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Hans汉斯

The G Expansion Method of Satisfying a Variable Coefficient Equation and New Exact Solutions of Rlw-Burgers Equation

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Abstract

Based on the basic idea of the (G'/G) expansion method, we construct a kind of New G method, and make the function G satisfy a class of variable coefficient Bernoulli equation. The RLW-Burgers equation is solved by this method, and several new explicit traveling wave solutions of the equation are obtained. It has been proved that this kind of satisfying variable coefficient equation G expansion method for solving nonlinear partial differential equations solutions is feasible and effective.

Keywords

RLW-Burgers Equation, Variable Coefficient Bernoulli Equation, G Expansion Method, Exact Solutions

满足变系数方程的*G*展开法及RLW-Burgers方程的新精确解

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摘　要

本文借鉴(G'/G)展开法的基本思路，构造了一类新的G展开法，并令其中的函数G满足一类变系数

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Bernoulli方程。用此法对RLW-Burgers方程进行了求解，得到了该方程的多个新的显式行波解。事实证明，这类满足变系数方程的*G*展开法对于求解非线性偏微分方程的精确解是有效可行的。

关键词

RLW-Burgers方程，变系数Bernoulli方程，*G*展开法，精确解

1. 引言

求解非线性偏微分方程的精确解是目前科技发展中解决非线性问题的关键。现在已有多种可以求得非线性偏微分方程精确解的常用方法，其中(G'/G)展开法[1]，是一种较为简单可行的计算方法。本文就是以(G'/G)展开法为基础，构造了一类$\left(\frac{m_1G+n_1G'}{m_2G+n_2G'}\right)$展开法，其中$m_i,n_i\ (i=1,2)$为互不相等的自由参数，并令其中的函数$G$满足一类变系数Bernoulli方程。通过此展开法，求得了RLW-Burgers方程的多个新的精确解。

RLW-Burgers方程

$$u_t+u_x-\theta u_{xx}+uu_x-\delta u_{xxt}=0 \tag{1}$$

其中$\theta>0,\delta>0$。该方程可以用来描述河渠中水波表面的传播性态，它的行波解的存在性及解的性质已由文献[2]和文献[3]给出；振荡激波解也由文献[4]得到；通过未知函数的变换及待定系数法，文献[5]得到了该方程的显式行波解；该方程的古典对称和势对称也在文献[6]中得到了研究，并通过推广的双曲函数法和Lie变换群得到了该方程的精确解；文献[7]通过改进的G展开法得到了方程的含参数的多种形式的精确解。

2. 满足变系数方程的 *G* 展开法

将非线性偏微分方程

$$P(u,u_x,u_t,u_{xx},u_{xt},u_{tt},\cdots)=0 \tag{2}$$

作行波变换，令$u(x,t)=u(\xi)$，$\xi=x-ct$，则化为常微分方程

$$P(u,u',u'',\cdots)=0,$$

设方程(2)的解为

$$u(\xi)=\sum_{i=0}^{l}a_i\left(\frac{m_1G+n_1G'}{m_2G+n_2G'}\right)^i \tag{3}$$

其中$a_i\ (i=0,1,2,\cdots)$为待定的常数，m_1,m_2,n_1,n_2为互不相等的自由参数，参数l可以通过齐次平衡法确定；其中函数$G(\xi)$满足一类变系数Bernoulli方程

$$G'(\xi)+p(\xi)G(\xi)+q(\xi)G^2(\xi)=0 \tag{4}$$

这里$p(\xi),q(\xi)$均为ξ的任意函数。通过借助符号计算软件Mathematica，我们可以得到方程(4)的解：

$$G(\xi)=\frac{\mathrm{e}^{\int_1^{\xi}-p(\tau)\mathrm{d}\tau}}{C_1-\int_1^{\xi}\left(-\mathrm{e}^{\int_1^{\varsigma}-p(\tau)\mathrm{d}\tau}q(\varsigma)\right)\mathrm{d}\varsigma} \tag{5}$$

这里 C_1 为积分常数。

将(3)式代入(2)式，并结合(4)式，合并 $\left(\frac{m_1G+n_1G'}{m_2G+n_2G'}\right)^i$ 的同幂次项，并令 $\left(\frac{m_1G+n_1G'}{m_2G+n_2G'}\right)^i$ 的各次幂的系数为零，从中求出 $a_i,p(\xi),q(\xi)$，再将求得的 $p(\xi),q(\xi)$ 代入(5)式，得到 $G(\xi)$ 函数，最后将 $G(\xi)$ 与 a_i 代入(3)式即得到方程(2)的解。

3. RLW-Burgers 方程新的精确解

设方程(1)有行波解 $u=u(\xi)=u(x-ct)$，其中 c 表示波速，从而方程(1)化为

$$(c+1)u'+uu'-\theta u''-c\delta u'''=0,$$

将上式两边关于 ξ 进行积分并化简，得

$$(c+1)u+\frac{1}{2}u^2-\theta u'-c\delta u''=M, \tag{6}$$

其中 M 为积分常数。设方程(1)的解能够表示成多项式 $u(\xi)=\sum_{i=0}^{l}a_i\left(\frac{m_1G+n_1G'}{m_2G+n_2G'}\right)^i$，且 $G=G(\xi)$ 满足一类变系数 Bernoulli 方程

$$G'(\xi)+p(\xi)G(\xi)+q(\xi)G(\xi)^2=0,$$

其中 $p(\xi),q(\xi)$ 为 ξ 的任意函数。利用齐次平衡法，我们有 $2l=l+2$，得 $l=2$。则方程(1)的解表示为

$$u(\xi)=a_2\left(\frac{m_1G+n_1G'}{m_2G+n_2G'}\right)^2+a_1\left(\frac{m_1G+n_1G'}{m_2G+n_2G'}\right)+a_0, \tag{7}$$

由方程(4)和(7)式可得

$$\begin{aligned}
u'(\xi)=&-\frac{2a_2\left(q(\xi)\left(-m_2^2+p(\xi)m_2n_2+n_2^2p'(\xi)\right)+n_2\left(m_2-p(\xi)n_2\right)q'(\xi)\right)}{q(\xi)(m_2n_1-m_1n_2)}\left(\frac{m_1G+n_1G'}{m_2G+n_2G'}\right)^3\\
&+\frac{1}{q(\xi)(m_2n_1-m_1n_2)}\Big(q(\xi)\Big(a_1\left(m_2^2-p(\xi)m_2n_2-n_2^2p'(\xi)\right)+2a_2\big(m_1\left(-2m_2+p(\xi)n_2\right)\\
&+n_1\left(p(\xi)m_2+2n_2p'(\xi)\right)\big)\Big)+\Big(a_1n_2\left(-m_2+p(\xi)n_2\right)+2a_2\left(m_2n_1+\left(m_1-2p(\xi)n_1\right)n_2\right)\Big)\\
&\times q'(\xi)\Big)\left(\frac{m_1G+n_1G'}{m_2G+n_2G'}\right)^2+\frac{1}{q(\xi)(m_2n_1-m_1n_2)}\Big(q(\xi)\Big(2a_2\left(m_1^2-p(\xi)m_1n_1-n_1^2p'(\xi)\right)\\
&+a_1\left(m_1\left(p(\xi)n_2-2m_2\right)+n_1\left(p(\xi)m_2+2n_2p'(\xi)\right)\right)\Big)\\
&+\Big(2a_2n_1\left(p(\xi)n_1-m_1\right)+a_1\left(m_2n_1+\left(m_1-2p(\xi)n_1\right)n_2\right)\Big)q'(\xi)\Big)\left(\frac{m_1G+n_1G'}{m_2G+n_2G'}\right)\\
&+\frac{a_1\left(q(\xi)\left(m_1^2-p(\xi)m_1n_1-n_1^2p'(\xi)\right)+n_1\left(-m_1+p(\xi)n_1\right)q'(\xi)\right)}{q(\xi)(m_2n_1-m_1n_2)}
\end{aligned}$$

$$
\begin{aligned}
u''(\xi) = {} & \frac{6a_2}{q(\xi)^2 (m_2 n_1 - m_1 n_2)^2} \Big(q(\xi)\big(m_2^2 - p(\xi) m_2 n_2 - n_2^2 p'(\xi)\big) + n_2 \big(-m_2 + p(\xi) n_2\big) q'(\xi)\Big)^2 \\
& \times \left(\frac{m_1 G + n_1 G'}{m_2 G + n_2 G'}\right)^4 + \frac{2}{q(\xi)^2 (m_2 n_1 - m_1 n_2)^2} \Big(n_2 \big(p(\xi) n_2 - m_2\big)\Big(a_1 n_2 \big(p(\xi) n_2 - m_2\big) \\
& + 2a_2 \big(2m_2 n_1 + (3m_1 - 5p(\xi) n_1) n_2\big)\Big) q'(\xi)^2 + q(\xi)^2 \Big(a_1 \big(-m_2^2 + p(\xi) m_2 n_2 + n_2^2 p'(\xi)\big)^2 \\
& - a_2 \Big(m_1 \Big(10 m_2^3 - 15 p(\xi) m_2^2 n_2 + m_2 n_2^2 \big(5p(\xi)^2 - 11 p'(\xi)\big) + n_2^3 \big(5 p(\xi) p'(\xi) - p''(\xi)\big)\Big) + n_1 \\
& \times \Big(5p(\xi)^2 m_2^2 n_2 - 5p(\xi)\big(m_2^3 - 3m_2 n_2^2 p'(\xi)\big) + n_2\big(-9m_2^2 p'(\xi) + 10 n_2^2 p'(\xi)^2 + m_2 n_2 p''(\xi)\big)\Big)\Big)\Big) \\
& + q(\xi)\Big(\Big(2a_1 n_2 \big(m_2 - p(\xi) n_2\big)\big(-m_2^2 + p(\xi) m_2 n_2 + n_2^2 p'(\xi)\big) + a_2 \Big(5\big(m_2 - p(\xi) n_2\big)\Big(m_2^2 n_1 \\
& + 3m_2 \big(m_1 - p(\xi) n_1\big) n_2 - p(\xi) m_1 n_2^2\Big) + 2n_2^2 \big(-7m_2 n_1 + (-3m_1 + 10 p(\xi) n_1) n_2\big) p'(\xi)\Big)\Big) q'(\xi) \\
& - a_2 n_2 \big(m_2 - p(\xi) n_2\big)\big(m_2 n_1 - m_1 n_2\big) q''(\xi)\Big)\Big) \left(\frac{m_1 G + n_1 G'}{m_2 G + n_2 G'}\right)^3 + \frac{1}{q(\xi)^2 (m_2 n_1 - m_1 n_2)^2} \\
& \times \Big(2\Big(a_2 \Big(m_2^2 n_1^2 + 2m_2 n_1 \big(4m_1 - 5p(\xi) n_1\big) n_2 + \big(3m_1^2 - 14 p(\xi) m_1 n_1 + 12 p(\xi)^2 n_1^2\big) n_2^2\Big) \\
& - a_1 n_2 \big(m_2 - p(\xi) n_2\big)\big(2m_1 n_2 + n_1(m_2 - 3p(\xi) n_2)\big)\Big) q'(\xi)^2 + q(\xi)^2 \Big(2a_2 \Big(m_1^2 \Big(12 m_2^2 \\
& - 12 p(\xi) m_2 n_2 + n_2^2 \big(2p(\xi)^2 - 5p'(\xi)\big)\Big) + n_1^2 \Big(2p(\xi)^2 m_2^2 + 12 p(\xi) m_2 n_2 p'(\xi) - 3p'(\xi) m_2^2 \\
& - 4n_2^2 p'(\xi)\big) + 2m_2 n_2 p''(\xi)\Big) + 2m_1 n_1 \big(4p(\xi)^2 - n_2^2 p'(\xi)\big) - n_2 \big(8m_2 p'(\xi) + n_2 p''(\xi)\big)\Big)\Big)\Big) \\
& - a_1 \Big(m_1 \Big(6m_2^3 - 9p(\xi) m_2^2 n_2 + m_2 n_2^2 \big(3p(\xi)^2 - 7p'(\xi)\big) + n_2^3 \big(3p(\xi)p'(\xi) - p''(\xi)\big)\Big)\Big) \\
& + n_1 \Big(3p(\xi)^2 m_2^2 n_2 - 3p(\xi)\big(m_2^3 - 3m_2 n_2^2 p'(\xi)\big) + n_2 \big(-5m_2^2 p'(\xi) + 6n_2^2 p'(\xi)^2 \Big(a_1 \Big(\Big(3(m_2 \\
& - p(\xi) n_2)\Big(m_2^2 n_1 + 3m_2 \big(m_1 - p(\xi) n_1\big) n_2 - p(\xi) m_1 n_2^2\Big) - 4n_2^2 \big(2m_2 n_1 + (m_1 - 3p(\xi) n_1) n_2\big) \\
& - n_2 \big(m_2 - p(\xi) n_2\big)\big(m_2 n_1 - m_1 n_2\big) q''(\xi)\Big) + 2a_2 \Big(-2\Big(2m_1^2 n_2 \big(3m_2 - 2p(\xi) n_2\big) + m_1 n_1 \Big(6m_2^2 \\
& - 16 p(\xi) m_2 n_2 + n_2^2 \big(6p(\xi)^2 - 7p'(\xi)\big)\Big)\Big) + n_1^2 \big(2p(\xi) m_2 \big(-2m_2 + 3p(\xi) n_2\big) + n_2 \big(-5m_2 \\
& + 12 p(\xi) n_1 - m_1 n_2\big)\big(m_2 n_1 + (m_1 - 2p(\xi) n_1) n_2\big) q''(\xi)\Big)\Big)\Big) \left(\frac{m_1 G + n_1 G'}{m_2 G + n_2 G'}\right)^2 + \frac{1}{q(\xi)^2 (m_2 n_1 - m_1 n_2)^2} \\
& \times (-2n_1)\Big(a_1 n_2 \big(-2m_2 n_1 - (m_1 - 3p(\xi) n_1) n_2\big) + 2a_2 n_1 \big(2m_1 n_2 + n_1 (m_2 - 3p(\xi) n_2)\big)\Big) q'(\xi)^2 \\
& + q(\xi)^2 \Big(-2a_2 \Big(m_1^3 \big(6m_2 - 3p(\xi) n_2\big) + m_1^2 n_1 \Big(3p(\xi)^2 n_2 - 9p(\xi) m_2 - 7n_2 p'(\xi)\Big) \\
& + n_1^3 \Big(3p(\xi) m_2 p'(\xi) + 6n_2 p'(\xi)^2 + m_2 p''(\xi)\Big) + m_1 n_1^2 \Big(3p(\xi)^2 m_2 - 5m_2 p'(\xi) + 9p(\xi) n_2 p'(\xi) \\
& - n_2 p''(\xi)\Big)\Big) + a_1 \Big(m_1^2 \Big(6m_2^2 - 6p(\xi) m_2 n_2 + n_2^2 \big(p(\xi)^2 - 3p'(\xi)\big)\Big) + n_1^2 \Big(p(\xi)^2 m_2^2 - m_2^2 p'(\xi) \\
& + 6p(\xi)p'(\xi) m_2 n_2 + 6n_2^2 p'(\xi)^2 + 2m_2 n_2 p''(\xi)\Big) + 2m_1 n_1 \Big(2p(\xi)^2 m_2 n_2 - 3p(\xi)\big(m_2^2 - n_2^2 p'(\xi)\big) \\
& - n_2 \big(4p'(\xi) m_2 + n_2 p''(\xi)\big)\Big)\Big)\Big)\Big) + q(\xi)\Big(2a_2 \Big(\Big(3\big(m_1 - p(\xi) n_1\big)\Big(m_2 n_1 \big(3m_1 - p(\xi) n_1\big) + m_1 (m_1 \\
& - 3p(\xi) n_1) n_2\Big) - 2n_1^2 \big(5m_1 n_2 + m_1 n_2\big) q''(\xi)\Big) + a_1 \Big(2\Big(m_1^2 n_2 \big(2p(\xi) n_2 - 3m_2\big) + n_1^2 \big(p(\xi) m_2 (2m_2
\end{aligned}
$$

$$-3p(\xi)n_2\big)+2n_2\left(m_2-3p(\xi)n_2\right)p'(\xi)\big)+m_1n_1\left(8p(\xi)m_2n_2-3m_2^2+n_2^2\left(4p'(\xi)-3p(\xi)^2\right)\right)\Big)$$

$$\times q'(\xi)+\left(m_2n_1-m_1n_2\right)\left(m_2n_1+\left(m_1-2p(\xi)n_1\right)n_2\right)q''(\xi)\Big)\Big)\Big)\left(\frac{m_1G+n_1G'}{m_2G+n_2G'}\right)$$

$$+\frac{1}{q(\xi)^2\left(m_2n_1-m_1n_2\right)^2}\left(2n_1-p(\xi)n_1\right)^2\left(a_2n_1-a_1n_2\right)q'(\xi)^2+q(\xi)^2\Big(2a_2\left(-m_1^2+p(\xi)m_1n_1\right.$$

$$\left.+n_1^2p'(\xi)\right)^2-a_1\Big(m_1^3\Big(m_1\left(2m_2\left(2m_2-p(\xi)n_2\right)+m_1^2n_1\left(-3p(\xi)m_2+p(\xi)^2n_2-3n_2p'(\xi)\right)\right.$$

$$+n_1^3\left(p(\xi)m_2p'(\xi)+2n_2p'(\xi)^2+m_2p''(\xi)\right)+m_1n_1^2\Big(p(\xi)^2m_2-m_2p'(\xi)+3p(\xi)n_2p'(\xi)$$

$$-n_2p''(\xi)\Big)\Big)\Big)+q(\xi)\left(m_1-p(\xi)n_1\right)\Big(4a_2n_1\left(-m_1^2+p(\xi)m_1n_1+n_1^2p'(\xi)\right)q'(\xi)$$

$$+a_1\Big(\Big(m_1^2n_2+3m_1n_1\left(m_2-p(\xi)n_2\right)-n_1^2\left(p(\xi)m_2+4p'(\xi)n_2\right)\Big)q'(\xi)$$

$$+n_1\left(-m_2n_1+m_1n_2\right)q''(\xi)\Big)\Big)\Big)$$

将上面u的各阶导数全部代入(6)式，并合并$\left(\dfrac{m_1G+n_1G'}{m_2G+n_2G'}\right)$的同幂次项，比较方程两端的系数，化简得

$\left(\dfrac{m_1G+n_1G'}{m_2G+n_2G'}\right)^0$:

$$q(\xi)^2\left(-2M+2(1+c)a_0+a_0^2\right)\left(m_2n_1-m_1n_2\right)^2-8c\delta q(\xi)a_2n_1\left(m_1-p(\xi)n_1\right)\Big(-m_1^2$$

$$+p(\xi)m_1n_1+n_1^2p'(\xi)\Big)q'(\xi)-2c\delta q(\xi)a_1\left(m_1-p(\xi)n_1\right)\Big(m_1^2n_2+3m_1n_1\left(m_2-p(\xi)n_2\right)$$

$$-n_1^2\left(p(\xi)m_2+4n_2p'(\xi)\right)\Big)q'(\xi)-4c\delta n_1\left(m_1-p(\xi)n_1\right)^2\left(a_2n_1-a_1n_2\right)q'(\xi)^2$$

$$-2c\delta q(\xi)^2\Big(2a_2\left(-m_1^2+p(\xi)m_1n_1+n_1^2p'(\xi)\right)^2-a_1\Big(m_1^3\left(2m_2-p(\xi)n_2\right)+m_1^2n_1$$

$$\times\left(-3p(\xi)m_2+p(\xi)^2n_2-3n_2p'(\xi)\right)+n_1^3\left(p(\xi)m_2p'(\xi)+2n_2p'(\xi)^2+m_2p''(\xi)\right)$$

$$+m_1n_1^2-n_2p''(\xi)\Big)\Big)\Big)+2q(\xi)a_1\left(m_2n_1-m_1n_2\right)\Big(\theta q(\xi)\left(-m_1^2+p(\xi)m_1n_1+n_1^2p'(\xi)\right)$$

$$+n_1\Big(m_1-p(\xi)\Big(p(\xi)^2m_2-m_2p'(\xi)+3p(\xi)n_2p'(\xi)n_1\Big)\left(\theta q'(\xi)+c\delta q''(\xi)\right)\Big)=0$$

$\left(\dfrac{m_1G+n_1G'}{m_2G+n_2G'}\right)^1$:

$$2\Big(2c\delta\left(m_1-p(\xi)n_1\right)\Big(a_1n_2\left(-2m_2n_1-\left(m_1-3p(\xi)n_1\right)n_2\right)+2a_2n_1\Big(2m_1n_2+n_1$$

$$\times\left(m_2-3p(\xi)n_2\right)\Big)\Big)q'(\xi)^2+q(\xi)^2\Big(2a_2\Big(m_1^3\left(6c\delta m_2+\left(\theta-3c\delta p(\xi)\right)n_2\right)$$

$$-m_1^2n_1\Big(\Big(\theta+9c\left(p(\xi)\left(\theta-3c\delta p(\xi)\right)+7c\delta p'(\xi)\right)\Big)+m_1n_1^2\Big(3c\delta p(\xi)^2m_2$$

$$-\left(5c\delta m_2+\theta n_2\right)p'(\xi)+p(\xi)\left(\theta m_2+9c\delta n_2p'(\xi)\right)-c\delta n_2p''(\xi)\Big)$$

$$+n_1^3\Big(6c\delta n_2p'(\xi)^2+m_2\Big(\left(\theta+3c\delta p(\xi)\right)p'(\xi)+p''(\xi)c\delta\Big)\Big)\Big)+a_1$$

$$\times\Big(m_1^2\Big(-6c\delta m_2^2-2\left(\theta-3c\delta p(\xi)\right)m_2n_2+n_2^2\Big(1+c+p(\xi)\left(\theta-c\delta p(\xi)\right)$$

$$+a_0+3c\delta p'(\xi)\Big)\Big)+2m_1n_1\Big(\left(\theta+3c\delta p(\xi)\right)m_2^2-m_2n_2\Big(1+c+a_0+2c\delta$$

$$\times\left(p(\xi)^2-2p'(\xi)\right)\Big)+n_2^2\Big(\left(-3c\delta p(\xi)+\theta\right)p'(\xi)+c\delta p''(\xi)\Big)\Big)+n_1^2$$

$$\times\left(a_0+m_2^2\left(1+c+c\delta p'(\xi)\right)-2m_2n_2\left(\left(\theta+3c\delta p(\xi)\right)p'(\xi)+c\delta p''(\xi)\right)\right)\Big)\Big)$$

$$+q(\xi)\Big(-2a_2\Big(\Big(\left(m_1-p(\xi)n_1\right)\left(-m_2n_1\left(-9c\delta m_1+\left(\theta+3c\delta p(\xi)\right)n_1\right)+m_1\right.$$

$$\left.\times\left(3c\delta m_1+\left(\theta-9c\delta p(\xi)\right)n_1\right)n_2\right)-2c\delta n_1^2\left(5m_1n_2+n_1\left(m_2-6p(\xi)n_2\right)\right)p'(\xi)\Big)$$

$$\times q'(\xi)+c\delta n_1\left(m_1-p(\xi)n_1\right)\left(-m_2n_1+m_1n_2\right)q''(\xi)\Big)+a_1\Big(m_1^2n_2$$

$$\times\left(\left(6c\delta m_2+\left(\theta-4c\delta p(\xi)\right)n_2\right)q'(\xi)+c\delta n_2q''(\xi)\right)+2m_1n_1\Big(\Big(3c\delta m_2^2$$

$$-8c\delta p(\xi)m_2n_2-n_2^2\left(p(\xi)\left(\theta-3c\delta p(\xi)\right)+4c\delta p'(\xi)\right)\Big)q'(\xi)$$

$$-c\delta p(\xi)q''(\xi)n_2^2\Big)+n_1^2\Big(\Big(m_2\left(-\left(\theta+4c\delta p(\xi)\right)m_2+2p(\xi)\left(\theta+3c\delta p(\xi)\right)n_2\right)$$

$$-4c\delta n_2\left(-3p(\xi)n_2+m_2\right)p'(\xi)\Big)q'(\xi)-c\delta m_2\left(-p(\xi)n_2+m_2\right)q''(\xi)\Big)\Big)\Big)=0$$

$$\left(\frac{m_1G+n_1G'}{m_2G+n_2G'}\right)^2:$$

$$\Big(4c\delta\Big(-a_2\Big(m_2^2n_1^2+2m_2n_1\left(4m_1-5p(\xi)n_1\right)n_2+\left(3m_1^2-14p(\xi)m_1n_1+12p(\xi)^2n_1^2\right)n_2^2\Big)$$

$$+a_1n_2\left(m_2-p(\xi)n_2\right)\left(2m_1n_2+n_1\left(m_2-3p(\xi)n_2\right)\right)\Big)q'(\xi)^2+q(\xi)^2\Big(2a_2\Big(m_1^2\Big(-cm_2^2 24\delta$$

$$-4\left(-6c\delta p(\xi)+\theta\right)m_2n_2+n_2^2\left(1+c+2p(\xi)\left(\theta-2c\delta p(\xi)\right)+a_0+10p'(\xi)c\delta\right)\Big)$$

$$+2m_1n_1\Big(2\left(\theta+6c\delta p(\xi)\right)m_2^2-m_2n_2\left(1+c+a_0+8c\delta\left(p(\xi)^2-2p'(\xi)\right)\right)+2n_2^2$$

$$\times\left(\left(\theta-6c\delta p(\xi)\right)p'(\xi)+p''(\xi)c\delta\right)\Big)+n_1^2\Big(-24c\delta n_2^2p'(\xi)^2+m_2^2\Big(1+c-2p(\xi)$$

$$\times\left(2c\delta p(\xi)+\theta\right)+a_0+6c\delta p'(\xi)\Big)-4m_2n_2\left(\left(\theta+6c\delta p(\xi)\right)p'(\xi)+c\delta p''(\xi)\right)\Big)\Big)$$

$$+a_1\Big(a_1m_1^2n_2^2+2m_1\Big(6c\delta m_2^3+\left(\theta-9c\delta p(\xi)\right)m_2^2n_2-m_2n_2\Big(a_1n_1+n_2\Big(p(\xi)\left(\theta-3c\delta p(\xi)\right)$$

$$+7c\delta p'(\xi)\Big)\Big)-n_2^3\left(\left(\theta-3c\delta p(\xi)\right)p'(\xi)+c\delta p''(\xi)\right)\Big)+n_1\Big(-2\left(\theta+3c\delta p(\xi)\right)m_2^3$$

$$+12c\delta n_2^3p'(\xi)^2+m_2^2\left(a_1n_1+2n_2(p(\xi)\left(\theta+3c\delta p(\xi)\right)-5c\delta p'(\xi)\right)\Big)+2m_2n_2^2$$

$$\times\left(\left(\theta+9c\delta p(\xi)\right)p'(\xi)+c\delta p''(\xi)\right)\Big)\Big)\Big)+2q(\xi)\Big(a_1\Big(\Big(-\left(m_2-p(\xi)n_2\right)\Big(3c\delta m_2^2n_1$$

$$+m_2\left(9c\delta m_1-\left(\theta+9c\delta p(\xi)\right)n_1\right)n_2+\left(\theta-3c\delta p(\xi)\right)m_1n_2^2\Big)+4c\delta n_2^2\Big(2m_2n_1$$

$$+\left(m_1-3p(\xi)n_1\right)n_2\Big)p'(\xi)\Big)q'(\xi)+c\delta n_2\left(m_2-p(\xi)n_2\right)\left(m_2n_1-m_1n_2\right)q''(\xi)\Big)$$

$$+2a_2\Big(m_1^2n_2\left(\left(12c\delta m_2+\left(\theta-8c\delta p(\xi)\right)n_2\right)q'(\xi)+c\delta n_2q''(\xi)\right)$$

$$+2m_1n_1\Big(\Big(6c\delta m_2^2-16c\delta p(\xi)m_2n_2-n_2^2\left(p(\xi)\left(\theta-6c\delta p(\xi)\right)+7c\delta p'(\xi)\right)\Big)q'(\xi)$$

$$-c\delta p(\xi)n_2^2q''(\xi)\Big)+n_1^2\Big(\Big(m_2\left(-\left(\theta+8c\delta p(\xi)\right)m_2+2p(\xi)\left(\theta+6p(\xi)c\delta\right)n_2\right)$$

$$+2c\delta n_2\left(-5m_2+12p(\xi)n_2\right)p'(\xi)\Big)q'(\xi)-c\delta m_2\left(m_2-2p(\xi)n_2\right)q''(\xi)\Big)\Big)\Big)\Big)=0$$

$$\left(\frac{m_1G+n_1G'}{m_2G+n_2G'}\right)^3:$$

$$\left(4c\delta n_2\left(m_2-p(\xi)n_2\right)\left(a_1n_2\left(-m_2+p(\xi)n_2\right)+2a_2\left(2m_2n_1+\left(3m_1-5p(\xi)n_1\right)n_2\right)\right)q'(\xi)^2\right.$$

$$+2q(\xi)^2(-2c\delta a_1\left(-m_2^2+p(\xi)m_2n_2+n_2^2p'(\xi)\right)^2+a_2\left(a_1m_1^2n_2^2+2m_1\left(10c\delta m_2^3\right.\right.$$

$$+\left(-15c\delta p(\xi)+\theta\right)m_2^2n_2-m_2n_2\left(a_1n_1+n_2\left(p(\xi)\left(\theta-5c\delta p(\xi)\right)+11c\delta p'(\xi)\right)\right)$$

$$-n_2^3\left(\left(\theta-5c\delta p(\xi)\right)p'(\xi)+c\delta p''(\xi)\right)\Big)+n_1\left(-2\left(\theta+5c\delta p(\xi)\right)m_2^3+20c\delta n_2^3p'(\xi)^2\right.$$

$$+m_2^2\left(a_1n_1+2n_2\left(p(\xi)\left(\theta+5p(\xi)c\delta\right)-9c\delta p'(\xi)\right)\right)+2m_2n_2^2\left(\left(\theta+15c\delta p(\xi)\right)p'(\xi)\right.$$

$$\left.\left.\left.+c\delta p''(\xi)\right)\right)\right)\Big)+4q(\xi)\left(2c\delta a_1n_2\left(p(\xi)n_2-m_2\right)\left(-m_2^2+p(\xi)m_2n_2+n_2^2p'(\xi)\right)q'(\xi)\right.$$

$$+a_2\Big(\Big(-\left(m_2-p(\xi)n_2\right)\left(5c\delta m_2^2n_1+m_2\left(15c\delta m_1-\left(\theta+15c\delta p(\xi)\right)n_1\right)n_2\right.$$

$$\left.+\left(-5p(\xi)c\delta+\theta\right)m_1n_2^2\right)+2c\delta n_2^2\left(7m_2n_1+\left(3m_1-10p(\xi)n_1\right)n_2\right)p'(\xi)\Big)q'(\xi)$$

$$\left.\left.+c\delta n_2\left(m_2-n_2p(\xi)\right)\left(m_2n_1-m_1n_2\right)q''(\xi)\right)\right)\Big)=0$$

$\left(\dfrac{m_1G+n_1G'}{m_2G+n_2G'}\right)^4$:

$$q(\xi)^2a_2\left(m_2n_1-m_1n_2\right)^2-12c\delta\Big(q(\xi)\left(m_2^2-p(\xi)m_2n_2-n_2^2p'(\xi)\right)$$

$$+n_2\left(-m_2+n_2p(\xi)\right)q'(\xi)\Big)^2=0$$ 。

用 Mathematica 软件对以上代数方程组进行求解，我们得到了以下五组解：

1) $a_2=\dfrac{3\left(10c\delta m_2+\left(\theta-10c\delta C_3\right)n_2\right)^2\left(-10c\delta m_2+\left(\theta+10c\delta C_3\right)n_2\right)^2}{2500c^3\delta^3\left(m_2n_1-m_1n_2\right)^2}$,

$$a_1=\frac{3\left(10c\delta m_1-\left(\theta+10c\delta C_3\right)n_1\right)\left(10c\delta m_2+\left(\theta-10c\delta C_3\right)n_2\right)^2\left(-10c\delta m_2+\left(\theta+10c\delta C_3\right)n_2\right)}{1250c^3\delta^3\left(m_2n_1-m_1n_2\right)^2},$$

$$\begin{aligned}a_0=&\frac{1}{2500c^3\delta^3\left(m_2n_1-m_1n_2\right)^2}\Big(3\theta^3n_1n_2\left(20c\delta m_2n_1+\left(-20c\delta m_1+\theta n_1\right)n_2\right)\\&-300c^2\delta^2\theta^2\left(m_2^2n_1^2-2C_3m_2n_1^2n_2+\left(m_1^2-2C_3m_1n_1+2C_3^2n_1^2\right)n_2^2\right)\\&+500c^3\delta^3\left(m_2n_1-m_1n_2\right)\Big(-n_1\left(\left(5+5c-12\theta C_3\right)m_2+12\theta C_3^2n_2\right)\\&+m_1\left(-12\theta m_2+\left(5+5c+12\theta C_3\right)n_2\right)\Big)\Big),\end{aligned}$$

$$M=\frac{18\theta^4}{625c^2\delta^2}-\frac{c^2}{2}-c-\frac{1}{2},\quad p(\xi)=C_3+\frac{\theta}{10c\delta}\tanh\left(\frac{\theta\left(\xi+5c\delta C_1\right)}{10c\delta}\right),$$

$$q(\xi)=C_1\mathrm{e}^{C_3\xi+C_2}\operatorname{sech}\left(\frac{\theta\left(\xi+5c\delta C_1\right)}{10c\delta}\right),$$

其中 $C_i\,(i=1,2,3)$ 均为任意常数。将上面求得的 $p(\xi),q(\xi)$ 代入(5)式，有

$$G(\xi)=\frac{\theta\,\mathrm{e}^{-C_3\xi}}{C_4\theta\cosh\left(\dfrac{\theta\left(\xi+5c\delta C_1\right)}{10c\delta}\right)+10c\delta C_1\mathrm{e}^{C_2}\sinh\left(\dfrac{\theta\left(\xi+5c\delta C_1\right)}{10c\delta}\right)},$$

其中 C_4 均为任意常数。再由(7)式，我们得到了 RLW-Burgers 方程的解为

$$u(\xi)=\frac{3\left(10c\delta m_2+(\theta-10c\delta C_3)n_2\right)^2\left(-10c\delta m_2+(\theta+10c\delta C_3)n_2\right)^2}{2500c^3\delta^3(m_2n_1-m_1n_2)^2}$$

$$\cdot\frac{\left(10c\delta\theta\left(C_1n_1\mathrm{e}^{C_2}+C_3C_4n_1-C_4m_1\right)+\left(C_4\theta^2n_1-100C_1c^2\delta^2\mathrm{e}^{C_2}(m_1-C_3n_1)\right)\tanh\left(\frac{\theta(\xi+5c\delta C_1)}{10c\delta}\right)\right)^2}{\left(10c\delta\theta\left(C_1n_2\mathrm{e}^{C_2}+C_3C_4n_2-C_4m_2\right)+\left(C_4\theta^2n_2-100C_1c^2\delta^2\mathrm{e}^{C_2}(m_2-C_3n_2)\right)\tanh\left(\frac{\theta(\xi+5c\delta C_1)}{10c\delta}\right)\right)^2}$$

$$+\frac{3\left(10c\delta m_1-(\theta+10c\delta C_3)n_1\right)\left(10c\delta m_2+(\theta-10c\delta C_3)n_2\right)^2\left(-10c\delta m_2+(\theta+10c\delta C_3)n_2\right)}{1250c^3\delta^3(m_2n_1-m_1n_2)^2}$$

$$\cdot\frac{10c\delta\theta\left(C_1n_1\mathrm{e}^{C_2}+C_3C_4n_1-C_4m_1\right)+\left(C_4\theta^2n_1-100C_1c^2\delta^2\mathrm{e}^{C_2}(m_1-C_3n_1)\right)\tanh\left(\frac{\theta(\xi+5c\delta C_1)}{10c\delta}\right)}{10c\delta\theta\left(C_1n_2\mathrm{e}^{C_2}+C_3C_4n_2-C_4m_2\right)+\left(C_4\theta^2n_2-100C_1c^2\delta^2\mathrm{e}^{C_2}(m_2-C_3n_2)\right)\tanh\left(\frac{\theta(\xi+5c\delta C_1)}{10c\delta}\right)}$$

$$+\frac{1}{2500c^3\delta^3(m_2n_1-m_1n_2)^2}\Big(3\theta^3n_1n_2\left(20c\delta m_2n_1+(-20c\delta m_1+\theta n_1)n_2\right)$$

$$-300c^2\delta^2\theta^2\left(m_2^2n_1^2-2C_3m_2n_1^2n_2+\left(m_1^2-2C_3m_1n_1+2C_3^2n_1^2\right)n_2^2\right)$$

$$+500c^3\delta^3(m_2n_1-m_1n_2)\Big(-n_1\left((5+5c-12\theta C_3)m_2+12\theta C_3^2n_2\right)$$

$$+m_1\left(-12\theta m_2+(5+5c+12\theta C_3)n_2\right)\Big)\Big)$$

这里的 $\xi=x-ct$。

2) $a_2=\frac{245\theta n_2(m_2n_1-m_1n_2)}{54n_1^3}$， $a_1=\frac{70\theta(m_2n_1-m_1n_2)}{27n_1^2}$， $a_0=\frac{10\theta}{27}\left(\frac{m_1}{n_1}-\frac{m_2}{n_2}\right)-1$，

$$c=\frac{44}{135}\theta\left(\frac{m_1}{n_1}-\frac{m_2}{n_2}\right),\quad \delta=-\frac{243n_1^2n_2^2}{352(m_2n_1-m_1n_2)^2},$$

$$M=\frac{37760\theta^2(m_2n_1-m_1n_2)^2+2376\theta n_1n_2(m_2n_1-m_1n_2)-3645n_1^2n_2^2}{7290n_1^2n_2^2},$$

$$p(\xi)=\frac{(-2m_2n_1+5m_1n_2)\mathrm{e}^{\frac{8m_1}{9n_1}\xi+24C_1m_2n_1^2n_2}+(2m_2n_1+7m_1n_2)\mathrm{e}^{\frac{8m_2}{9n_2}\xi+24C_1m_1n_1n_2^2}}{3n_1n_2\mathrm{e}^{\frac{8m_1}{9n_1}\xi+24C_1m_2n_1^2n_2}+9n_1n_2\mathrm{e}^{\frac{8m_2}{9n_2}\xi+24C_1m_1n_1n_2^2}},$$

$$q(\xi)=\frac{C_2\mathrm{e}^{\left(\frac{5m_1}{3n_1}+\frac{2m_2}{9n_2}\right)\xi}}{\mathrm{e}^{\frac{8m_1}{9n_1}\xi+24C_1m_2n_1^2n_2}+3\mathrm{e}^{\frac{8m_2}{9n_2}\xi+24C_1m_1n_1n_2^2}},$$

其中 $C_i(i=1,2,3)$ 均为任意常数。将上面求得的 $p(\xi),q(\xi)$ 代入(5)式，有

$$G(\xi)=\frac{8(m_2n_1-m_1n_2)\mathrm{e}^{\frac{1}{9}\left(-\frac{7m_1}{n_1}+\frac{6m_2}{n_2}\right)\xi}}{9C_2n_1n_2\mathrm{e}^{\frac{8m_2}{9n_2}\left(\xi-27C_1n_1^2n_2^2\right)}+8C_3(m_2n_1-m_1n_2)\left(\mathrm{e}^{\frac{8m_1}{9n_1}\xi+24C_1m_2n_1^2n_2}+3\mathrm{e}^{\frac{8m_2}{9n_2}\xi+24C_1m_1n_1n_2^2}\right)},$$

再由(7)式，我们得到了 RLW-Burgers 方程的解为

$$u(\xi)=\frac{490\theta(m_2n_1-m_1n_2)\left(3C_2n_1n_2\mathrm{e}^{\frac{8m_2}{9n_2}\xi}-8C_3(m_2n_1-m_1n_2)\left(\mathrm{e}^{\frac{8m_1}{9n_1}\xi+48C_1m_2n_1^2n_2}-\mathrm{e}^{\frac{8m_2}{9n_2}\xi+24C_1n_1n_2(m_2n_1+m_1n_2)}\right)\right)^2}{27n_1n_2\left(21C_2n_1n_2\mathrm{e}^{\frac{8m_2}{9n_2}\xi}+8C_3(m_2n_1-m_1n_2)\left(5\mathrm{e}^{\frac{8m_1}{9n_1}\xi+48C_1m_2n_1^2n_2}+7\mathrm{e}^{\frac{8m_2}{9n_2}\xi+24C_1n_1n_2(m_2n_1+m_1n_2)}\right)\right)^2}$$

$$-\frac{140\theta(m_2n_1-m_1n_2)\left(3C_2n_1n_2\mathrm{e}^{\frac{8m_2}{9n_2}\xi}-8C_3(m_2n_1-m_1n_2)\left(\mathrm{e}^{\frac{8m_1}{9n_1}\xi+48C_1m_2n_1^2n_2}-\mathrm{e}^{\frac{8m_2}{9n_2}\xi+24C_1n_1n_2(m_2n_1+m_1n_2)}\right)\right)}{27n_1n_2\left(21C_2n_1n_2\mathrm{e}^{\frac{8m_2}{9n_2}\xi}+8C_3(m_2n_1-m_1n_2)\left(5\mathrm{e}^{\frac{8m_1}{9n_1}\xi+48C_1m_2n_1^2n_2}+7\mathrm{e}^{\frac{8m_2}{9n_2}\xi+24C_1n_1n_2(m_2n_1+m_1n_2)}\right)\right)},$$

$$+\frac{10\theta}{27}\left(\frac{m_1}{n_1}-\frac{m_2}{n_2}\right)-1$$

这里的 $\xi=x-\frac{44}{135}\theta\left(\frac{m_1}{n_1}-\frac{m_2}{n_2}\right)t$。

3) $a_2=\frac{12(m_2-C_1n_2)^2\left(5c\delta(m_2-C_1n_2)\mp\theta n_2\right)^2}{25c\delta(m_2n_1-m_1n_2)^2}$,

$$a_1=\frac{6}{125c^2\delta^2(m_2n_1-m_1n_2)^2}\left(10c\delta(m_1-C_1n_1)-(1\pm1)\theta n_1\right)(m_2-C_1n_2)\times\left(5c\delta(m_2-C_1n_2)\mp\theta n_2\right)\left(10c\delta(m_2-C_1n_2)+(1\mp1)\theta n_2\right),$$

$$\begin{aligned}a_0=&\frac{1}{125c^2\delta^2(m_2n_1-m_1n_2)^2}\Big(n_1^2\Big(5c\delta\Big(25c^2\delta\big(12C_1^2\delta-1\big)\pm6\theta^2+5c\delta\big(-5+12C_1\theta(1\pm1)\big)\Big)m_2^2\\&-12(5c\delta C_1+\theta)^2\big(10c\delta C_1+(-1\pm1)\theta\big)m_2n_2+60c\delta C_1^2(5c\delta C_1\pm\theta)^2n_2^2\Big)\\&-m_1n_1\Big(300c^2\delta^2\big(10c\delta C_1+(1\pm1)\theta\big)m_2^2-10c\delta\Big(25c^2\delta\big(1+24C_1^2\delta\big)+6\theta^2+5c\delta(5\pm24C_1\theta)\Big)m_2n_2\\&+3\big(10c\delta C_1+(-1\pm1)\theta\big)^2\big(10c\delta C_1+(1\pm1)\theta\big)n_2^2\Big)+5c\delta m_1^2\Big(300c^2\delta^2m_2^2\\&-60c\delta\big(10c\delta C_1+(-1\pm1)\theta\big)m_2n_2+\Big(25c^2\delta\big(12\delta C_1^2-1\big)+5c\delta\big(-5+12\theta C_1(-1\pm1)\big)\mp6\theta^2\Big)n_2^2\Big)\Big)\end{aligned},$$

$$M=\frac{18\theta^4}{625c^2\delta^2}-\frac{c^2}{2}-c-\frac{1}{2},\quad p(\xi)=C_1\pm\frac{\theta}{5c\delta},\quad q(\xi)=C_2\mathrm{e}^{C_1\xi},$$

其中 $C_i\,(i=1,2,3)$ 均为任意常数。将上面求得的 $p(\xi),q(\xi)$ 代入(5)式，有

$$G(\xi)=\frac{\theta\,\mathrm{e}^{-C_1\xi}}{C_3\theta\,\mathrm{e}^{\pm\frac{\theta}{5c\delta}\xi}\mp5c\delta C_2},$$

再由(7)式，我们得到了 RLW-Burgers 方程的解为

$$u(\xi)=\frac{12(m_2-C_1n_2)^2}{25c\delta(m_2n_1-m_1n_2)^2}\frac{\left(C_3\theta\mathrm{e}^{\pm\frac{\theta}{5c\delta}\xi}\left(-5c\delta(m_1-C_1n_1)\pm\theta n_1\right)\pm25c^2\delta^2C_2(m_1-C_1n_1)\right)^2}{\left(C_3\theta\mathrm{e}^{\pm\frac{\theta}{5c\delta}\xi}\left(-5c\delta(m_2-C_1n_2)\pm\theta n_2\right)\pm25c^2\delta^2C_2(m_2-C_1n_2)\right)^2}$$

$$
\begin{aligned}
&\times\left(5c\delta\left(m_2-C_1n_2\right)\mp\theta n_2\right)^2+\frac{6}{125c^2\delta^2\left(m_2n_1-m_1n_2\right)^2}\left(10c\delta\left(m_1-C_1n_1\right)-\left(1\pm1\right)\theta n_1\right)\\
&\times\left(m_2-C_1n_2\right)\left(5c\delta\left(m_2-C_1n_2\right)\mp\theta n_2\right)\left(10c\delta\left(m_2-C_1n_2\right)+\left(1\mp1\right)\theta n_2\right)\\
&\times\frac{\left(C_3\theta\mathrm{e}^{\pm\frac{\theta}{5c\delta}\xi}\left(-5c\delta\left(m_1-C_1n_1\right)\pm\theta n_1\right)\pm25c^2\delta^2C_2\left(m_1-C_1n_1\right)\right)}{\left(C_3\theta\mathrm{e}^{\pm\frac{\theta}{5c\delta}\xi}\left(-5c\delta\left(m_2-C_1n_2\right)\pm\theta n_2\right)\pm25c^2\delta^2C_2\left(m_2-C_1n_2\right)\right)}\\
&+\frac{1}{125c^2\delta^2\left(m_2n_1-m_1n_2\right)^2}\Big(n_1^2\Big(5c\delta\Big(25c^2\delta\left(12C_1^2\delta-1\right)\pm6\theta^2+5c\delta\left(-5+12C_1\theta\left(1\pm1\right)\right)\Big)m_2^2\\
&-12\left(5c\delta C_1+\theta\right)^2\left(10c\delta C_1+\left(-1\pm1\right)\theta\right)m_2n_2+60c\delta C_1^2\left(5c\delta C_1\pm\theta\right)^2n_2^2\Big)\\
&-m_1n_1\Big(300c^2\delta^2\left(10c\delta C_1+\left(1\pm1\right)\theta\right)m_2^2-10c\delta\Big(25c^2\delta\left(1+24C_1^2\delta\right)+6\theta^2\\
&+5c\delta\left(5\pm24C_1\theta\right)\Big)m_2n_2+3\left(10c\delta C_1+\left(-1\pm1\right)\theta\right)^2\left(10c\delta C_1+\left(1\pm1\right)\theta\right)n_2^2\Big)\\
&+5c\delta m_1^2\Big(300c^2\delta^2m_2^2-60c\delta\left(10c\delta C_1+\left(-1\pm1\right)\theta\right)m_2n_2\\
&+\left(25c^2\delta\left(12\delta C_1^2-1\right)+5c\delta\left(-5+12\theta C_1\left(-1\pm1\right)\right)\mp6\theta^2\right)n_2^2\Big)\Big)
\end{aligned}
$$

这里的 $\xi=x-ct$。

4) $a_2=-\dfrac{2\left(2\theta\left(m_2n_1-m_1n_2\right)-n_1n_2\right)^2}{5n_1^4}$， $a_1=0$， $a_0=\dfrac{2\left(\theta^2-\delta\right)}{15\delta}$， $c=-\dfrac{2\theta^2}{15\delta}$， $M=0$，

$$
p(\xi)=\frac{\left(2\theta m_1+n_1\right)\mathrm{e}^{\frac{\xi}{2\theta}}-m_1\,\mathrm{e}^{C_1n_1^2}}{2n_1\theta\,\mathrm{e}^{\frac{\xi}{2\theta}}-n_1\,\mathrm{e}^{C_1n_1^2}},\quad q(\xi)=\frac{C_2\,\mathrm{e}^{\left(\frac{1}{2\theta}+\frac{m_1}{n_1}\right)\xi}}{-2\theta\,\mathrm{e}^{\frac{\xi}{2\theta}}+\mathrm{e}^{C_1n_1^2}},
$$

其中 $C_i\left(i=1,2,3\right)$ 均为任意常数。将上面求得的 $p(\xi),q(\xi)$ 代入(5)式，有

$$
G(\xi)=\frac{\mathrm{e}^{-\frac{m_1}{n_1}\xi}}{C_2+C_3\,\mathrm{e}^{C_1n_1^2}-2\theta C_3\,\mathrm{e}^{\frac{\xi}{2\theta}}},
$$

再由(7)式，我们得到了 RLW-Burgers 方程的解为

$$
u(\xi)=-\frac{2C_3^2\left(2\theta\left(m_2n_1-m_1n_2\right)-n_1n_2\right)^2\mathrm{e}^{\frac{\xi}{\theta}}}{5\left(\left(C_2+C_3\,\mathrm{e}^{C_1n_1^2}\right)\left(m_2n_1-m_1n_2\right)-C_3\left(2\theta\left(m_2n_1-m_1n_2\right)-n_1n_2\right)\mathrm{e}^{\frac{\xi}{2\theta}}\right)^2}+\frac{2\left(\theta^2-\delta\right)}{15\delta},
$$

这里的 $\xi=x+\dfrac{2\theta^2}{15\delta}t$。

5) $a_2=\dfrac{2\left(2\theta\left(m_2n_1-m_1n_2\right)+n_1n_2\right)^2}{5n_1^4}$， $a_1=0$， $a_0=-\dfrac{2\left(\theta^2+14\delta\right)}{15\delta}$， $c=\dfrac{2\theta^2}{15\delta}$， $M=0$，

$$
p(\xi)=\frac{m_1\,\mathrm{e}^{\frac{\xi}{2\theta}}-\left(2\theta m_1-n_1\right)\mathrm{e}^{C_1n_1^2}}{n_1\,\mathrm{e}^{\frac{\xi}{2\theta}}-2n_1\theta\,\mathrm{e}^{C_1n_1^2}},\quad q(\xi)=\frac{C_2\,\mathrm{e}^{\frac{m_1}{n_1}\xi}}{\mathrm{e}^{\frac{\xi}{2\theta}}-2\theta\,\mathrm{e}^{C_1n_1^2}},
$$

其中 $C_i\left(i=1,2,3\right)$ 均为任意常数。将上面求得的 $p\left(\xi\right),q\left(\xi\right)$ 代入(5)式，有

$$G\left(\xi\right)=\frac{\mathrm{e}^{\frac{1}{2}\left(\frac{1}{\theta}-\frac{2m_1}{n_1}\right)\xi}}{C_3\,\mathrm{e}^{\frac{\xi}{2\theta}}-2\theta C_3\,\mathrm{e}^{C_1n_1^2}-2\theta C_2},$$

再由(7)式，我们得到了 RLW-Burgers 方程的解为

$$u\left(\xi\right)=\frac{2\left(2\theta\left(m_2n_1-m_1n_2\right)+n_1n_2\right)^2\left(C_2+C_3\,\mathrm{e}^{C_1n_1^2}\right)^2}{5\left(\left(2\theta\left(m_2n_1-m_1n_2\right)+n_1n_2\right)\left(C_2+C_3\,\mathrm{e}^{C_1n_1^2}\right)-C_3\left(m_2n_1-m_1n_2\right)\mathrm{e}^{\frac{\xi}{2\theta}}\right)^2}-\frac{2\left(\theta^2+14\delta\right)}{15\delta},$$

这里的 $\xi=x-\dfrac{2\theta^2}{15\delta}t$。

4. 总结

本文构造了一类满足变系数 Bernoulli 方程的，且含有自由参数 $m_i,n_i\left(i=1,2\right)$ 的 $\left(\dfrac{m_1G+n_1G'}{m_2G+n_2G'}\right)$ 展开法。用此方法对 RLW-Burgers 方程进行了求解，得到了该方程的多个新的显式行波解。事实证明，该方法不仅可以求解非线性偏微分方程，而且可以得到方程更为丰富的精确解。

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