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Matched Pair and Manin Triple of Hom-Malcev Algebra

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Abstract

In this paper, we study matched pair and Manin triple of Hom-Malcev algebra. First, we give the definition of matched pair of Hom-Malcev algebra and the method of getting a new Hom-Malcev algebra on the direct sum of two Hom-Malcev algebras, we also study the method of constructing a new Hom-Malcev algebra on the dual space of Hom-Malcev algebra. Then, we give the definition of Manin triple of Hom-Malcev algebra, and we give the relation between matched pair and Manin triple of Hom-Malcev algebra.

Keywords

Hom-Malcev Algebra, Matched Pair, Manin Triple

Hom-Malcev代数的配对和Manin Triple

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摘 要

本文主要研究了**Hom-Malcev**代数的配对和**Manin triple**。首先给出**Hom-Malcev**代数的配对的定义以及在两个**Hom-Malcev**代数的直和上构造**Hom-Malcev**代数的方法，研究在**Hom-Malcev**代数的对偶空间上构造**Hom-Malcev**代数的方法。然后给出**Hom-Malcev**代数的**Manin triple**的定义，并给出**Hom-Malcev**代数的配对和**Manin triple**之间的关系。

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关键词

Hom-Malcev代数，配对，Manin Triple

1. 引言

作为李代数的推广，Malcev 代数不仅和李代数之间存在着密切的联系，和交错代数之间也有着密切的关系。就像李群在单位元的切空间是李代数一样，局部解析的 Moufang 群的切空间也是一个 Malcev 代数[1]。在[2]中，作者不仅给出了 Malcev 代数的定义，还发现每一个交错代数都是可容许的 Malcev 代数。Hom-Malcev 代数的定义是由 Yau 在[3]中给出的，并证明了每一个 Hom-交错代数也都是可容许的 Hom-Malcev 代数。此后，许多学者对 Hom-Malcev 代数进行了研究，例如，在[4]中，作者证明了 Hom-Malcev 代数上的几个恒等式。在[5]中，作者研究了 Hom-李代数上的 Hom-Yang-Baxter 方程以及 Hom-李双代数。在[6]中，作者给出了 Malcev 代数上的 Yang-Baxter 方程以及 Malcev 双代数。因此，可以考虑的一个问题是在 Hom-Malcev 代数上是否也会存在类似于 Hom-Yang-Baxter 方程的方程以及 Hom-Malcev 双代数。

2. Hom-Malcev 代数的定义和表示

定义 2.1 [3]：设 M 是域 F 上的线性空间，$\alpha: M \to M$ 为代数同态，如果 M 中有二元双线性运算 $[\ ]: M \times M \to M$，对于 $\forall x, y, z, w \in M$，有

$$[x,y] = -[y,x], \tag{2.1}$$

$$\begin{aligned}\alpha([[x,z],[y,w]]) = {}& [[[x,y],\alpha(z)],\alpha^2(w)] + [[[y,z],\alpha(w)],\alpha^2(x)] \\ & + [[[z,w],\alpha(x)],\alpha^2(y)] + [[[w,x],\alpha(y)],\alpha^2(z)],\end{aligned} \tag{2.2}$$

则称 $(M,[\ ],\alpha)$ 为域 F 上的 Hom-Malcev 代数。

定义 2.2 [7]：设 $(M,[\ ],\alpha)$ 为 Hom-Malcev 代数，V 为线性空间，$\rho: M \to End(V)$ 为线性映射，$\sigma \in End(V)$，如果对于 $\forall x, y, z \in M$，有

$$\sigma(\rho(x)) = \rho(\alpha(x))\sigma, \tag{2.3}$$

$$\begin{aligned}&\sigma(\rho([x,z])\rho(y)) - \rho([[x,y],\alpha(z)])\sigma^2 + \rho(\alpha^2(x))\rho([y,z])\sigma \\ &- \rho(\alpha^2(y))\rho(\alpha(x))\rho(z) + \rho(\alpha^2(z))\rho(\alpha(y))\rho(x) = 0,\end{aligned} \tag{2.4}$$

则称 (ρ, V, σ) 为 $(M,[\ ],\alpha)$ 的表示。

定义 2.3 [7]：设 $(M,[\ ],\alpha)$ 为 Hom-Malcev 代数，$ad: M \to End(M)$ 为 $(M,[\ ],\alpha)$ 的伴随表示，如果对于 $\forall x, y, z \in M$，有

$$\alpha(ad\alpha(x)) = adx\alpha, \tag{2.5}$$

$$
\begin{aligned}
& ady\left(ad[x,z]\alpha\right)+\alpha^2\left(ad\left[[x,y],\alpha(z)\right]\right)+\alpha\left(ad[y,z]\left(ad\alpha^2(x)\right)\right) \\
& +adz\left(ad\alpha(x)\left(ad\alpha^2(y)\right)\right)-adx\left(ad\alpha(y)\left(ad\alpha^2(z)\right)\right)=0,
\end{aligned} \tag{2.6}
$$

则称这个 Hom-Malcev 代数为相容的 Hom-Malcev 代数。

3. Hom-Malcev 代数的配对

定义 3.1：设$(M_1,[\ ]_1,\alpha_1)$和$(M_2,[\ ]_2,\alpha_2)$为 Hom-Malcev 代数，若(ρ_1,M_2,α_2)和(ρ_2,M_1,α_1)分别为$(M_1,[\ ]_1,\alpha_1)$和$(M_2,[\ ]_2,\alpha_2)$的表示，对于$\forall x_1,y_1,z_1\in M_1$， $x_2,y_2,z_2\in M_2$，有

$$
\begin{aligned}
& \left[\left[\rho_2(x_2)x_1,\alpha_1(y_1)\right]_1,\alpha_1^2(z_1)\right]_1-\left[\rho_2(\alpha_2(x_2))[y_1,z_1]_1,\alpha_1^2(x_1)\right]_1-\left[\rho_2(\rho_1(x_1)x_2)\alpha_1(y_1),\alpha_1^2(z_1)\right]_1 \\
& -\left[\left[\rho_2(x_2)z_1,\alpha_1(x_1)\right]_1,\alpha_1^2(y_1)\right]_1+\rho_2\left(\rho_1(\alpha_1(y_1))(\rho_1(x_1)x_2)\right)\alpha_1^2(z_1)+\alpha_1\left(\left[[x_1,z_1]_1,\rho_2(x_2)y_1\right]_1\right) \\
& -\rho_2\left(\alpha_2^2(x_2)\right)\left[[x_1,y_1]_1,\alpha_1(z_1)\right]_1+\alpha_1\left(\rho_2(\rho_1(y_1)x_2)[x_1,z_1]_1\right)+\left[\rho_2(\rho_1(z_1)x_2)\alpha_1(x_1),\alpha_1^2(y_1)\right]_1 \\
& +\rho_2\left(\rho_1([y_1,z_1]_1)\alpha_2(x_2)\right)\alpha_1^2(x_1)-\rho_2\left(\rho_1(\alpha_1(x_1))(\rho_1(z_1)x_2)\right)\alpha_1^2(y_1)=0,
\end{aligned} \tag{3.1}
$$

$$
\begin{aligned}
& \left[\left[\rho_1(x_1)x_2,\alpha_2(y_2)\right]_2,\alpha_2^2(z_2)\right]_2-\left[\rho_1(\alpha_1(x_1))[y_2,z_2]_2,\alpha_2^2(x_2)\right]_2-\left[\rho_1(\rho_2(x_2)x_1)\alpha_2(y_2),\alpha_2^2(z_2)\right]_2 \\
& -\left[\left[\rho_1(x_1)z_2,\alpha_2(x_2)\right]_2,\alpha_2^2(y_2)\right]_2+\rho_1\left(\rho_2(\alpha_2(y_2))(\rho_2(x_2)x_1)\right)\alpha_2^2(z_2)+\alpha_2\left(\left[[x_2,z_2]_2,\rho_1(x_1)y_2\right]_2\right) \\
& -\rho_1\left(\alpha_1^2(x_1)\right)\left[[x_2,y_2]_2,\alpha_2(z_2)\right]_2+\left[\rho_1(\rho_2(z_2)x_1)\alpha_2(x_2),\alpha_2^2(y_2)\right]_2+\alpha_2\left(\rho_1(\rho_2(y_2)x_1)[x_2,z_2]_2\right) \\
& +\rho_1\left(\rho_2([y_2,z_2]_2)\alpha_1(x_1)\right)\alpha_2^2(x_2)-\rho_1\left(\rho_2(\alpha_2(x_2))(\rho_2(z_2)x_1)\right)\alpha_2^2(y_2)=0,
\end{aligned} \tag{3.2}
$$

$$
\begin{aligned}
& \rho_2\left(\rho_1(\alpha_1(x_1))[x_2,y_2]_2\right)\alpha_1^2(y_1)-\alpha_1\left(\rho_2(\rho_1(x_1)x_2)(\rho_2(y_2)y_1)\right)+\rho_2\left(\alpha_2^2(x_2)\right)\left[\rho_2(y_2)(x_1),\alpha_1(y_1)\right]_1 \\
& -\left[\rho_2(\alpha_2(y_2))(\rho_2(x_2)y_1),\alpha_1^2(x_1)\right]_1-\left[\rho_2([x_2,y_2]_2)\alpha_1(x_1),\alpha_1^2(y_1)\right]_1+\alpha_1\left(\rho_2(\rho_1(y_1)y_2)(\rho_2(x_2)x_1)\right) \\
& -\rho_2\left(\left[\rho_1(y_1)x_2,\alpha_2(y_2)\right]_2\right)\alpha_1^2(x_1)-\rho_2\left(\alpha_2^2(y_2)\right)\left(\rho_2(\alpha_2(x_2))[x_1,y_1]_1\right)+\alpha_1\left(\left[\rho_2(x_2)x_1,\rho_2(y_2)y_1\right]_1\right) \\
& -\rho_2\left(\alpha_2^2(x_2)\right)\left(\rho_2(\rho_1(x_1)y_2)\alpha_1(y_1)\right)+\rho_2\left(\rho_1(\rho_2(x_2)y_1)\alpha_2(y_2)\right)\alpha_1^2(x_1)=0,
\end{aligned} \tag{3.3}
$$

$$
\begin{aligned}
& \rho_1\left(\rho_2(\alpha_2(x_2))[x_1,y_1]_1\right)\alpha_2^2(y_2)-\alpha_2\left(\rho_1(\rho_2(x_2)x_1)(\rho_1(y_1)y_2)\right)+\rho_1\left(\alpha_1^2(x_1)\right)\left[\rho_1(y_1)x_2,\alpha_2(y_2)\right]_2 \\
& -\left[\rho_1(\alpha_1(y_1))(\rho_1(x_1)y_2),\alpha_2^2(x_2)\right]_2-\left[\rho_1([x_1,y_1]_1)\alpha_2(x_2),\alpha_2^2(y_2)\right]_2+\alpha_2\left(\rho_1(\rho_2(y_2)y_1)(\rho_1(x_1)x_2)\right) \\
& -\rho_1\left(\left[\rho_2(y_2)x_1,\alpha_1(y_1)\right]_1\right)\alpha_2^2(x_2)-\rho_1\left(\alpha_1^2(y_1)\right)\left(\rho_1(\alpha_1(x_1))[x_2,y_2]_2\right)+\alpha_2\left(\left[\rho_1(x_1)x_2,\rho_1(y_1)y_2\right]_2\right) \\
& -\rho_1\left(\alpha_1^2(x_1)\right)\left(\rho_1(\rho_2(x_2)y_1)\alpha_2(y_2)\right)+\rho_1\left(\rho_2(\rho_1(x_1)y_2)\alpha_1(y_1)\right)\alpha_2^2(x_2)=0,
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
& \rho_2\left(\alpha_2^2(x_2)\right)\left[\rho_2(y_2)y_1,\alpha_1(x_1)\right]_1-\rho_2\left(\left[\rho_1(x_1)y_2,\alpha_2(x_2)\right]_2\right)\alpha_1^2(y_1)-\left[\rho_2(\alpha_2(x_2))(\rho_2(y_2)x_1),\alpha_1^2(y_1)\right]_1 \\
& +\rho_2\left(\alpha_2^2(y_2)\right)\left[\rho_2(x_2)x_1,\alpha_1(y_1)\right]_1-\left[\rho_2(\alpha_2(y_2))(\rho_2(x_2)y_1),\alpha_1^2(x_1)\right]_1-\rho_2\left(\left[\rho_1(y_1)x_2,\alpha_2(y_2)\right]_2\right)\alpha_1^2(x_1) \\
& -\rho_2\left(\alpha_2^2(y_2)\right)\left(\rho_2(\rho_1(x_1)x_2)\alpha_1(y_1)\right)-\rho_2\left(\alpha_2^2(x_2)\right)\left(\rho_2(\rho_1(y_1)y_2)\alpha_1(x_1)\right)+\alpha_1\left(\rho_2([x_2,y_2]_2)[x_1,y_1]_1\right) \\
& +\rho_2\left(\rho_1(\rho_2(x_2)y_1)\alpha_2(y_2)\right)\alpha_1^2(x_1)+\rho_2\left(\rho_1(\rho_2(y_2)x_1)\alpha_2(x_2)\right)\alpha_1^2(y_1)=0,
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
& \rho_1\left(\alpha_1^2(x_1)\right)\left[\rho_1(y_1)y_2,\alpha_2(x_2)\right]_2-\rho_1\left(\left[\rho_2(x_2)y_1,\alpha_1(x_1)\right]_1\right)\alpha_2^2(y_2)-\left[\rho_1(\alpha_1(x_1))(\rho_1(y_2)x_2),\alpha_2^2(y_2)\right]_2 \\
& +\rho_1\left(\alpha_1^2(y_1)\right)\left[\rho_1(x_1)x_2,\alpha_2(y_2)\right]_2-\left[\rho_1(\alpha_1(y_1))(\rho_1(x_1)y_2),\alpha_2^2(x_2)\right]_2-\rho_1\left(\left[\rho_2(y_2)x_1,\alpha_1(y_1)\right]_1\right)\alpha_2^2(x_2) \\
& -\rho_1\left(\alpha_1^2(y_1)\right)\left(\rho_1(\rho_2(x_2)x_1)\alpha_2(y_2)\right)-\rho_1\left(\alpha_1^2(x_1)\right)\left(\rho_1(\rho_2(y_2)y_1)\alpha_2(x_2)\right)+\alpha_2\left(\rho_1([x_1,y_1]_1)[x_2,y_2]_2\right) \\
& +\rho_1\left(\rho_2(\rho_1(x_1)y_2)\alpha_1(y_1)\right)\alpha_2^2(x_2)+\rho_1\left(\rho_2(\rho_1(y_1)x_2)\alpha_1(x_1)\right)\alpha_2^2(y_2)=0,
\end{aligned} \tag{3.6}
$$

则称 $(M_1, M_2, \rho_1, \rho_2)$ 为这两个 Hom-Malcev 代数的配对。

定理 3.2： 设 $(M_1, [\ ]_1, \alpha_1)$，$(M_2, [\ ]_2, \alpha_2)$ 为 Hom-Malcev 代数，$\rho_1 : M_1 \to End(M_2)$ 和 $\rho_2 : M_2 \to End(M_1)$ 为线性映射，在 $M_1 \oplus M_2$ 上定义二元反对称双线性运算 $[\ ] : (M_1 \oplus M_2) \times (M_1 \oplus M_2) \to (M_1 \oplus M_2)$，对于 $\forall x_1, y_1 \in M_1, x_2, y_2 \in M_2$，有

$$[x_1 + x_2, y_1 + y_2] = [x_1, y_1]_1 + \rho_2(x_2) y_1 - \rho_2(y_2) x_1 + [x_2, y_2]_2 + \rho_1(x_1) y_2 - \rho_1(y_1) x_2,$$

并定义

$$(\alpha_1 + \alpha_2)(x_1 + x_2) = \alpha_1(x_1) + \alpha_2(x_2),$$

则 $(M_1 \oplus M_2, [\ ], \alpha_1 + \alpha_2)$ 为 Hom-Malcev 代数当且仅当 $(M_1, M_2, \rho_1, \rho_2)$ 为 $(M_1, [\ ]_1, \alpha_1)$ 和 $(M_2, [\ ]_2, \alpha_2)$ 这两个 Hom-Malcev 代数的配对。

证明： $(M_1 \oplus M_2, [\ ], \alpha_1 + \alpha_2)$ 为 Hom-Malcev 代数当且仅当对于 $\forall x_1, y_1, z_1, w_1 \in M_1$，$x_2, y_2, z_2, w_2 \in M_2$，

$$(\alpha_1 + \alpha_2)([x_1 + x_2, y_1 + y_2]) = [(\alpha_1 + \alpha_2)(x_1 + x_2), (\alpha_1 + \alpha_2)(y_1 + y_2)], \tag{3.7}$$

$$\begin{aligned}
&(\alpha_1 + \alpha_2)([[x_1 + x_2, z_1 + z_2], [y_1 + y_2, w_1 + w_2]]) \\
&= [[[x_1 + x_2, y_1 + y_2], (\alpha_1 + \alpha_2)(z_1 + z_2)], (\alpha_1 + \alpha_2)^2 (w_1 + w_2)] \\
&\quad + [[[y_1 + y_2, z_1 + z_2], (\alpha_1 + \alpha_2)(w_1 + w_2)], (\alpha_1 + \alpha_2)^2 (x_1 + x_2)] \\
&\quad + [[[z_1 + z_2, w_1 + w_2], (\alpha_1 + \alpha_2)(x_1 + x_2)], (\alpha_1 + \alpha_2)^2 (y_1 + y_2)] \\
&\quad + [[[w_1 + w_2, x_1 + x_2], (\alpha_1 + \alpha_2)(y_1 + y_2)], (\alpha_1 + \alpha_2)^2 (z_1 + z_2)]
\end{aligned} \tag{3.8}$$

成立。

(3.7)成立等价于(2.3)成立，(3.8)成立等价于这 16 种情况下(3.8)成立：

1) $x_2, y_2, z_2, w_2 \in M_2$，$x_1 = y_1 = z_1 = w_1 = 0$；
2) $w_1 \in M_1$，$x_2, y_2, z_2 \in M_2$，$x_1 = y_1 = z_1 = w_2 = 0$；
3) $z_1 \in M_1$，$x_2, y_2, w_2 \in M_2$，$x_1 = y_1 = z_2 = w_1 = 0$；
4) $z_1, w_1 \in M_1$，$x_2, y_2 \in M_2$，$x_1 = y_1 = z_2 = w_2 = 0$；
5) $y_1 \in M_1$，$x_2, z_2, w_2 \in M_2$，$x_1 = y_2 = z_1 = w_1 = 0$；
6) $y_1, w_1 \in M_1$，$x_2, z_2 \in M_2$，$x_1 = y_2 = z_1 = w_2 = 0$；
7) $y_1, z_1 \in M_1$，$x_2, w_2 \in M_2$，$x_1 = y_2 = z_2 = w_1 = 0$；
8) $y_1, z_1, w_1 \in M_1$，$x_2 \in M_2$，$x_1 = y_2 = z_2 = w_2 = 0$；
9) $x_1 \in M_1$，$y_2, z_2, w_2 \in M_2$，$x_2 = y_1 = z_1 = w_1 = 0$；
10) $x_1, w_1 \in M_1$，$y_2, z_2 \in M_2$，$x_2 = y_1 = z_1 = w_2 = 0$；
11) $x_1, z_1 \in M_1$，$y_2, w_2 \in M_2$，$x_2 = y_1 = z_2 = w_1 = 0$；
12) $x_1, z_1, w_1 \in M_1$，$y_2 \in M_2$，$x_2 = y_1 = z_2 = w_2 = 0$；
13) $x_1, y_1 \in M_1$，$z_2, w_2 \in M_2$，$x_2 = y_2 = z_1 = w_1 = 0$；
14) $x_1, y_1, w_1 \in M_1$，$z_2 \in M_2$，$x_2 = y_2 = z_1 = w_2 = 0$；
15) $x_1, y_1, z_1 \in M_1$，$w_2 \in M_2$，$x_2 = y_2 = z_2 = w_1 = 0$；
16) $x_1, y_1, z_1, w_1 \in M_1$，$x_2 = y_2 = z_2 = w_2 = 0$。

其中，情况 1)下(3.8)成立 $\Leftrightarrow$ $(M_2, [\ ]_2, \alpha_2)$ 为 Hom-Malcev 代数，情况 2) 3) 5) 9)下(3.8)成立 $\Leftrightarrow$ (2.4) (3.1)成立，情况 8) 12) 14) 15)下(3.8)成立 $\Leftrightarrow$ (2.4) (3.2)成立，情况 4) 7) 10) 13)下(3.8)成立 $\Leftrightarrow$ (3.3) (3.4)成

立，情况 6) 11)下(3.8)成立 $\Leftrightarrow$ (3.5) (3.6)成立，情况 16)下(3.8)成立 $\Leftrightarrow$ $\left(M_1,[\ ]_1,\alpha_1\right)$ 为 Hom-Malcev 代数。

定理 3.3： 设 $\left(M,[\ ]_M,\alpha\right)$ 为 Hom-Malcev 代数，$\Delta: M\to M\otimes M$ 为线性映射，在 M^* 上定义 $\left[a^*,b^*\right]_{M^*}=\Delta^*\left(a^*\otimes b^*\right)$（$\forall a^*,b^*\in M^*$），则

1) $\left(M^*,[\ ]_{M^*},\alpha^*\right)$ 为 Hom-Malcev 代数当且仅当 Δ 满足以下两个条件：

$$\Delta=-\tau\Delta, \tag{3.9}$$

$$\begin{aligned}&(1\otimes\tau\otimes1)(\Delta\otimes\Delta)\Delta\alpha\\&=(\Delta\otimes\alpha\otimes1)\left(\Delta\otimes\alpha^2\right)\Delta+(1\otimes1\otimes\tau)(1\otimes\tau\otimes1)(\tau\otimes1\otimes1)(\Delta\otimes\alpha\otimes1)\left(\Delta\otimes\alpha^2\right)\Delta\\&\quad+(1\otimes\tau\otimes1)(\tau\otimes1\otimes1)(1\otimes1\otimes\tau)(1\otimes\tau\otimes1)(\Delta\otimes\alpha\otimes1)\left(\Delta\otimes\alpha^2\right)\Delta\\&\quad+(\tau\otimes1\otimes1)(1\otimes\tau\otimes1)(1\otimes1\otimes\tau)(\Delta\otimes\alpha\otimes1)\left(\Delta\otimes\alpha^2\right)\Delta,\end{aligned} \tag{3.10}$$

2) $\left(M^*,[\ ]_{M^*},\alpha^*\right)$ 为相容的 Hom-Malcev 代数当且仅当 Δ 满足(2.9)和(2.10)且

$$(\alpha\otimes1)\Delta\alpha=(1\otimes\alpha)\Delta, \tag{3.11}$$

$$\begin{aligned}&(\tau\otimes1\otimes1)(1\otimes\Delta\otimes\tau)(1\otimes\Delta)\Delta\\&=\left(1\otimes1\otimes\alpha^2\otimes1\right)(1\otimes\alpha\otimes\Delta)(1\otimes\Delta)\Delta-(\Delta\otimes\alpha\otimes1)(\Delta\otimes1)\Delta\alpha^2\\&\quad-(\tau\otimes1\otimes1)(1\otimes\tau\otimes1)\left(1\otimes1\otimes\alpha^2\otimes1\right)(1\otimes\alpha\otimes\Delta)(1\otimes\Delta)\Delta\\&\quad-(\tau\otimes1\otimes1)(1\otimes\tau\otimes1)\left(1\otimes1\otimes\alpha^2\otimes1\right)(\Delta\otimes\Delta)\Delta\alpha,\end{aligned} \tag{3.12}$$

证明： 1) $\left(M^*,[\ ]_{M^*},\alpha^*\right)$ 为 Hom-Malcev 代数当且仅当对于 $\forall a^*,b^*,c^*,d^*\in M^*$

$$\left[a^*,b^*\right]_{M^*}=-\left[b^*,a^*\right]_{M^*},$$

$$\begin{aligned}&\alpha^*\left(\left[\left[a^*,c^*\right]_{M^*},\left[b^*,d^*\right]_{M^*}\right]_{M^*}\right)\\&=\left[\left[\left[a^*,b^*\right]_{M^*},\alpha^*\left(c^*\right)\right]_{M^*},\alpha^{*2}\left(d^*\right)\right]_{M^*}\\&\quad+\left[\left[\left[d^*,a^*\right]_{M^*},\alpha^*\left(b^*\right)\right]_{M^*},\alpha^{*2}\left(c^*\right)\right]_{M^*}\\&\quad+\left[\left[\left[b^*,c^*\right]_{M^*},\alpha^*\left(d^*\right)\right]_{M^*},\alpha^{*2}\left(a^*\right)\right]_{M^*}\\&\quad+\left[\left[\left[c^*,d^*\right]_{M^*},\alpha^*\left(a^*\right)\right]_{M^*},\alpha^{*2}\left(b^*\right)\right]_{M^*}\end{aligned}$$

成立。因此，$\forall x\in M$，

$$\left\langle\left[a^*,b^*\right]_{M^*}+\left[b^*,a^*\right]_{M^*},x\right\rangle=0\Leftrightarrow\left\langle a^*\otimes b^*,\Delta(x)\right\rangle+\left\langle\tau\left(a^*\otimes b^*\right),\Delta(x)\right\rangle=0$$

等价于(3.9)成立。

$$\begin{aligned}&\Big\langle\alpha^*\left(\left[\left[a^*,c^*\right]_{M^*},\left[b^*,d^*\right]_{M^*}\right]_{M^*}\right)-\left[\left[\left[a^*,b^*\right]_{M^*},\alpha^*\left(c^*\right)\right]_{M^*},\alpha^{*2}\left(d^*\right)\right]_{M^*}\\&-\left[\left[\left[d^*,a^*\right]_{M^*},\alpha^*\left(b^*\right)\right]_{M^*},\alpha^{*2}\left(c^*\right)\right]_{M^*}+\left[\left[\left[b^*,c^*\right]_{M^*},\alpha^*\left(d^*\right)\right]_{M^*},\alpha^{*2}\left(a^*\right)\right]_{M^*}\\&+\left[\left[\left[c^*,d^*\right]_{M^*},\alpha^*\left(a^*\right)\right]_{M^*},\alpha^{*2}\left(b^*\right)\right]_{M^*},x\Big\rangle=0\end{aligned}$$

$$\Leftrightarrow \left\langle a^* \otimes c^* \otimes b^* \otimes d^*, (\Delta \otimes \Delta)\Delta(\alpha(x))\right\rangle$$
$$-\left\langle a^* \otimes b^* \otimes c^* \otimes d^*, (\Delta \otimes \alpha \otimes 1)(\Delta \otimes \alpha^2)\Delta(x)\right\rangle$$
$$-\left\langle b^* \otimes c^* \otimes d^* \otimes a^*, (\Delta \otimes \alpha \otimes 1)(\Delta \otimes \alpha^2)\Delta(x)\right\rangle$$
$$-\left\langle c^* \otimes d^* \otimes a^* \otimes b^*, (\Delta \otimes \alpha \otimes 1)(\Delta \otimes \alpha^2)\Delta(x)\right\rangle$$
$$-\left\langle d^* \otimes a^* \otimes b^* \otimes c^*, (\Delta \otimes \alpha \otimes 1)(\Delta \otimes \alpha^2)\Delta(x)\right\rangle = 0$$

等价于(3.10)成立。

2) $\left(M^*, [\ ]_{M^*}, \alpha^*\right)$ 为相容的 Hom-Malcev 代数当且仅当 1)且对于 $\forall a^*, b^*, c^*, d^* \in M^*$。

$$\alpha^*\left(ad_{M^*}\alpha^*(a^*)(b^*)\right) = ad_{M^*}a^*\left(\alpha^*(b^*)\right),$$

$$ad_{M^*}b^*\left(ad_{M^*}\left[a^*, c^*\right]_{M^*}\left(\alpha^*(d^*)\right)\right) - ad_{M^*}a^*\left(ad_{M^*}\alpha^*(b^*)\left(ad_{M^*}\alpha^{*2}(c^*)(d^*)\right)\right)$$
$$+\alpha^*\left(ad_{M^*}\left[b^*, c^*\right]_{M^*}\left(ad_{M^*}\alpha^{*2}(a^*)(d^*)\right)\right) + ad_{M^*}c^*\left(ad_{M^*}\alpha^*(a^*)\left(ad_{M^*}\alpha^{*2}(b^*)(d^*)\right)\right)$$
$$+\alpha^{*2}\left(ad_{M^*}\left[\left[a^*, b^*\right]_{M^*}, \alpha^*(c^*)\right]_{M^*}(d^*)\right) = 0$$

成立。因此，$\forall x \in M$，

$$\left\langle \alpha^*\left(ad_{M^*}\alpha^*(a^*)(b^*)\right) - ad_{M^*}a^*\left(\alpha^*(b^*)\right), x\right\rangle = 0 \Leftrightarrow \left\langle\left[\alpha^*(a^*), b^*\right]_{M^*}, \alpha(x)\right\rangle - \left\langle\left[a^*, \alpha^*(b^*)\right]_{M^*}, x\right\rangle = 0$$

等价于(3.11)成立。

$$\Big\langle ad_{M^*}b^*\left(ad_{M^*}\left[a^*, c^*\right]_{M^*}\left(\alpha^*(d^*)\right)\right) - ad_{M^*}a^*\left(ad_{M^*}\alpha^*(b^*)\left(ad_{M^*}\alpha^{*2}(c^*)(d^*)\right)\right)$$
$$+\alpha^*\left(ad_{M^*}\left[b^*, c^*\right]_{M*}\left(ad_{M^*}\alpha^{*2}(a^*)(d^*)\right)\right) + ad_{M^*}c^*\left(ad_{M^*}\alpha^*(a*)\left(ad_{M^*}\alpha^{*2}(b^*)(d^*)\right)\right)$$
$$+\alpha^{*2}\left(ad_{M^*}\left[\left[a^*, b^*\right]_{M^*}, \alpha^*(c^*)\right]_{M^*}(d^*)\right), x\Big\rangle = 0$$
$$\Leftrightarrow \left\langle b^* \otimes a^* \otimes c^* \otimes d^*, (1 \otimes \Delta \otimes \alpha)(1 \otimes \Delta)\Delta(x)\right\rangle$$
$$+\left\langle c^* \otimes a^* \otimes b^* \otimes d^*, (1 \otimes 1 \otimes \alpha^2 \otimes 1)(1 \otimes \alpha \otimes \Delta)(1 \otimes \Delta)\Delta(x)\right\rangle$$
$$+\left\langle b^* \otimes c^* \otimes a^* \otimes d^*, (1 \otimes 1 \otimes \alpha^2 \otimes 1)(\Delta \otimes \Delta)\Delta(\alpha(x))\right\rangle$$
$$+\left\langle a^* \otimes b^* \otimes c^* \otimes d^*, (\Delta \otimes \alpha \otimes 1)(\Delta \otimes 1)\Delta(\alpha^2(x))\right\rangle$$
$$-\left\langle a^* \otimes b^* \otimes c^* \otimes d^*, (1 \otimes 1 \otimes \alpha^2 \otimes 1)(1 \otimes \alpha \otimes \Delta)(1 \otimes \Delta)\Delta(x)\right\rangle = 0$$

等价于(3.12)成立。

定理 3.4： 设 $\left(M, [\ ]_M, \alpha\right)$ 为相容的 Hom-Malcev 代数，线性映射 $\Delta: M \to M \otimes M$ 满足(3.9)~(3.12)，在 $M \oplus M^*$ 上定义二元反对称双线性运算 $[\ ]: \left(M \oplus M^*\right) \times \left(M \oplus M^*\right) \to \left(M \oplus M^*\right)$，对于 $\forall x, y \in M$，$a^*, b^* \in M^*$，有

$$\left[x + a^*, y + b^*\right] = [x, y]_M + ad^*_{M^*}(a^*)y - ad^*_{M^*}(b^*)x + \left[a^*, b^*\right]_{M^*} + ad^*_{M^*}(x)b^* - ad^*_{M^*}(y)a^*,$$

并定义

$$\left(\alpha + \alpha^*\right)\left(x + a^*\right) = \alpha(x) + \alpha^*(a^*),$$

则 $\left(M \oplus M^*, [\ ], \alpha + \alpha^*\right)$ 是 Hom-Malcev 代数当且仅当对于 $\forall x, y, z, \in M$，$a^*, b^*, c^* \in M^*$，Δ 满足

$$\begin{aligned}
&(\alpha\otimes 1)\left(ad_M[y,z]_M\otimes 1\right)\Delta\left(\alpha^2(x)\right)-\left(1\otimes ad_M\alpha^2(z)\right)\left(1\otimes ad_M\alpha(y)\right)\Delta(x)+\left(ad_M y\otimes\alpha\right)\Delta\left([x,z]_M\right)\\
&+\left(1\otimes ad_M\alpha^2(y)\right)\left(1\otimes ad_M\alpha(x)\right)\Delta(z)+\left(\alpha^2\otimes 1\right)\Delta\left[[x,y]_M,\alpha(z)\right]_M-\left(\alpha\otimes ad_M\alpha^2(x)\right)\Delta\left([y,z]_M\right)\\
&-(1\otimes\alpha)\left(1\otimes ad_M[x,z]_M\right)\Delta(y)-\left(ad_M z\otimes ad_M\alpha^2(y)\right)\Delta\left(\alpha(x)\right)+\left(ad_M x\otimes ad_M\alpha^2(z)\right)\Delta\left(\alpha(y)\right)\\
&-\left(ad_M x\otimes 1\right)\left(ad_M\alpha(y)\otimes 1\right)\Delta\left(\alpha^2(z)\right)+\left(ad_M z\otimes 1\right)\left(ad_M\alpha(x)\otimes 1\right)\Delta\left(\alpha^2(y)\right)=0,
\end{aligned}\tag{3.13}$$

$$\begin{aligned}
&\left(1\otimes ad_{M^*}\alpha^{*2}(b^*)\right)\left(1\otimes ad_{M^*}\alpha^*(a^*)\right)\Delta^*(c^*)+\left(\alpha^{*2}\otimes 1\right)\Delta^*\left[[a^*,b^*]_{M^*},\alpha^*(c^*)\right]_{M^*}\\
&+\left(ad_{M^*}a^*\otimes ad_{M^*}\alpha^{*2}(c^*)\right)\Delta^*\left(\alpha^*(b^*)\right)-\left(ad_{M^*}c^*\otimes ad_{M^*}\alpha^{*2}(b^*)\right)\Delta^*\left(\alpha^*(a^*)\right)\\
&-\left(1\otimes ad_{M^*}\alpha^{*2}(c^*)\right)\left(1\otimes ad_{M^*}\alpha^*(b^*)\right)\Delta^*(a^*)-\left(1\otimes\alpha^*\right)\left(1\otimes ad_{M^*}[a^*,c^*]_{M^*}\right)\Delta^*(b^*)\\
&-\left(\alpha^*\otimes ad_{M^*}\alpha^{*2}(a^*)\right)\Delta^*\left([b^*,c^*]_{M^*}\right)+\left(\alpha^*\otimes 1\right)\left(ad_{M^*}[b^*,c^*]_{M^*}\otimes 1\right)\Delta^*\left(\alpha^{*2}(a^*)\right)\\
&+\left(ad_{M^*}c^*\otimes 1\right)\left(ad_{M^*}\alpha^*(a^*)\otimes 1\right)\Delta^*\left(\alpha^{*2}(b^*)\right)+\left(ad_{M^*}b^*\otimes\alpha^*\right)\Delta^*\left([a^*,c^*]_{M^*}\right)\\
&-\left(ad_{M^*}a^*\otimes 1\right)\left(ad_{M^*}\alpha^*(b^*)\otimes 1\right)\Delta^*\left(\alpha^{*2}(c^*)\right)=0,
\end{aligned}\tag{3.14}$$

$$\begin{aligned}
&\left\langle b^*\otimes c^*,\left(\alpha^2\otimes 1\right)\Delta\left(ad^*_{M^*}\alpha^*(a^*)\left([x,y]_M\right)\right)\right\rangle-\left\langle b^*\otimes c^*,(1\otimes\alpha)\left(1\otimes ad_M\left(ad^*_{M^*}\alpha^*(x)\right)\right)\Delta(y)\right\rangle\\
&+\left\langle b^*\otimes c^*,\left(ad_M y\otimes\alpha\right)\Delta\left(ad^*_{M^*}\alpha^*(x)\right)\right\rangle-\left\langle b^*\otimes c^*,(\alpha\otimes 1)\left(ad^*_{M^*}\left(ad^*_M y(a^*)\right)\otimes 1\right)\Delta\left(\alpha^2(x)\right)\right\rangle\\
&-\left\langle b^*\otimes c^*,\left(ad^*_{M^*}a^*\otimes 1\right)\left(ad_M\alpha(x)\otimes 1\right)\Delta\left(\alpha^2(y)\right)\right\rangle+\left\langle b^*\otimes c^*,\left(ad^*_{M^*}a^*\otimes ad_M\alpha^2(y)\right)\Delta\left(\alpha(x)\right)\right\rangle\\
&-\left\langle b^*\otimes c^*,\left(ad_M x\otimes ad^*_{M^*}\alpha^{*2}(a^*)\right)\Delta\left(\alpha(y)\right)\right\rangle+\left\langle b^*\otimes c^*,(1\otimes\alpha)\left(1\otimes ad^*_{M^*}\left(ad^*_M x(a^*)\right)\right)\Delta(y)\right\rangle\\
&-\left\langle b^*\otimes c^*,\left(\alpha\otimes ad_M\alpha^2(x)\right)\Delta\left(ad^*_{M^*}a^*(y)\right)\right\rangle+\left\langle b^*\otimes c^*,\left(1\otimes ad^*_{M^*}\alpha^{*2}(a^*)\right)\left(1\otimes ad_M\alpha(y)\right)\Delta(x)\right\rangle\\
&+\left\langle b^*\otimes c^*,(\alpha\otimes 1)\left(ad_M\left(ad^*_{M^*}a^*(y)\right)\otimes 1\right)\Delta\left(\alpha^2(x)\right)\right\rangle=0,
\end{aligned}\tag{3.15}$$

$$\begin{aligned}
&\left\langle y\otimes z,\left(ad^*_M x\otimes ad_{M^*}\alpha^{*2}(b^*)\right)\Delta^*\left(\alpha^*(a^*)\right)\right\rangle-\left\langle y\otimes z,\left(1\otimes\alpha^*\right)\left(1\otimes ad_{M^*}\left(ad^*_M x(a^*)\right)\right)\Delta^*(b^*)\right\rangle\\
&+\left\langle y\otimes z,\left(ad_{M^*}b^*\otimes\alpha^*\right)\Delta^*\left(ad^*_M x(a^*)\right)\right\rangle-\left\langle y\otimes z,\left(\alpha^*\otimes 1\right)\left(ad^*_M\left(ad^*_{M^*}b^*(x)\right)\otimes 1\right)\Delta^*\left(\alpha^{*2}(a^*)\right)\right\rangle\\
&-\left\langle y\otimes z,\left(ad_{M^*}a^*\otimes ad^*_M\alpha^2(x)\right)\Delta^*\left(\alpha^*(b^*)\right)\right\rangle+\left\langle y\otimes z,\left(1\otimes\alpha^*\right)\left(1\otimes ad^*_M\left(ad^*_{M^*}a^*(x)\right)\right)\Delta^*(b^*)\right\rangle\\
&-\left\langle y\otimes z,\left(ad^*_M x\otimes 1\right)\left(ad_{M^*}\alpha^*(a^*)\otimes 1\right)\Delta^*\left(\alpha^{*2}(b^*)\right)\right\rangle+\left\langle y\otimes z,\left(\alpha^{*2}\otimes 1\right)\Delta^*\left(ad^*_M\alpha(x)\left([a^*,b^*]_{M^*}\right)\right)\right\rangle\\
&-\left\langle y\otimes z,\left(ad^*_M x\otimes 1\right)\left(ad_{M^*}\alpha^*(a^*)\otimes 1\right)\Delta^*\left(\alpha^{*2}(b^*)\right)\right\rangle+\left\langle y\otimes z,\left(\alpha^{*2}\otimes 1\right)\Delta^*\left(ad^*_M\alpha(x)\left([a^*,b^*]_{M^*}\right)\right)\right\rangle\\
&+\left\langle y\otimes z,\left(\alpha^*\otimes 1\right)\left(ad_{M^*}\left(ad^*_M x(b^*)\right)\otimes 1\right)\Delta^*\left(\alpha^{*2}(a^*)\right)\right\rangle=0,
\end{aligned}\tag{3.16}$$

$$\begin{aligned}
&\left\langle a^*\otimes b^*,\left(\alpha^2\otimes 1\right)\left(ad^*_{M^*}c^*\otimes 1\right)\left(ad_M\alpha(x)\otimes 1\right)\Delta(y)\right\rangle-\left\langle a^*\otimes b^*,\left(\alpha^2\otimes 1\right)\left(ad^*_{M^*}c^*\otimes ad_M y\right)\Delta\left(\alpha(x)\right)\right\rangle\\
&-\left\langle a^*\otimes b^*,\left(ad_M y\otimes\alpha\right)\Delta\left(ad^*_{M^*}c^*\left(\alpha^2(x)\right)\right)\right\rangle-\left\langle a^*\otimes b^*,\left(1\otimes\alpha^2\right)\left(1\otimes ad^*_{M^*}c^*\right)\left(1\otimes ad_M\alpha(y)\right)\Delta(x)\right\rangle\\
&+\left\langle a^*\otimes b^*,\left(1\otimes\alpha^2\right)\left(ad_M x\otimes ad^*_{M^*}c^*\right)\Delta\left(\alpha(y)\right)\right\rangle-\left\langle a^*\otimes b^*,(\alpha\otimes 1)\left(ad_M\left(ad^*_{M^*}c^*\left(\alpha^2(y)\right)\right)\otimes 1\right)\Delta(x)\right\rangle\\
&+\left\langle a^*\otimes b^*,(1\otimes\alpha)\left(1\otimes ad_M\left(ad^*_{M^*}c^*\left(\alpha^2(x)\right)\right)\right)\Delta(y)\right\rangle+\left\langle a^*\otimes b^*,\left(\alpha\otimes ad_M x\right)\Delta\left(ad^*_{M^*}c^*\left(\alpha^2(y)\right)\right)\right\rangle\\
&+\left\langle a^*\otimes b^*,(\alpha\otimes 1)\left(ad^*_{M^*}\left(ad^*_M\alpha^2(y)(c^*)\right)\otimes 1\right)\Delta(x)\right\rangle-\left\langle a^*\otimes b^*,\Delta\left(ad^*_{M^*}\alpha^*(c^*)\left([x,y]_M\right)\right)\right\rangle\\
&-\left\langle a^*\otimes b^*,(1\otimes\alpha)\left(1\otimes ad^*_{M^*}\left(ad^*_M\alpha^2(x)(c^*)\right)\right)\Delta(y)\right\rangle=0,
\end{aligned}\tag{3.17}$$

$$\begin{aligned}&\left\langle x\otimes y,\left(\alpha^{*2}\otimes 1\right)\left(ad_M^* z\otimes 1\right)\left(ad_{M^*}\alpha^*\left(a^*\right)\otimes 1\right)\Delta^*\left(b^*\right)\right\rangle-\left\langle x\otimes y,\left(\alpha^{*2}\otimes 1\right)\left(ad_M^* z\otimes ad_{M^*}b^*\right)\Delta^*\left(\alpha^*\left(a^*\right)\right)\right\rangle\\&-\left\langle x\otimes y,\left(ad_{M^*}b^*\otimes\alpha^*\right)\Delta^*\left(ad_M^* z\left(\alpha^{*2}\left(a^*\right)\right)\right)\right\rangle-\left\langle x\otimes y,\left(1\otimes\alpha^{*2}\right)\left(1\otimes ad_M^* z\right)\left(1\otimes ad_{M^*}\alpha^*\left(b^*\right)\right)\Delta^*\left(a^*\right)\right\rangle\\&+\left\langle x\otimes y,\left(1\otimes\alpha^{*2}\right)\left(ad_{M^*}a^*\otimes ad_M^* z\right)\Delta^*\left(\alpha^*\left(b^*\right)\right)\right\rangle-\left\langle x\otimes y,\left(\alpha^*\otimes 1\right)\left(ad_{M^*}\left(ad_M^* z\left(\alpha^{*2}\left(b^*\right)\right)\right)\otimes 1\right)\Delta^*\left(a^*\right)\right\rangle\\&+\left\langle x\otimes y,\left(\alpha^*\otimes 1\right)\left(ad_M^*\left(ad_{M^*}^*\alpha^{*2}\left(b^*\right)(z)\right)\otimes 1\right)\Delta^*\left(a^*\right)\right\rangle+\left\langle x\otimes y,\left(\alpha^*\otimes ad_{M^*}a^*\right)\Delta*\left(ad_M^* z\left(\alpha^{*2}\left(b^*\right)\right)\right)\right\rangle\\&+\left\langle x\otimes y,\left(1\otimes\alpha^*\right)\left(1\otimes ad_{M^*}\left(ad_M^* z\left(\alpha^{*2}\left(a^*\right)\right)\right)\right)\Delta^*\left(b^*\right)\right\rangle-\left\langle x\otimes y,\Delta^*\left(ad_M^*\alpha(z)\left(\left[a^*,b^*\right]_{M^*}\right)\right)\right\rangle\\&-\left\langle x\otimes y,\left(1\otimes\alpha^*\right)\left(1\otimes ad_{M^*}\left(ad_{M^*}^*\alpha^{*2}\left(a^*\right)(z)\right)\right)\Delta^*\left(b^*\right)\right\rangle=0,\end{aligned}\tag{3.18}$$

证明： 由定理 3.2 可知，$\left(M\oplus M^*,[\ ],\alpha+\alpha^*\right)$是 Hom-Malcev 代数当且仅当$\left(M,M^*,ad_M^*,ad_{M^*}^*\right)$为$\left(M,[\ ]_M,\alpha\right)$和$\left(M^*,[\ ]_{M^*},\alpha^*\right)$这两个 Hom-Malcev 代数的配对，当且仅当$\forall x,y,z,\in M$，$\forall a^*,b^*,c^*\in M^*$满足(3.1)~(3.6)即可。

定义 3.5： 设$\left(M,[\ ]_M,\alpha\right)$为相容的 Hom-Malcev 代数，$\Delta:M\to M\otimes M$为线性映射，若$\Delta$满足(3.9)~(3.18)，则称$\left(M,M^*,\Delta\right)$为 Hom-Malcev 双代数。

4. Hom-Malcev 代数的 Manin triple

定义 4.1： 设$\left(M,[\ ],\alpha\right)$是 Hom-Malcev 代数，若$M_+$和$M_-$都为$M$的子代数，$M=M_++M_-$，$M$上存在一个非退化的、对称的双线性函数$B(\cdot,\cdot)$保持不变性，即$\forall x,y,z\in M$，有

$$B\left(\left[x,y\right],z\right)=B\left(x,\left[y,z\right]\right),\tag{4.1}$$

$$B\left(\alpha\left(x\right),y\right)=B\left(x,\alpha\left(y\right)\right),\tag{4.2}$$

且M_+和M_-关于$B(\cdot,\cdot)$都是迷向的，即$B\left(M_+,M_+\right)=B\left(M_-,M_-\right)=0$，则称$\left(M,M_+,M_-\right)$为 Hom-Malcev 代数$\left(M,[\ ],\alpha\right)$的 Manin triple。

定理 4.2： 设$\left(M,[\ ]_M,\alpha\right)$，$\left(M^*,[\ ]_{M^*},\alpha^*\right)$为相容的 Hom-Malcev 代数，$B(\cdot,\cdot)$为$M\oplus M^*$上的双线性函数，$B\left(x+a^*,y+b^*\right)=\left\langle x,b^*\right\rangle+\left\langle y,a^*\right\rangle$，在$M\oplus M^*$上定义运算

$$\left[x+a^*,y+b^*\right]=\left[x,y\right]_M+ad_M^* x\left(b^*\right)-ad_M^* y\left(a^*\right)+\left[a^*,b^*\right]_{M^*}+ad_{M^*}^* a^*\left(y\right)-ad_{M^*}^* b^*\left(x\right),$$

并定义

$$\left(\alpha+\alpha^*\right)\left(x+a^*\right)=\alpha\left(x\right)+\alpha^*\left(a^*\right),$$

其中，$\forall x,y\in M$，$\forall a^*,b^*\in M^*$，则$\left(M,M^*,ad_M^*,ad_{M^*}^*\right)$是 Hom-Malcev 代数的配对的充分必要条件为$\left(M\oplus M^*,M,M^*\right)$是 Hom-Malcev 代数的 Manin triple。

证明： 必要性。根据定理 3.2 可知，若$\left(M,M^*,ad_M^*,ad_{M^*}^*\right)$是配对，则$\left(M\oplus M^*,[\ ],\alpha+\alpha^*\right)$是 Hom-Malcev 代数。显然，$\left(M,[\ ]_M,\alpha\right)$和$\left(M^*,[\ ]_{M^*},\alpha^*\right)$都为$\left(M\oplus M^*,[\ ],\alpha+\alpha^*\right)$的子代数。

$\forall y+b^*\in M\oplus M^*$，取$x+a^*\in M\oplus M^*$，若$B\left(x+a^*,y+b^*\right)=0$，则当$b^*=0$时，

$$B\left(x+a^*,y\right)=\left\langle x,0\right\rangle+\left\langle y,a^*\right\rangle=0+\left\langle y,a^*\right\rangle=0,$$

可以得到$a^*=0$。同理，当$y=0$时，可以得到$x=0$，因此$x+a^*=0$，$B(\cdot,\cdot)$是非退化的。

$$B\left(x+a^*,y+b^*\right)=\left\langle x,b^*\right\rangle+\left\langle y,a^*\right\rangle=\left\langle y,a^*\right\rangle+\left\langle x,b^*\right\rangle=B\left(y+b^*,x+a^*\right),$$

可知$B(\cdot,\cdot)$是对称的。

$$
\begin{aligned}
&B\left(\left[x+a^*,y+b^*\right],z+c^*\right)-B\left(x+a^*,\left[y+b^*,z+c^*\right]\right)\\
&=\left\langle [x,y]_M,c^*\right\rangle+\left\langle ad^*_{M^*}a^*(y),c^*\right\rangle-\left\langle ad^*_{M^*}b^*(x),c^*\right\rangle+\left\langle \left[a^*,b^*\right]_{M^*},z\right\rangle\\
&\quad+\left\langle ad^*_M x(b*),z\right\rangle-\left\langle ad^*_M y\left(a^*\right),z\right\rangle-\left\langle x,\left[b^*,c^*\right]_{M^*}\right\rangle-\left\langle x,ad^*_M y\left(c^*\right)\right\rangle\\
&\quad+\left\langle x,ad^*_M z\left(b^*\right)\right\rangle-\left\langle a^*,\left[b^*,c^*\right]_{M^*}\right\rangle-\left\langle a^*,ad^*_{M^*}b^*(z)\right\rangle+\left\langle a^*,ad^*_{M^*}c^*\right\rangle,
\end{aligned}
$$

由 $\left\langle ad^*_{M^*}a^*(y),c^*\right\rangle=-\left\langle y,ad_{M^*}a^*\left(c^*\right)\right\rangle=-\left\langle y,\left[a^*,c^*\right]_{M^*}\right\rangle$，可知(4.1)成立。

$$
\begin{aligned}
&B\left(\left(\alpha+\alpha^*\right)\left(x+a^*\right),y+b^*\right)-B\left(x+a^*,\left(\alpha+\alpha^*\right)\left(y+b^*\right)\right)\\
&=\left\langle \alpha(x),b^*\right\rangle+\left\langle \alpha^*\left(a^*\right),y\right\rangle-\left\langle x,\alpha^*\left(b^*\right)\right\rangle-\left\langle a^*,\alpha(y)\right\rangle
\end{aligned}
$$

由 $\left\langle \alpha(x),b^*\right\rangle=\left\langle x,\alpha^*\left(b^*\right)\right\rangle$，可知(4.2)成立。因此，$B(\cdot,\cdot)$ 是不变的。

由 $B(x,y)=B(x+0,y+0)=\langle x,0\rangle+\langle y,0\rangle=0$，可知 M 关于 $B(\cdot,\cdot)$ 都是迷向的。同理，M^* 关于 $B(\cdot,\cdot)$ 也是迷向的。综上所述，$\left(M\oplus M^*,M,M^*\right)$ 是 Manin triple。

充分性。若 $\left(M\oplus M^*,M,M^*\right)$ 是 Manin triple，由定义 4.1 可知，$\left(M\oplus M^*,[\ ],\alpha+\alpha^*\right)$ 是 Hom-Malcev 代数，因此，由定理 3.2 可知，$\left(M,M^*,ad^*_M,ad^*_{M^*}\right)$ 是配对。

定理 4.3： 设 $\left(M,[\ ]_M,\alpha\right)$，$\left(M^*,[\ ]_{M^*},\alpha^*\right)$ 为相容的 Hom-Malcev 代数，则下列三个条件是等价的。

1) $\left(M,M^*,\Delta\right)$ 为 Hom-Malcev 双代数。

2) $\left(M,M^*,ad^*_M,ad^*_{M^*}\right)$ 是 Hom-Malcev 代数的配对。

3) $\left(M\oplus M^*,M,M^*\right)$ 是 Hom-Malcev 代数的 Manin triple。

证明： 由定理 4.2 和定义 3.5 可推出。

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