(t+\tau_1))>0,$$

$$\left(I(\xi,x_2(\xi))-I(\xi,x_1(\xi))\right)>0,\left(Q(\xi,x_2(\xi))-Q(\xi,x_1(\xi))\right)>0,$$

且由(H3)可得

$$\begin{aligned}\gamma_0(x_2)-\gamma_0(x_1)=&\,h_0(x_2(0),x_2(1))-h_0(x_1(0),x_1(1))\\&+(m_1x_2(0)+n_1x_2(1))-\begin{pmatrix}m_1x_1(0)+n_1x_1(1)\\ \le 0,\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\gamma_1(x_2)-\gamma_1(x_1)=&\,h_1\left({}_t^cD_{\xi^-}^{\alpha-1}x_2(0),{}_{\xi^+}^{\ c}D_t^{\beta-1}x_2(1)\right)-h_1\left({}_t^cD_{\xi^-}^{\alpha-1}x_2(0),{}_{\xi^+}^{\ c}D_t^{\beta-1}x_2(1)\right)\\&+m_2\,{}_t^cD_{\xi^-}^{\alpha-1}x_2(0)+n_2\,{}_{\xi^+}^{\ c}D_t^{\beta-1}x_2(1)-\left(m_2\,{}_t^cD_{\xi^-}^{\alpha-1}x_1(0)+n_2\,{}_{\xi^+}^{\ c}D_t^{\beta-1}x_1(1)\right)\\&\le 0,\end{aligned}$$

$$\begin{aligned}\Delta_2^{x_2}-\Delta_2^{x_1}=&-\frac{1}{m_1+n_1}\Big(n_1\left(I(\xi,x_2(\xi))-I(\xi,x_1(\xi))\right)\Big)\\&+n_1\left(1-\xi+\frac{n_2(1-\xi)^{2-\beta}}{\Delta_1\Gamma(3-\beta)}\right)\left(Q(\xi,x_2(\xi))-Q(\xi,x_1(\xi))\right)\\&-\frac{n_1}{\Delta_1}\left(\gamma_1(x_2)-\gamma_1(x_1)\right)-\left(\gamma_0(x_2)-\gamma_0(x_1)\right)\ge 0,\end{aligned}$$

$$\begin{aligned}&Tx_2(t)-Tx_1(t)\\&=\int_0^\xi G_1(t,s)\left(f_1(s,x_2(s),x_2(s+\tau_1))-f_1(s,x_1(s),x_1(s+\tau_1))\right)\mathrm{d}s+\Delta_2^{x_2}-\Delta_2^{x_1}\\&\quad+\int_\xi^1 g_2(t,s)\left(f_2(t,x_2(t),x_2(t-\tau_2))-f_2(t,x_1(t),x_1(t-\tau_2))\right)\mathrm{d}s\\&\quad+\frac{t}{\Delta_1}\left(\frac{\left(Q(\xi,x_2(\xi))-Q(\xi,x_1(\xi))\right)n_2(1-\xi)^{2-\beta}}{\Gamma(3-\beta)}-\gamma_1(x_2)-\gamma_1(x_1)\right)\\&>\int_0^\xi G_1(t,s)\left(f_1(s,x_2(s),x_2(s+\tau_1))-f_1(s,x_1(s),x_1(s+\tau_1))\right)\mathrm{d}s\\&>0.\end{aligned}$$

同理当 $[a,b]\subset(\xi,1]$ 时，

$$\left(f_2(t,x_2(t),x_2(t-\tau_2))-f_2(t,x_1(t),x_1(t-\tau_2))\right)>0,$$

$$\begin{aligned}\Delta_3^{x_2}-\Delta_3^{x_1}=&-\frac{1}{m_1+n_1}\Big(-m_1\left(I(\xi,x_2(\xi))-I(\xi,x_1(\xi))\right)\Big)\\&+\left(m_1\xi+n_1+\frac{n_1n_2}{\Delta_1\Gamma(3-\beta)}\right)\left(Q(\xi,x_2(\xi))-Q(\xi,x_1(\xi))\right)\\&-\frac{n_1}{\Delta_1}\left(\gamma_1(x_2)-\gamma_1(x_1)-\left(\gamma_0(x_2)-\gamma_0(x_1)\right)\right)\\&\ge 0,\end{aligned}$$

$$
\begin{aligned}
&Tx_2(t)-Tx_1(t)\\
&=\int_\xi^1 G_2(t,s)\big(f_2(s,x_2(s),x_2(s-\tau_2))-f_2(s,x_1(s),x_1(s-\tau_2))\big)\mathrm{d}s+\Delta_3^{x_2}-\Delta_3^{x_1}\\
&\quad+\int_0^\xi g_1(t,s)\big(f_1(t,x_2(t),x_2(t+\tau_1))-f_1(t,x_1(t),x_1(t+\tau_1))\big)\mathrm{d}s\\
&\quad+\frac{t}{\Delta_1}\left(\frac{\big(Q(\xi,x_2(\xi))-Q(\xi,x_1(\xi))\big)m_2}{\Gamma(3-\beta)}\xi^{2-\alpha}-\gamma_1(x_2)-\gamma_1(x_1)\right)\\
&>\int_\xi^1 G_2(t,s)\big(f_2(s,x_2(s),x_2(s-\tau_2))-f_2(s,x_1(s),x_1(s-\tau_2))\big)\mathrm{d}s\\
&>0.
\end{aligned}
$$

综上所述，对任意 $t\in[0,1]$ 有 $(Tx_2)(t)>(Tx_1)(t)$，则 T 为强增算子。

定义 2： 令 $\alpha,\beta\in E$。称 α 为边值问题(1)的一个下解，若 α 满足

$$
\begin{cases}
{}_t^cD_{\xi^-}^{\alpha}\alpha(t)\le f_1(t,\alpha(t),\alpha(t+\tau_1)), & t\in[0,\xi],\\
{}_{\xi^+}^{\ c}D_t^{\beta}\alpha(t)\le f_2(t,\alpha(t),\alpha(t-\tau_2)), & t\in(\xi,1],\\
\Delta\alpha(\xi)\le I(\xi,u(\xi)),\ \Delta\alpha'(\xi)\le Q(\xi,\alpha(\xi)),\\
h_0(\alpha(0),\alpha(1))\le 0,\ h_1\left({}_t^cD_{\xi^-}^{\alpha-1}\alpha(0),{}_{\xi^+}^{\ c}D_t^{\beta-1}\alpha(1)\right)\ge 0.
\end{cases}
$$

称 β 为边值问题(1)的一个下解，若 β 满足

$$
\begin{cases}
{}_t^cD_{\xi^-}^{\alpha}\beta(t)\ge f_1(t,\beta(t),\beta(t+\tau_1)), & t\in[0,\xi],\\
{}_{\xi^+}^{\ c}D_t^{\beta}\beta(t)\ge f_2(t,\beta(t),\beta(t-\tau_2)), & t\in(\xi,1],\\
\Delta\beta(\xi)\ge I(\xi,\beta(\xi)),\quad \Delta\beta'(\xi)\ge Q(\xi,\beta(\xi)),\\
h_0(\beta(0),\beta(1))\ge 0,\ h_1\left({}_t^cD_{\xi^-}^{\alpha-1}\beta(0),{}_{\xi^+}^{\ c}D_t^{\beta-1}\beta(1)\right)\le 0.
\end{cases}
$$

4. 主要结论

定理 1： 假设(H1)、(H2)、(H3)成立，且边值问题(1)存在两个下解 α_1,α_2 和两个上解 β_1,β_2，且 α_2,β_1 不是边值问题(1)的解，

$$\alpha_1\prec\beta_1\prec\alpha_2\prec\beta_2.$$

则边值问题(1)至少存在三个不同的解 u_1，u_2，u_3 满足：

$$\alpha_1\preceq u_1\prec\prec\beta_1,\alpha_2\prec\prec u_2\preceq\beta_2,\alpha_2\npreceq u_3\npreceq\beta_1.$$

证明：令算子 T 在 $[\alpha_1,\beta_1]$ 上，$T|_{[\alpha_1,\beta_1]}$ 也记作 T。由引理 5 和引理 6 可得，$T:[\alpha_1,\beta_1]\to P$ 是一个全连续强增算子。

通过定义算子 T 可得，

$$\begin{cases} {}_{t}^{c}D_{\xi^-}^{\alpha}\left(T\alpha_1\right)(t)=f_1\left(t,\alpha_1(t),\alpha_1(t+\tau_1)\right), \quad t\in(0,\xi), \\ {}_{\xi^+}^{\ c}D_{t}^{\beta}\left(T\alpha_1\right)(t)=f_2\left(t,\alpha_1(t),\alpha_1(t-\tau_2)\right), \quad t\in(\xi,1), \\ \Delta\left(T\alpha_1\right)(\xi)=I\left(\xi,\alpha_1(\xi)\right), \quad \Delta\left(T\alpha_1\right)'(\xi)=Q\left(\xi,\alpha_1(\xi)\right), \\ m_1\left(T\alpha_1\right)(0)+n_1\left(T\alpha_1\right)(1)=h_0\left(\alpha_1(0),\alpha_1(1)\right)+m_1\alpha_1(0)+n_1\alpha_1(1), \\ m_2\,{}_{t}^{c}D_{\xi^-}^{\alpha-1}\left(T\alpha_1\right)(0)+n_2\,{}_{\xi^+}^{\ c}D_{t}^{\beta-1}\left(T\alpha_1\right)(1) \\ =h_1\left({}_{t}^{c}D_{\xi^-}^{\alpha-1}\alpha_1(0),\ {}_{\xi^+}^{\ c}D_{t}^{\beta-1}\alpha_1(1)\right)+m_2\,{}_{t}^{c}D_{\xi^-}^{\alpha-1}\alpha_1(0)+n_2\,{}_{\xi^+}^{\ c}D_{t}^{\beta-1}\alpha_1(1). \end{cases}$$

令 $\alpha(t)=\left(T\alpha_1\right)(t)-\alpha_1(t)$。由于 α_1 是边值问题(1)的一个下解，则

$${}_{t}^{c}D_{\xi^-}^{\alpha}\alpha(t)={}_{t}^{c}D_{\xi^-}^{\alpha}\left(T\alpha_1\right)(t)-{}_{t}^{c}D_{\xi^-}^{\alpha}\alpha_1(t)=f_1\left(t,\alpha_1(t),\alpha_1(t+\tau_1)\right)-{}_{t}^{c}D_{\xi^-}^{\alpha}\alpha_1(t)\ge 0,\ \ t\in(0,\xi),$$

$${}_{\xi^+}^{\ c}D_{t}^{\beta}\alpha(t)={}_{\xi^+}^{\ c}D_{t}^{\beta}\left(T\alpha_1\right)(t)-{}_{\xi^+}^{\ c}D_{t}^{\beta}\alpha_1(t)=f_2\left(t,\alpha_1(t),\alpha_1(t-\tau_2)\right)-{}_{\xi^+}^{\ c}D_{t}^{\beta}\alpha_1(t)\ge 0,\ \ t\in(\xi,1),$$

且

$$\begin{aligned} m_1\alpha(0)+n_1\alpha(1)&=m_1\left(\left(T\alpha_1\right)(0)-\alpha_1(0)\right)+n_1\left(\left(T\alpha_1\right)(1)-\alpha_1(1)\right) \\ &=\left(m_1\left(T\alpha_1\right)(0)+n_1\left(T\alpha_1\right)(1)\right)-\left(m_1\alpha_1(0)+n_1\alpha_1(1)\right) \\ &=h_0\left(\alpha_1(0),\alpha_1(1)\right)+m_1\alpha_1(0)+n_1\alpha_1(1)-\left(m_1\alpha_1(0)+n_1\alpha_1(1)\right) \\ &=h_0\left(\alpha_1(0),\alpha_1(1)\right) \\ &\le 0. \end{aligned}$$

$$\begin{aligned} & m_2\,{}_{t}^{c}D_{\xi^-}^{\alpha-1}\alpha(0)+n_2\,{}_{\xi^+}^{\ c}D_{t}^{\beta-1}\alpha(1) \\ &=m_2\,{}_{t}^{c}D_{\xi^-}^{\alpha-1}\left(\left(T\alpha_1\right)(0)-\alpha_1(0)\right)+n_2\,{}_{\xi^+}^{\ c}D_{t}^{\beta-1}\left(\left(T\alpha_1\right)(1)-\alpha_1(1)\right) \\ &=m_2\,{}_{t}^{c}D_{\xi^-}^{\alpha-1}\left(T\alpha_1\right)(0)+n_2\,{}_{\xi^+}^{\ c}D_{t}^{\beta-1}\left(T\alpha_1\right)(1)-\left(m_2\,{}_{t}^{c}D_{\xi^-}^{\alpha-1}\alpha_1(0)+n_2\,{}_{\xi^+}^{\ c}D_{t}^{\beta-1}\alpha_1(1)\right) \\ &=h_1\left({}_{t}^{c}D_{\xi^-}^{\alpha-1}\alpha_1(0),\ {}_{\xi^+}^{\ c}D_{t}^{\beta-1}\alpha_1(1)\right)+m_2\,{}_{t}^{c}D_{\xi^-}^{\alpha-1}\alpha_1(0)+n_2\,{}_{\xi^+}^{\ c}D_{t}^{\beta-1}\alpha_1(1) \\ &\quad-\left(m_2\,{}_{t}^{c}D_{\xi^-}^{\alpha-1}\alpha_1(0)+n_2\,{}_{\xi^+}^{\ c}D_{t}^{\beta-1}\alpha_1(1)\right)=h_1\left({}_{t}^{c}D_{\xi^-}^{\alpha-1}\alpha_1(0),\ {}_{\xi^+}^{\ c}D_{t}^{\beta-1}\alpha_1(1)\right)\ge 0. \end{aligned}$$

$$\Delta\alpha(\xi)=\Delta\left(T\alpha_1\right)(\xi)-\Delta\alpha_1(\xi)=I\left(\xi,\alpha_1(\xi)\right)-\Delta\alpha_1(\xi)\ge 0,$$

$$\Delta\alpha'(\xi)=\Delta\left(T\alpha_1\right)'(\xi)-\Delta\alpha_1'(\xi)=Q\left(\xi,\alpha_1(\xi)\right)-\Delta\alpha_1'(\xi)\ge 0.$$

由引理 4 可知，$\alpha(t)\ge 0$。

因此，$\alpha_1\preceq T\alpha_1$。同理可得，$\alpha_2\preceq T\alpha_2$。

由于 α_2 是边值问题(1)的一个下解但不是边值问题(1)的解，则 $\left(T\alpha_2\right)\ne\alpha_2$。因此，

$$\alpha_2\prec T\alpha_2.$$

类似可得，

$$T\beta_1\prec\beta_1, T\beta_2\preceq\beta_2.$$

由引理 1 可知，算子 T 至少有三个不动点 $x_1, x_2, x_3 \in [\alpha_1, \beta_2]$ 使得

$$\alpha_1 \preceq x_1 \prec\prec \beta_1, \alpha_2 \prec\prec x_2 \preceq \beta_2, \alpha_2 \npreceq x_2 \npreceq \beta_2.$$

因此，边值问题(1)至少有三个不同解。

证毕。

参考文献

[1] Bashir, A., Sotiris, K.N. and Ahmed, A. (2019) Fractional Order Differential Systems Involving Right Caputo and Left Riemann-Liouville Fractional Derivatives with Nonlocal Coupled Conditions. Boundary Value Problems, **2019**, Article Number: 109. https://doi.org/10.1186/s13661-019-1222-0

[2] Song, S. and Cui, Y. (2020) Existence of Solutions for Integral Boundary Value Problems of Mixed Fractional Differential Equations under Resonance. *Boundary Value Problems*, **2020**, Article Number: 23. https://doi.org/10.1186/s13661-020-01332-5

[3] Liu, X. and Jia, M. (2019) Solvability and Numerical Simulations for BVPs of Fractional Coupled Systems Involving Left and Right Fractional Derivatives. *Applied Mathematics and Computers*, **353**, 230-242. https://doi.org/10.1016/j.amc.2019.02.011

[4] Bai, Z. (2010) On Solutions of Some Fractional M-Point Boundary Value Problems at Resonance. *Electronic Journal of Qualitative Theory of Differential Equations*, **37**, 1-15. https://doi.org/10.14232/ejqtde.2010.1.37

[5] Cabada, A. and Wang, G. (2012) Positive Solutions of Nonlinear Fractional Differential Equations with Integral Boundary Value Conditions. *Journal of Mathematical Analysis*, **389**, 403-411. https://doi.org/10.1016/j.jmaa.2011.11.065

[6] 张秀云, 寇春海. 分数阶泛函微分方程解的存在唯一性[J]. 东华大学学报(自然科学版), 2007, 33(4): 452-454.

[7] 张海, 郑祖庥, 蒋威. 非线性分数阶泛函微分方程解的存在性[J]. 数学物理学报, 2011, 31(2): 289-297.

[8] 翁佩萱. 四阶泛函微分方程边值问题的上下解方法[J]. 华南师范大学学报, 2000(3): 1-6.

[9] Jia, M. and Liu, X. (2014) Multiplicity of Solutions for Integral Boundary Value Problems of Fractional Differential Equations with Upper and Lower Solutions. *Applied Mathematics and Computation*, **232**, 313-323. https://doi.org/10.1016/j.amc.2014.01.073

[10] Mitkowski, W. (2013) The Application of Fractional Order Differential Calculus for the Description of Temperature Profiles in a Granular Layer. *Theory and Applications of Non-integer Order Systems*, **257**, 243-248. https://doi.org/10.1007/978-3-319-00933-9_22

[11] 白占兵. 分数阶微分方程边值问题理论及应用[M]. 北京: 中国科学技术出版社, 2012.

[12] Podlubny, I. (1999) Fractional Differential Equations. Academic Press, San Diego.