A New Half-Discrete Hilbert’s Inequality

Zitian Xie¹, Zheng Zeng²
¹Department of Mathematics, Zhaoqing University, Zhaoqing
²Shaoguan University, Shaoguan
Email: gdzqxzt@163.com

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Abstract: In this paper, by introducing some parameters and estimating the weight function, we give a new half-discrete Hilbert-type inequality with a best constant factor. The equivalent inequality forms is considered.

Keywords: Half-Discrete; Hilbert’s Inequality; Hölder’s Inequality

1. 引言

设 $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, $a_n, b_n > 0$, 且 $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, 及 $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, 则有如下含最佳常数因子的 Hardy-Hilbert 积分不等式[1]:

$$\sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} \right) < \frac{\pi}{\sin (\pi/p)} \left( \sum_{n=1}^{\infty} a_n^p \right)^{1/p} \left( \sum_{n=1}^{\infty} b_n^q \right)^{1/q}$$

$$\sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} \right)^p < \left( \frac{\pi}{\sin (\pi/p)} \right)^p \left( \sum_{n=1}^{\infty} a_n^p \right)$$

近年来，人们陆续对不等式(1)(2)作了大量推广[2-16]。2011 年杨必成教授给出以下半离散 Hilbert 型不等式[2]:

设 $p > 1, \frac{1}{p} + \frac{1}{q} = 1, p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_1 > 0, 1 \geq \lambda_2 > 0, p > 1, \lambda_1 + \lambda_2 = \lambda$, 且 $0 < \int_0^\infty x^{\nu(1-\lambda_1)-1} f^p(x)dx < \infty$，

$$0 < \sum_{n=1}^{\infty} n^{\nu(1-\lambda_1)-1} < \infty,$$,

则
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\[
\sum_{n=1}^{\infty} a_n \int_0^\infty \frac{f(x)}{(x+n)^\mu} dx = \int_0^\infty f(x) \sum_{n=1}^{\infty} a_n (x+n)^{-\mu} dx < B(\lambda_1, \lambda_2) \left( \int_0^\infty x^{\theta(1-\lambda_1)-1} f(x) dx \right)^{\theta\mu} \cdot \left( \sum_{n=1}^{\infty} x^{\theta(1-\lambda_2)-1} \right)^{\theta\mu}
\]

其中 \( B(\lambda_1, \lambda_2) \) 为 \( \beta \) 函数。

我们应用权函数，将给出一个 –3\( \mu \) 齐次的有最佳常数因子的半离散 Hilbert 型不等式。同时给出他的等价式。

以下我们总假设 \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, 0 \leq \mu \leq \frac{2}{3} \)。

### 2. 一些引理

**引理 1** 定义权系数及权函数 \( W(n) \) 和 \( \tilde{W}(x) \) 如下

\[
W(n) = \int_n^\infty \frac{1}{\max\{x, n\}^{\mu} (x + a^2 n^{\mu})(x + b^2 n^{\mu})} dx
\]

\[
\tilde{W}(x) = \sum_{n=1}^{\infty} \frac{1}{\max\{x, n\}^{\mu} (x + a^2 n^{\mu})(x + b^2 n^{\mu})} dx
\]

则有

\[
W(n) = K = \frac{h}{\mu}; \tilde{W}(x) < K;
\]

其中

\[
h = \int_0^\infty \frac{u^{\frac{1}{2}} du}{\max\{u, 1\}(u + a^2)(u + b^2)} = \int_0^\infty \frac{u^{\frac{1}{2}} du}{\max\{u, 1\}(1 + a^2 u)(1 + b^2 u)}
\]

\[
= \begin{cases}
\pi & \text{当} a = b \text{时}, \\
\frac{\pi}{2a^3} - \frac{1}{2a^2} + \frac{1}{a} \left(\frac{1}{a^2} - 1\right) \arctan \frac{1}{a}, & \text{当} a \neq b \text{时},
\end{cases}
\]

**证明**：首先我们易有

\[
\int_0^\infty \frac{u^{\frac{1}{2}} du}{\max\{u, 1\}(u + a^2)(u + b^2)} = \int_0^\infty \frac{u^{\frac{1}{2}} du}{\max\{u, 1\}(1 + a^2 u)(1 + b^2 u)}
\]

设 \( x = nu^{\frac{1}{\mu}} \)，则

\[
W(n) = \int_0^\infty \frac{u^{\frac{1}{2}} du}{\max\{u, 1\}(u + a^2)(u + b^2)} + \int_0^\infty \frac{u^{\frac{1}{2}} du}{\max\{u, 1\}(u + a^2)(u + b^2)}
\]

\[
= \int_0^\infty \frac{u^{\frac{1}{2}} du}{u(u + a^2)(u + b^2)} + \int_0^\infty \frac{2u^{\frac{1}{2}} du}{u(u + a^2)(u + b^2)} = \int_0^1 \frac{2u^{\frac{1}{2}} du}{u(u + a^2)(u + b^2)} + \int_1^\infty \frac{2u^{\frac{1}{2}} du}{u(u + a^2)(u + b^2)}
\]

\[
= \begin{cases}
\frac{\pi}{ab(a + b)} + \frac{2}{a^2 - b^2} \left(\frac{a^2 + 1}{a} \arctan \frac{1}{a} - \frac{b^2 + 1}{b} \arctan \frac{1}{b}\right), & \text{当} a \neq b \text{时};
\end{cases}
\]

\[
\frac{\pi}{2a^2} - \frac{1}{2a} + \frac{1}{a} \left(\frac{1}{a^2} - 1\right) \arctan \frac{1}{a}, & \text{当} a = b \text{时};
\]
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又 \[
\frac{1}{\max\{x^\alpha, n^\alpha\}} \left( x^\alpha + a^2 n^\alpha \right) \frac{1}{n^\frac{3\mu}{2}} \frac{x^\frac{3\mu}{2}}{n^\frac{3\mu}{2}} \] 关于 \( n \) 严格单调下降, 于是

\[
\tilde{W}(x) = \int_0^\infty \frac{1}{\max\{x^\alpha, y^\alpha\}} \left( x^\alpha + a^2 y^\alpha \right) \left( x^\alpha + b^2 y^\alpha \right) \frac{1}{y^\frac{3\mu}{2}} \frac{y^\frac{3\mu}{2}}{y^\frac{3\mu}{2}} dy = \frac{u^{\frac{3\mu}{2}} du}{\mu} = K
\]

引理获证。

**引理 2** 设 \( p > 1, a_n \geq 0 \), \( f(x) \) 在 \((0, \infty)\) 非负可测, 且

\[
0 < \int_0^\infty x^{\left( \frac{3\mu}{2} \right)-1} f^p(x) dx < \infty, \quad 0 < \sum_{n=1}^\infty { \left( \frac{3\mu}{2} \right)-1} \, a_n^p < \infty,
\]

则有如下不等式:

\[
J_1 := \sum_{n=1}^\infty n^{\frac{3\mu}{2}-1} \left( \int_0^\infty \frac{f(x)}{\max\{x^\alpha, n^\alpha\}} \left( x^\alpha + a^2 n^\alpha \right) \left( x^\alpha + b^2 n^\alpha \right) \frac{1}{n^\frac{3\mu}{2}} \frac{x^\frac{3\mu}{2}}{n^\frac{3\mu}{2}} dx \right)^p \leq K^p \sum_{n=1}^\infty \int_0^\infty x^{\left( \frac{3\mu}{2} \right)-1} f^p(x) dx
\]

\[
J_2 := \sum_{n=1}^\infty n^{\frac{3\mu}{2}-1} \left( \int_0^\infty \frac{a_n}{\max\{x^\alpha, n^\alpha\}} \left( x^\alpha + a^2 n^\alpha \right) \left( x^\alpha + b^2 n^\alpha \right) \frac{1}{n^\frac{3\mu}{2}} \frac{x^\frac{3\mu}{2}}{n^\frac{3\mu}{2}} dx \right)^p \leq K^p \sum_{n=1}^\infty n^{\left( \frac{3\mu}{2} \right)-1} a_n^p
\]

**证明**：由带权 Hölder 不等式及引理 1，有

\[
\left( \int_0^\infty \frac{f(x)}{\max\{x^\alpha, n^\alpha\}} \left( x^\alpha + a^2 n^\alpha \right) \left( x^\alpha + b^2 n^\alpha \right) \frac{1}{n^\frac{3\mu}{2}} \frac{x^\frac{3\mu}{2}}{n^\frac{3\mu}{2}} dx \right)^p \leq \left( \int_0^\infty \frac{1}{\max\{x^\alpha, n^\alpha\}} \left( x^\alpha + a^2 n^\alpha \right) \left( x^\alpha + b^2 n^\alpha \right) \frac{1}{n^\frac{3\mu}{2}} \frac{x^\frac{3\mu}{2}}{n^\frac{3\mu}{2}} dx \right)^{\frac{p}{p-1}} \left( \int_0^\infty f^p(x) dx \right)^{\frac{1}{p-1}} \left( \int_0^\infty \frac{1}{\max\{x^\alpha, n^\alpha\}} \left( x^\alpha + a^2 n^\alpha \right) \left( x^\alpha + b^2 n^\alpha \right) \frac{1}{n^\frac{3\mu}{2}} \frac{x^\frac{3\mu}{2}}{n^\frac{3\mu}{2}} dx \right)^{\frac{1}{p-1}}
\]

\[
= \left[ W(n) \right]^{\frac{p}{p-1}} n^{-\frac{3\mu}{2}} \int_0^\infty \frac{1}{\max\{x^\alpha, n^\alpha\}} \left( x^\alpha + a^2 n^\alpha \right) \left( x^\alpha + b^2 n^\alpha \right) \frac{1}{n^\frac{3\mu}{2}} \frac{x^\frac{3\mu}{2}}{n^\frac{3\mu}{2}} f^p(x) dx
\]

\[
= K^p n^{-\frac{3\mu}{2}} \int_0^\infty \frac{1}{\max\{x^\alpha, n^\alpha\}} \left( x^\alpha + a^2 n^\alpha \right) \left( x^\alpha + b^2 n^\alpha \right) \frac{1}{n^\frac{3\mu}{2}} \frac{x^\frac{3\mu}{2}}{n^\frac{3\mu}{2}} f^p(x) dx
\]

\[
J_1 \leq K^{p-1} \sum_{n=1}^\infty \left( \int_0^\infty \frac{1}{\max\{x^\alpha, n^\alpha\}} \left( x^\alpha + a^2 n^\alpha \right) \left( x^\alpha + b^2 n^\alpha \right) \frac{1}{n^\frac{3\mu}{2}} \frac{x^\frac{3\mu}{2}}{n^\frac{3\mu}{2}} f^p(x) dx \right)^{\frac{p}{p-1}}
\]

\[
= K^p \int_0^\infty \frac{1}{\max\{x^\alpha, n^\alpha\}} \left( x^\alpha + a^2 n^\alpha \right) \left( x^\alpha + b^2 n^\alpha \right) \frac{1}{n^\frac{3\mu}{2}} \frac{x^\frac{3\mu}{2}}{n^\frac{3\mu}{2}} f^p(x) dx \leq K^p \int_0^\infty x^{\left( \frac{3\mu}{2} \right)-1} f^p(x) dx
\]
故(4)成立。类似地有，
\[
\left( \sum_{n=1}^{\infty} \max \left\{ x^{\mu}, n^{\mu} \right\} \left( x^{\mu} + a^{2} n^{\mu} \right) \left( x^{\mu} + b^{2} n^{\mu} \right) \right)^{q} = \sum_{n=1}^{\infty} \frac{1}{\max \left\{ x^{\mu}, n^{\mu} \right\} \left( x^{\mu} + a^{2} n^{\mu} \right) \left( x^{\mu} + b^{2} n^{\mu} \right)} \left( \frac{1}{x^{1/2}} \right)^{p} \left( \frac{n}{x^{1/2}} \right)^{p} a_{n}^{q} \]
\leq \left[ \int_{0}^{\infty} x^{-1/2} \sum_{n=1}^{\infty} \frac{1}{\max \left\{ x^{\mu}, n^{\mu} \right\} \left( x^{\mu} + a^{2} n^{\mu} \right) \left( x^{\mu} + b^{2} n^{\mu} \right)} \left( \frac{1}{x^{1/2}} \right)^{p} \left( \frac{n}{x^{1/2}} \right)^{p} a_{n}^{q} \right] \leq K^{-1} \sum_{n=1}^{\infty} \sum_{\max \left\{ x^{\mu}, n^{\mu} \right\} \left( x^{\mu} + a^{2} n^{\mu} \right) \left( x^{\mu} + b^{2} n^{\mu} \right)} n^{1-\mu/2} a_{n}^{q}
\]
及类似地，
\[
J_{x} = \int_{0}^{\infty} x^{-1/2} \sum_{n=1}^{\infty} \frac{1}{\max \left\{ x^{\mu}, n^{\mu} \right\} \left( x^{\mu} + a^{2} n^{\mu} \right) \left( x^{\mu} + b^{2} n^{\mu} \right)} \left( \frac{1}{x^{1/2}} \right)^{p} \left( \frac{n}{x^{1/2}} \right)^{p} a_{n}^{q} \]
有(5)成立。

引理 3 设 \( p > 0, \epsilon \) 充分小，定义 \( \tilde{f}(x) = 0, x \in (0, 1), \tilde{f}(x) = x^{\mu - 1/2 - \epsilon}, x \in (1, \infty) \)；及 \( \tilde{a}_{n} = n^{\mu - 1/2 - \epsilon}, n \in \mathbb{N} \)。则
\[
I(\epsilon) := \epsilon \left( \int_{1}^{\infty} x^{\mu - 1/2} \tilde{f}(x) \, dx \right) \left( \sum_{n=1}^{\infty} n^{\mu - 1/2} \tilde{a}_{n}^{q} \right)^{1/q} = 1 + o(1) \left( \epsilon \to 0^{+} \right) \tag{6}
\]
\[
\hat{I}(\epsilon) := \epsilon \left[ \sum_{n=1}^{\infty} \frac{n^{\mu - 1/2 - \epsilon}}{\sum_{n=1}^{\infty} \frac{n^{\mu - 1/2 - \epsilon}}{\max \left\{ x^{\mu}, n^{\mu} \right\} \left( x^{\mu} + a^{2} n^{\mu} \right) \left( x^{\mu} + b^{2} n^{\mu} \right)}} \right] \, dx \geq K + o(1) \left( \epsilon \to 0^{+} \right) \tag{7}
\]
证明 易有，
\[
I(\epsilon) = \epsilon \left[ \int_{1}^{\infty} x^{-1/2 - \epsilon} \, dx \right] \left( \sum_{n=1}^{\infty} n^{\mu - 1/2 - \epsilon} \right)^{1/q}
\]
注意及右边最后项有以下双边不等式
\[
\frac{1}{\epsilon} = \int_{1}^{\infty} x^{-1/2 - \epsilon} \, dx < \sum_{n=1}^{\infty} n^{\mu - 1/2 - \epsilon} = 1 + \sum_{n=2}^{\infty} n^{\mu - 1/2 - \epsilon} < 1 + \int_{1}^{\infty} x^{-1/2 - \epsilon} \, dx = 1 + \frac{1}{\epsilon}
\]
13(6). 又设 \( y = xt^{\mu/\nu} \)

\[
I(\varepsilon) = \varepsilon \int_1^\infty \frac{3\mu - \varepsilon}{x^{\frac{3\mu - \varepsilon}{\nu}}} \left[ \sum_{n=1}^N \max \{ x^n, n^\mu \} \right] \left( x^n + a^n x^n + b^n x^n + c^n x^n \right) dx
\]

故有式（6）。又设 \( y = xt^{\mu/\nu} \)

\[
I(\varepsilon) = \varepsilon \int_1^\infty \frac{3\mu - \varepsilon}{x^{\frac{3\mu - \varepsilon}{\nu}}} \left[ \sum_{n=1}^N \max \{ x^n, n^\mu \} \right] \left( x^n + a^n x^n + b^n x^n + c^n x^n \right) dx
\]

又设 \( y = xt^{\mu/\nu} \)

\[
I(\varepsilon) = \varepsilon \int_1^\infty \frac{3\mu - \varepsilon}{x^{\frac{3\mu - \varepsilon}{\nu}}} \left[ \sum_{n=1}^N \max \{ x^n, n^\mu \} \right] \left( x^n + a^n x^n + b^n x^n + c^n x^n \right) dx
\]

\[
= \frac{E}{\mu} \int_1^\infty \frac{1}{x^{\frac{\mu}{\nu}} \max \{ 1, t \}} \left( 1 + a^2 t \right) \left( 1 + b^2 t \right) dt
\]

\[
= \frac{E}{\mu} \int_1^\infty \frac{1}{x^{\frac{\mu}{\nu}} \max \{ 1, t \}} \left( 1 + a^2 t \right) \left( 1 + b^2 t \right) dx - \frac{E}{\mu} \int_1^\infty \frac{1}{x^{\frac{\mu}{\nu}} \max \{ 1, t \}} \left( 1 + a^2 t \right) \left( 1 + b^2 t \right) dx
\]

\[
= \frac{1}{\mu} \int_0^\infty t^{\frac{\mu}{2}} dt - \frac{E}{\mu} \int_1^\infty \frac{1}{x^{\frac{\mu}{\nu}} \max \{ 1, t \}} \left( 1 + a^2 t \right) \left( 1 + b^2 t \right) dx
\]

\[
= K + \frac{1}{\mu} \int_0^\infty t^{\frac{\mu}{2}} dt - \frac{E}{\mu} \int_1^\infty \frac{1}{x^{\frac{\mu}{\nu}} \max \{ 1, t \}} \left( 1 + a^2 t \right) \left( 1 + b^2 t \right) dx
\]

其中 \( \lim_{\varepsilon \to 0} \eta(\varepsilon) = 0 \) 知(7)成立，引理得证。

### 3. 主要结果

定理：设 \( p > 1, a_n \geq 0, f(x) \) 在 \((0, \infty)\) 非负可测，且 \( 0 < \int_0^\infty x^\left(\frac{1-3\mu}{2}\right)^{-1} f^p(x) dx < \infty \)，\( 0 < \sum_{n=1}^N a_n \) \( \infty \) \( a_n^q < \infty \)。则有如下等价不等式：

\[
I = \sum_{n=1}^N a_n \int_0^\infty \frac{f(x)}{\max \{ x^n, n^\mu \}} \left( x^n + a^n x^n + b^n x^n + c^n x^n \right) dx
\]

\[
= \int_0^\infty f(x) \sum_{n=1}^N \frac{a_n}{\max \{ x^n, n^\mu \}} \left( x^n + a^n x^n + b^n x^n + c^n x^n \right) dx
\]

\[
< K \left\{ \int_0^\infty x^\left(\frac{1-3\mu}{2}\right)^{-1} f^p(x) dx \right\} \left( \sum_{n=1}^N \frac{a_n}{\max \{ x^n, n^\mu \}} \right)^{\frac{1}{q}}
\]

\[
J_1 = \sum_{n=1}^N a_n \left( \int_0^\infty \frac{f(x)}{\max \{ x^n, n^\mu \}} x^n + a^n x^n + b^n x^n + c^n x^n dx \right)^{\frac{1}{p}} < K^p \left\{ \int_0^\infty x^\left(\frac{1-3\mu}{2}\right)^{-1} f^p(x) dx \right\}
\]

\[
J_2 = \sum_{n=1}^N a_n \left( \int_0^\infty \frac{f(x)}{\max \{ x^n, n^\mu \}} x^n + a^n x^n + b^n x^n + c^n x^n dx \right)^{\frac{1}{q}} < K^q \sum_{n=1}^N a_n
\]

这里常数因子 \( K \) 由引理 1 定义，且 \( K^p \) 及 \( K^q \) 均为最佳值。
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证明 由逐项积分定理，I 有两种表示，由 Hölder 不等式，有

\[ I := \sum_{n=1}^{\infty} \frac{3^{\frac{1}{2}}}{n^{2}} \int_{0}^{\infty} \frac{f(x)}{\max \{ x^\mu, n^\mu \}} \left( x^\mu + a^2 n^\mu \right) dx \left[ \frac{1}{n^{p-2} a_n} \right] \leq J_1 \left[ \sum_{n=1}^{\infty} \frac{3^{\frac{1}{2}}}{n^{2}} \right] \frac{1}{a_n^p} \]

由式(9)得式(8)。反之，设(8)成立，取

\[ a_n = n^{-\gamma} \left( \int_{0}^{\infty} \frac{f(x)}{\max \{ x^\mu, n^\mu \}} \left( x^\mu + a^2 n^\mu \right) dx \right)^{p-1} \]

则由式(8)，有

\[ \sum_{n=1}^{\infty} \frac{3^{\frac{1}{2}}}{n^{2}} a_n^p = J_1 = I < K \left[ \int_{1}^{\infty} x^{\left( \frac{3\mu}{2} - 1 \right)} f^p(x) dx \right] \left\{ \sum_{n=1}^{\infty} \frac{3^{\frac{1}{2}}}{n^{2}} a_n^p \right\}^{\frac{1}{p}} \]

易由条件知 $J_1 < \infty$，如 $J_1 = 0$，则式(9)自然成立，如 $J_1 > 0$ 式(8)条件都具备，上式取严格不等号，且有(9)成立。可知(8)与(9)两式等价，由条件，上式取严格不等号。类似不难证明，(8)与(10)两式等价。

类似地，设(8)成立，取

\[ f(x) = x^{\frac{3\mu}{2} - 1} \left( \sum_{n=1}^{\infty} \frac{a_n}{\max \{ x^\mu, n^\mu \}} \left( x^\mu + a^2 n^\mu \right) \right)^{p-1} \]

由式(8)有

\[ \int_{1}^{\infty} x^{\left( \frac{3\mu}{2} - 1 \right)} f^p(x) dx = J_2 = I < K \left[ \int_{1}^{\infty} x^{\left( \frac{3\mu}{2} - 1 \right)} f^p(x) dx \right] \left\{ \sum_{n=1}^{\infty} \frac{3^{\frac{1}{2}}}{n^{2}} a_n^p \right\}^{\frac{1}{p}} \]

易由条件知 $J_2 < \infty$，如 $J_2 = 0$，则式(10)自然成立，如 $J_2 > 0$ 则式(8)条件都具备，上式取严格不等号，且有(10)成立。式(8)与(10)两式等价。故式(8)，(9)与式(10)等价。

设有常数 $\tilde{K}$，使 $\tilde{K}$ 代替式(8)中的常数因子 $K$ 后不等式(8)仍然成立。取 $\tilde{f}(x)$ 和 $\tilde{a}_n$ 代入，$\tilde{f}(x)$ 和 $\tilde{a}_n$ 如引理 3 所定义，有

\[ \varepsilon \int_{0}^{\infty} \tilde{f}(x) \sum_{n=1}^{\infty} \frac{\tilde{a}_n}{\max \{ x^\mu, n^\mu \}} \left( x^\mu + a^2 n^\mu \right) dx < \varepsilon K \left[ \int_{1}^{\infty} x^{\left( \frac{3\mu}{2} - 1 \right)} \tilde{f}^p(x) dx \right] \left\{ \sum_{n=1}^{\infty} \frac{3^{\frac{1}{2}}}{n^{2}} \tilde{a}_n^p \right\}^{\frac{1}{p}} \]

由引理 3 并令 $\varepsilon$ 充分小，得 $K(1 + o(1)) < \tilde{K}(1 + \tilde{a}(1))$，则再令 $\varepsilon \to 0^+$，有 $\tilde{K} \leq \hat{K}$，与 $\tilde{K} < K$ 矛盾，即式(8)中的常数因子 $K$ 为最佳的。又由等价性易知(9)和式(10)中的 $K^\mu$ 和 $K^\nu$ 也为最佳值。证毕。

### 参考文献 (References)