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关于3D不可压Navier-Stokes方程H^1正则性的注记

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摘 要

这篇论文主要研究了3D不可压Navier-Stokes方程解的H^1正则性. 首先, 本论文给出并详细证明了3D不可压Navier-Stokes方程解的局部适定性引理. 其次, 应用上述解的局部适定性引理, 第一, 可以严格证明解在小初始数据情形时的全局正则性. 第二, 对于最大可能的所有U_0和最大可能的所有F, 证明出3D不可压Navier-Stokes方程解的H^1 正则性. 本论文强调不仅对最大可能U_0和某一固定F这一情形, 解具有H^1正则性, 而且对最大可能U_0和最大可能的F这一情形, 解同样具有H^1正则性。

关键词

H^1正则性，局部适定性引理，3D不可压Navier-Stokes方程

A Note on the H^1 Regularity of 3D Incompressible Navier-Stokes Equation

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Abstract

In this paper, we mainly study the H^1 regularity of the solution of 3D incompressible Navier-Stokes equations. Firstly, the local well-fit lemma for solutions of 3D incompressible Navier-Stokes equations is given and proved in detail. Secondly, by applying the local well-fit lemma of the solution mentioned above, firstly, the global regularity of the solution in the case of small initial data can be proved strictly. Second, the H^1 regularity of the solution of 3D incompressible Navier-Stokes equations is proved for all possible U_0 and all possible F. In this paper, it is emphasized that the solution is H^1 regular not only for the maximum possible U_0 and a fixed F, but also for the maximum possible U_0 and the maximum possible F.

Keywords

H^1 **Regularity, The Local Well-Fit Lemma, 3D Incompressible Navier-Stokes Equations**

1. 引言

Navier-Stokes方程是求解流体流动问题的基本模型，该方程被广泛用于航空科学、气象学、石油工业、等离子体物理学等的理论研究. 其中，三维空间中的N-S方程组光滑解的存在性问题被设定为七个千禧年大奖难题之一：即对所有有U_0和F，方程是否具有一个全局正则解，因此三维N-S方程解的正则性问题是偏微分方程理论中的一个突出问题. 与3D不可压Navier-Stokes方程不同，Ladyzhenskaya [1]和Temam [2] [3] [4]都得出了2D不可压Navier-Stokes 方程对所有U_0和F，方程存在一个全局正则解. 对于3D 不可压Navier-Stokes方程，当初始数据U_0和F充分小时，方程存在一个全局正则解，参见Temam [4].

对最大可能$U_0(A_\epsilon^{\frac{1}{2}}v_0||^2 \leq \eta_1^{-2}, ||A_\epsilon^{\frac{1}{2}}\omega_0||^2 \leq \epsilon^p\eta_3^{-2})$和某一固定$F$这一情形，3D不可压Navier-Stokes 方程在窄区域上的解具有H^1正则性，具体我们可以参考文献 [1]. 本论文受该文献启发，同样在窄区域上考虑3D不可压Navier-Stokes方程，我们可以得到对于最大可能的$U_0(A_\epsilon^{\frac{1}{2}}v_0||^2 \leq$

$\eta_1^{-2}, ||A_\epsilon^{\frac{1}{2}}\omega_0||^2 \le \epsilon^p\eta_3^{-2}$)和最大可能的$F(||MP_\epsilon f||_\infty^2 \le \eta_2^{-2}, ||(I-M)P_\epsilon f||_\infty^2 \le \epsilon^{\frac{r}{3}}\eta_4^{-2})$，解具有$H^1$正则性.

本文第一节给出相关的预备知识, 具体内容可以参考文献 [1] [2] [5] [6]. 第二节, 给出必要的假设条件和我们在证明过程遇到的重要不等式. 第三节, 证明3D 不可压Navier-Stokes 方程解的局部适定性引理, 局部适定性引理在本文中起着非常重要的作用. 其次, 我们严格证明了解在小初始数据情形时的全局正则性, 最后, 结合引理(4.2)和引理(4.3), 我们得出对于最大可能的$U_0(A_\epsilon^{\frac{1}{2}}v_0||^2 \le \eta_1^{-2}, ||A_\epsilon^{\frac{1}{2}}\omega_0||^2 \le \epsilon^p\eta_3^{-2})$和最大可能的$F(||MP_\epsilon f||_\infty^2 \le \eta_2^{-2}, ||(I-M)P_\epsilon f||_\infty^2 \le \epsilon^{\frac{r}{3}}\eta_4^{-2})$, 解具有$H^1$ 正则性.

2. 预备知识

本文在一个有界区域$\Omega \subset \mathbb{R}^3$上考虑3D不可压Navier-Stokes方程

$$\begin{cases} U_t - \nu\Delta U + (U\cdot V) + \nabla P = F, \\ \nabla\cdot U = 0, \end{cases} \tag{2.1}$$

其中Ω 是窄区域, 即$\Omega = \Omega_\epsilon = Q_2 \times (0,\epsilon)$, Q_2 是$\mathbb{R}^3$中的一个有界区域, $\epsilon > 0$ 是一个小参数. 特别的, 假设$Q_2 = (0,l_1)\times(0,l_2)$, 其中$l_1, l_2 > 0, \epsilon \le l_2 \le l_1, 0 < \epsilon \le 1$, 并且我们还假设(2.1)的解$U$ 满足下述周期边界条件

$$\begin{cases} U(y + l_ie_i, t) = U(y,t), \quad i = 1,2, \\ U(y + \epsilon y_3) = U(y,t), \end{cases}$$

其中$\{e_1, e_2, e_3\}$ 是$\mathbb{R}^3$ 中的标准正交基. 另外, 假设F 和U_0 满足

$$\int_{\Omega_\epsilon} F dy = \int_{\Omega_\epsilon} U_0 dy = 0.$$

为了约化这个问题, 通过非退化变量替换, 可以将区域变化方程固定的问题转化为区域固定方程变化的问题. 其次, 定义零散度零均值空间, 将约化的方程抽象成零散度零均值空间上的发展方程发展方程.

$$u' + \nu A_\epsilon u + B_\epsilon(u,u) = P_\epsilon f. \tag{2.2}$$

这部分具体内容可以参考文献 [5]

3. 假设条件和重要不等式

首先, 假设

$$\begin{cases} ||A_\epsilon^{\frac{1}{2}}v_0||^2 \le \eta_1^{-2}, \\ ||A_\epsilon^{\frac{1}{2}}w_0||^2 \le \epsilon^p\eta_3^{-2}. \end{cases} \qquad \begin{cases} ||MP_\epsilon f||_\infty^2 \le \eta_2^{-2}, \\ ||(I-M)P_\epsilon f||_\infty^2 \le \epsilon^{\frac{r}{3}}\eta_4^{-2}. \end{cases} \tag{3.1}$$

其次, 下面假设条件记为(H1)

(1) $-1 \le p \le 0, \quad -2 \le r \le 0,$

(2) 当$\epsilon \to 0$ 时，$\epsilon^{\frac{1}{4}}\eta_i^{-1} \to 0, \quad i = 1, 2.$

(3) 当$\epsilon \to 0$ 时，$\epsilon^{\frac{1}{8}}\eta_i^{-1} \to 0, \quad i = 3, 4.$

(4) 对于$0 < \epsilon \le 1, \quad \epsilon^{\frac{1}{4}}Q(\epsilon)$ 是有界的.

其中

$$Q(\epsilon) = |\ln(2C_5^2\nu^{-2}\epsilon^{2+\frac{r}{3}-p}\eta_4^{-2}\eta_3^2)|,$$

且当$0 < \epsilon \le 1$时，$\eta_i(\epsilon)$时有界单调函数,$i = 1, 2, 3, 4.$

下面假设条件记为H(a,b)

(5) 固定$a > 0$, 那么当$\epsilon \to 0^+$, 时，有

$$\begin{cases} \epsilon^{\frac{5}{8}}\eta^{-2}e^{a\eta^{-4}} \to 0, \\ \eta^{-2} \to 0, \end{cases} \tag{3.2}$$

其中

$$\eta^{-2} \overset{def}{=} \max(4\eta_1^{-2} + k_1^2\eta_3^{-4} + k_2^2\epsilon^{2+\frac{r}{3}}\eta_4^{-2}, 1),$$

且对于$0 < \epsilon \le 1, \quad \epsilon^{\frac{5}{8}}e^{2a\eta_2^{-4}}$ 是有界的, (常数k_1和k_2在引理(4.2)有定义)

(6) 固定$b > 0$，那么对于任意的$\lambda, 0 < \lambda < 1$, 存在$\epsilon_4 = \epsilon_4(b, \lambda) > 0$, 使得

$$\eta_2^{-2}e^{b\eta_2^{-4}} \le \lambda(4\eta_1^{-2} + k_1^2\eta_3^{-4}), \quad 0 < \epsilon < \epsilon_4.$$

(7) 当$\epsilon \to 0$，函数$\epsilon^{4+\frac{2r}{3}}\eta_4^{-4}(\ln \eta^{-4} + 1)$ 是有界的.

下面给出重要估计式：

$$\lambda_1||u||^2 \le ||A_\epsilon^{\frac{1}{2}}u||^2, \qquad u \in D(A_\epsilon^{\frac{1}{2}}), \tag{3.3}$$

$$||\omega||^2 \le C_5^2\epsilon^2||A_\epsilon^{\frac{1}{2}}\omega||^2, \qquad \omega \in V_\epsilon^1, M\omega = 0, \tag{3.4}$$

其中C_5不依赖ϵ.

4. 主要结果及其证明

引理 4.1 (解的局部适定性) 设$u_0 \in D(A_\epsilon^{\frac{1}{2}})$ 和$f \in L^\infty(0, T; H_\epsilon)$, 那么存在一个$T_*, 0 < T_* < \infty$, 使得(2.3) 在$(0, T_*)$ 上存在一个唯一解u, 满足$u \in C([0, T_*], V_\epsilon^1) \cap L^2(0, T_*; V_\epsilon^2)$ 和$u_t \in L^2(0, T_*; H_\epsilon)$. 如果$N > 1$ 并且$||A_\epsilon^{\frac{1}{2}}u_0||^2 + ||P_\epsilon f||_\infty^2 \le NR_0^2$, 那么存在$T^N$, 使得$||A_\epsilon^{\frac{1}{2}}u(t)||^2 \le NR_0^2, 0 \le T^N \le T_*$.

证明. (2.2) 式与$A_\epsilon u$作内积, 使用重要估计式 [5]和Young不等式可以得到

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||A_\epsilon^{\frac{1}{2}}u||^2+\nu||A_\epsilon u||^2 &\le |(P_\epsilon f,A_\epsilon u)|+|(B_\epsilon(u,u),A_\epsilon u)|\\ &\le ||P_\epsilon f||||A_\epsilon u||+C_8(\Omega)||A_\epsilon^{\frac{1}{2}}u||^{\frac{3}{2}}||A_\epsilon u||^{\frac{1}{2}}\\ &\le \frac{\nu}{4}||A_\epsilon u||^2+\frac{1}{\nu}||P_\epsilon f||+\frac{27}{4\nu^3}C_8^4||A_\epsilon^{\frac{1}{2}}u||^6+\frac{\nu}{4}||A_\epsilon u||^2,\end{aligned}$$

即有

$$\frac{\mathrm{d}}{\mathrm{d}t}||A_\epsilon^{\frac{1}{2}}u||^2+\nu||A_\epsilon u||^2\le\frac{2}{\nu}||P_\epsilon f||_\infty+\frac{27}{2\nu^3}C_8^4||A_\epsilon^{\frac{1}{2}}u||^6. \tag{4.1}$$

从0到t积分, 得

$$||A_\epsilon^{\frac{1}{2}}u(t)||^2\le||A_\epsilon^{\frac{1}{2}}u_0||^2+\frac{2}{\nu}||P_\epsilon f||_\infty+\frac{27}{2\nu^3}C_8^4\int_0^t||A_\epsilon^{\frac{1}{2}}u||^6ds. \tag{4.2}$$

再应用特殊形式的Gronwall不等式 [7], 得

$$||A_\epsilon^{\frac{1}{2}}u(t)||^2\le(||A_\epsilon^{\frac{1}{2}}u_0||^2+\frac{2}{\nu}||P_\epsilon f||_\infty t)[1-2\int_0^t\frac{27}{2\nu^3}C_8^4||A_\epsilon^{\frac{1}{2}}u(t)||^2+\frac{2}{\nu}||P_\epsilon f||_\infty ds]^{-\frac{1}{2}}.$$

我们令

$$T^{***}=\sup\left\{t\in[0,T]\middle|\quad 2\int_0^t\frac{27}{2\nu^3}C_8^4||A_\epsilon^{\frac{1}{2}}u(t)||^2+\frac{2}{\nu}||P_\epsilon f||_\infty ds<1\right\},$$

因此有

$$||A_\epsilon^{\frac{1}{2}}u(t)||^2\le||A_\epsilon^{\frac{1}{2}}u_0||^2+\frac{2}{\nu}||P_\epsilon f||_\infty t.$$

已知$||A_\epsilon^{\frac{1}{2}}u_0||^2+||P_\epsilon f||_\infty^2\le NR_0^2$，那么存在$T^{N'}>0$，使得

$$||A_\epsilon^{\frac{1}{2}}u_0||^2+\frac{2}{\nu}||P_\epsilon f||_\infty^2T^{N'}\le NR_0^2,$$

最后, 取

$$T^N=\min\{T^{N'},T^{***}\},$$

所以存在T^N, 使得

$$||A_\epsilon^{\frac{1}{2}}u(t)||^2\le NR_0^2,\quad [0,T^N].$$

定理 4.1 *(小数据正则性)*

在任意一个有界区域$\Omega\subset\mathbb{R}^3$上考虑下述问题

$$U^{'}+\nu AU+B(U,U)=PF, \tag{4.3}$$

其中P 表示$L^2(\Omega)$ 到零散度空间的正交投影算子, (4.3) 满足$Direchlet$边界条件, 且F 不依赖t, 如果(4.17)的初始数据充分小, 那么(4.17)有一个全局正则解.

证明. (4.3)式与AU作内积, 使用重要估计式 [5]和Young不等式可以得到

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||A^{\frac{1}{2}}U||^2_{L^2(\Omega)}+\nu||AU||^2_{L^2(\Omega)} &\le |(PF,AU)|+|(B(U,U),AU)|\\ &\le ||PF||_{L^2(\Omega)}||AU||_{L^2(\Omega)}+C_8(\Omega)||A^{\frac{1}{2}}U||^{\frac{3}{2}}_{L^2(\Omega)}||AU||^{\frac{1}{2}}_{L^2(\Omega)}\\ &\le \frac{\nu}{4}||AU||^2_{L^2(\Omega)}+\frac{1}{\nu}||PF||_{L^2(\Omega)}+\frac{27}{4\nu^3}C_8^4||A^{\frac{1}{2}}U||^6_{L^2(\Omega)}+\frac{\nu}{4}||AU||^2_{L^2(\Omega)},\end{aligned}$$

即有

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}||A^{\frac{1}{2}}U||^2_{L^2(\Omega)}+\lambda_1\nu||A^{\frac{1}{2}}U||^2_{L^2(\Omega)} &\le \frac{\mathrm{d}}{\mathrm{d}t}||A^{\frac{1}{2}}U||^2_{L^2(\Omega)}+\nu||AU||^2_{L^2(\Omega)}\\ &\le \frac{2}{\nu}||PF||_{L^2(\Omega)}+\frac{27}{2\nu^3}C_8^4||A^{\frac{1}{2}}U||^6_{L^2(\Omega)}.\end{aligned}$$

令$R_0^2=||A^{\frac{1}{2}}U_0||^2_{L^2(\Omega)}+||PF||^2_{L^2(\Omega)}$, 取定$N>\max(1,\frac{4}{\lambda_1\nu^2})$, 对上述$N>1$, 有$||A^{\frac{1}{2}}U_0||^2_{L^2(\Omega)}+||PF||^2_{L^2(\Omega)}\le NR_0^2$, 利用Lamma(4.1), 存在$T^N$ 使得

$$||A^{\frac{1}{2}}U(t)||^2_{L^2(\Omega)}\le NR_0^2,\quad 0\le t\le T^N. \tag{4.4}$$

不妨假设$[0,T^\infty)$是使得(4.4)式成立的最大时间区间, 下述讨论我们在$[0,T^\infty)$上进行

$$\frac{\mathrm{d}}{\mathrm{d}t}||A^{\frac{1}{2}}U||^2_{L^2(\Omega)}+\lambda_1\nu||A^{\frac{1}{2}}U||^2_{L^2(\Omega)}\le\frac{2}{\nu}||PF||^2_{L^2(\Omega)}+\frac{27}{2\nu^3}C_8^4||A^{\frac{1}{2}}U||^6_{L^2(\Omega)},$$

由于(4.3)的初始数据充分小, 我们可以假设

$$\frac{27}{2\nu^3}C_8^4NR_0^2\le\frac{\lambda_1\nu}{2},$$

得到

$$\frac{\mathrm{d}}{\mathrm{d}t}||A^{\frac{1}{2}}U||^2_{L^2(\Omega)}+\frac{\lambda_1\nu}{2}||A^{\frac{1}{2}}U||^2_{L^2(\Omega)}\le\frac{2}{\nu}||PF||^2_{L^2(\Omega)}. \tag{4.5}$$

应用Gronwall不等式, 得

$$\begin{aligned}||A^{\frac{1}{2}}U||^2_{L^2(\Omega)} &\le e^{-\frac{\lambda_1\nu t}{2}}(||A^{\frac{1}{2}}U_0||^2_{L^2(\Omega)}+\int_0^t\frac{2}{\nu}||PF||^2_{L^2(\Omega)}ds)\\ &\le \max(1,\frac{4}{\lambda_1\nu^2})(||A^{\frac{1}{2}}U_0||^2_{L^2(\Omega)}+||PF||^2_{L^2(\Omega)})\\ &< NR_0^2,\end{aligned} \tag{4.6}$$

最后, 应用反证法来证明$T_{\max}=\infty$.

首先, 否定第一种情形：$T^\infty = T_{\max} < \infty$, 此时, $[0, T^\infty) = [0, T_{\max})$, 因为$T_{\max} < \infty$, 所以

$$\lim_{t \to T_{\max}^-} ||A^{\frac{1}{2}} U(t)||^2_{L^2(\Omega)} = \infty.$$

即当取$M = NR_0^2$ 时, $\exists \delta > 0, \forall t \in (0, T_{\max})$, 只要$T_{\max} - \delta < t < T_{\max}$, 就有

$$||A^{\frac{1}{2}} U(t)||^2 > NR_0^2,$$

即$\exists t_0 \in (T_{\max} - \delta, T_{\max}) = (T^\infty - \delta, T^\infty)$, 使得

$$||A^{\frac{1}{2}} U(t)||^2_{L^2(\Omega)} > NR_0^2,$$

这与$[0, T^\infty)$的定义矛盾.

其次, 否定第二种情形：$T^\infty < T_{\max} < \infty$, 由于$||A^{\frac{1}{2}} U(t)||^2_{L^2(\Omega)}$, 在$[0, T_{\max})$ 上关于t连续, 所以$||A^{\frac{1}{2}} U(t)||^2_{L^2(\Omega)}$ 也在$[0, T^\infty)$上连续并且

$$||A^{\frac{1}{2}} U(t)||^2_{L^2(\Omega)} \leq NR_0^2, \quad [0, T^\infty) .$$

由于$||A^{\frac{1}{2}} U(t)||^2_{L^2(\Omega)}$ 在T^∞ 连续, 所以

$$||A^{\frac{1}{2}} U(t)||^2_{L^2(\Omega)} \leq NR_0^2, \quad [0, T^\infty] .$$

从(4.6) 知

$$||A^{\frac{1}{2}} U(t)||^2_{L^2(\Omega)} < NR_0^2, \quad [0, T^\infty]. \tag{4.7}$$

对(4.7)式利用在T^∞的连续性, 得

$$||A^{\frac{1}{2}} U(t)||^2_{L^2(\Omega)} \leq NR_0^2, \qquad t \in (T^\infty, T^\infty + \delta).$$

这与$[0, T^\infty)$的定义矛盾.

综上所述, $T_{\max} = \infty$, 证毕.

引理 4.2 设条件*H1*和(3.1) 成立, 那么存在$k_1, k_2, \epsilon_1 > 0$, 对于$0 < \epsilon \leq \epsilon_1$, 存在$T_1 = T_1(\epsilon) > 0$, 使得$u(t) \in D(A_\epsilon^{\frac{1}{2}})$, $0 \leq t \leq T_1$ 且成立

$$\begin{cases} ||A_\epsilon^{\frac{1}{2}} v(T_1)||^2 \leq 4\eta_1^{-2} + k_1^2 \eta_3^{-4}, \\ \\ ||A_\epsilon^{\frac{1}{2}} w(T_1)||^2 \leq k_2^2 \epsilon^{2+\frac{r}{3}} \eta_4^{-2}. \end{cases} \tag{4.8}$$

证明. 定义

$$R_0^2=\eta_1^{-2}+\epsilon^p\eta_3^{-2}+\eta_3^{-4}+2\eta_2^{-2}+2\epsilon^{\frac{r}{3}}\eta_4^{-2},$$

因为

$$\begin{aligned}||A_\epsilon^{\frac{1}{2}}u_0||^2+||P_\epsilon f||_\infty^2&=||A_\epsilon^{\frac{1}{2}}v_0||^2+||A_\epsilon^{\frac{1}{2}}\omega_0||^2+||(M+I-M)P_\epsilon f||_\infty^2\\&\le ||A_\epsilon^{\frac{1}{2}}v_0||^2+||A_\epsilon^{\frac{1}{2}}\omega_0||^2+2||MP_\epsilon f||_\infty^2+2||(I-M)P_\epsilon f||_\infty^2\\&\le \eta_1^{-2}+\epsilon^p\eta_3^{-2}+2\eta_2^{-2}+2\epsilon^{\frac{r}{3}}\eta_4^{-2}\\&\le \eta_1^{-2}+\epsilon^p\eta_3^{-2}+2\eta_2^{-2}+2\epsilon^{\frac{r}{3}}\eta_4^{-2}+\eta_3^{-4}\\&=R_0^2,\end{aligned}$$

选取$N=\max\{4,\frac{7}{2}D_2\}+1>1$, 应用Lemma4.1, 存在$T^N>0$, 使得

$$||A_\epsilon^{\frac{1}{2}}u(t)||^2\le NR_0^2,\quad [0,T^N]. \tag{4.9}$$

记$[0,T^\infty)$ 表示使得(4.23) 成立的最大时间区间. 如果$T^\infty<\infty$, 那么

$$||A_\epsilon^{\frac{1}{2}}u(t)||^2=NR_0^2. \tag{4.10}$$

接下来, 把t 限制在$[0,T^\infty)$上进行讨论
因为

$$(I-M)B_\epsilon(v,v)=B_\epsilon(v,v)-MB_\epsilon(v,v)=B_\epsilon(v,v)-B_\epsilon(v,v)=0,$$

那么关于ω的分量方程为

$$\frac{\mathrm{d}\omega}{\mathrm{dt}}+\nu A_\epsilon\omega=(I-M)P_\epsilon f-(I-M)(B_\epsilon(\omega,v)+B_\epsilon(v,\omega)+B_\epsilon(\omega,\omega)). \tag{4.11}$$

(4.11) 式与$A_\epsilon\omega$作内积, 得

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}\omega}{\mathrm{dt}}||A_\epsilon^{\frac{1}{2}}\omega||^2+\nu||A_\epsilon\omega||^2\le&|((I-M)P_\epsilon f,A_\epsilon\omega)|\\&+|(b_\epsilon(\omega,v,A_\epsilon w))|\\&+|(b_\epsilon(v,\omega,A_\epsilon w))|\\&+|(b_\epsilon(\omega,\omega,A_\epsilon w))|,\end{aligned}$$

利用Young不等式得

$$\frac{1}{2}\frac{\mathrm{d}\omega}{\mathrm{dt}}||A_\epsilon^{\frac{1}{2}}\omega||^2+\nu||A_\epsilon\omega||^2\leq|((I-M)P_\epsilon f,A_\epsilon\omega)|+|(b_\epsilon(\omega,v,A_\epsilon w))|$$

$$+|(b_\epsilon(v,\omega,A_\epsilon w))|+|(b_\epsilon(\omega,\omega,A_\epsilon w))|$$

$$\leq\frac{\nu}{2}||A_\epsilon\omega||^2+\frac{1}{2\nu}||(I-M)P_\epsilon f||_\infty^2$$

$$+C_3\epsilon^{\frac{5}{32}}||A_\epsilon^{\frac{1}{2}}\omega||^{\frac{15}{32}}||A_\epsilon^{\frac{1}{2}}v||||A_\epsilon^{\frac{1}{2}}\omega||^{\frac{49}{32}}$$

$$+C_4\epsilon^{\frac{1}{4}}||A_\epsilon^{\frac{1}{2}}v||||A_\epsilon^{\frac{1}{2}}\omega||^{\frac{1}{2}}||A_\epsilon\omega||^{\frac{3}{2}}$$

$$+C_2\epsilon^{\frac{1}{2}}||A_\epsilon^{\frac{1}{2}}\omega||^{\frac{3}{2}}||A_\epsilon\omega||^{\frac{3}{2}},$$

由于$M\omega=0$, 使用(3.4)得

$$\frac{\mathrm{d}}{\mathrm{dt}}||A_\epsilon^{\frac{1}{2}}\omega||^2+\nu||A_\epsilon\omega||^2\leq\frac{1}{\nu}||(I-M)P_\epsilon f||\infty^2$$

$$+2C_5^{\frac{15}{32}}C_3\epsilon^{\frac{5}{8}}||A_\epsilon^{\frac{1}{2}}v||||A_\epsilon\omega||^2$$

$$+2C_5^{\frac{1}{2}}C_4\epsilon^{\frac{3}{4}}||A_\epsilon^{\frac{1}{2}}v||||A_\epsilon\omega||^2$$

$$+2C_5^{\frac{1}{2}}C_2\epsilon||A_\epsilon^{\frac{1}{2}}\omega||||A_\epsilon\omega||^2,$$

注意到

$$||A_\epsilon^{\frac{1}{2}}\omega||\leq||A_\epsilon^{\frac{1}{2}}u||,\quad ||A_\epsilon^{\frac{1}{2}}v||\leq||A_\epsilon^{\frac{1}{2}}u||,$$

所以我们得到

$$\frac{\mathrm{d}}{\mathrm{dt}}||A_\epsilon^{\frac{1}{2}}\omega||^2+(\nu-D_1\epsilon^{\frac{5}{8}}||A_\epsilon^{\frac{1}{2}}u||)||A_\epsilon\omega||^2\leq\frac{1}{\nu}||(I-M)P_\epsilon f||_\infty^2,\tag{4.12}$$

其中$D_1=2C_5^{\frac{15}{32}}C_3++2C_5^{\frac{1}{2}}C_4+2C_5^{\frac{1}{2}}C_2$.

对于$0\leq t<T^\infty$, 利用假设条件$H1$ 得

$D_1\epsilon^{\frac{5}{8}}||A_\epsilon^{\frac{1}{2}}u||\leq D_1\epsilon^{\frac{5}{8}}N^{\frac{1}{2}}R_0=D_1\epsilon^{\frac{5}{8}}N^{\frac{1}{2}}(\eta_1^{-1}+\epsilon^{\frac{p}{2}}\eta_3^{-1}+\eta_3^{-2}+\sqrt{2}\eta_2^{-1}+\sqrt{2}\epsilon^{\frac{r}{6}}\eta_4^{-1})\to 0,\qquad \epsilon\to 0.$

因此, $\exists\epsilon_2>0$, 使得

$$D_1\epsilon^{\frac{5}{8}}N^{\frac{1}{2}}R_0\leq\frac{\nu}{2},\qquad 0<\epsilon\leq\epsilon_2.\tag{4.13}$$

结合(4.12) 和(4.13)得

$$\frac{\mathrm{d}}{\mathrm{dt}}||A_\epsilon^{\frac{1}{2}}\omega||^2+\frac{\nu}{2}||A_\epsilon\omega||^2\leq\frac{1}{\nu}||(I-M)P_\epsilon f||_\infty^2,\tag{4.14}$$

利用不等式(3.4)得

$$\frac{\mathrm{d}}{\mathrm{dt}}||A_\epsilon^{\frac{1}{2}}\omega||^2+\frac{\nu C_5^{-2}\epsilon^{-2}}{2}||A_\epsilon^{\frac{1}{2}}\omega||^2\leq\frac{1}{\nu}||(I-M)P_\epsilon f||_\infty^2,\tag{4.15}$$

用Gronwall不等式, 得到

$$||A_\epsilon^{\frac{1}{2}}\omega||^2 \le e^{-\int_0^t \frac{\nu C_5^{-2}\epsilon^{-2}}{2}ds}\left[\quad ||A_\epsilon^{\frac{1}{2}}\omega_0||^2 + \int_0^t \frac{1}{\nu}||(I-M)P_\epsilon f||_\infty^2 \quad\right]$$
$$\le e^{\frac{\nu C_5^{-2}\epsilon^{-2}}{2}}t||A_\epsilon^{\frac{1}{2}}\omega_0||^2 + \frac{2C_5^2\epsilon^2}{\nu^2}||(I-M)P_\epsilon f||_\infty^2, \tag{4.16}$$

为了使得

$$e^{\frac{\nu C_5^{-2}\epsilon^{-2}}{2}}t||A_\epsilon^{\frac{1}{2}}\omega_0||^2 = \frac{2C_5^2\epsilon^2}{\nu^2}||(I-M)P_\epsilon f||_\infty^2,$$

即

$$e^{\frac{\nu C_5^{-2}\epsilon^{-2}}{2}}t\epsilon^p\eta_3^{-2} = \frac{2C_5^2\epsilon^2}{\nu^2}\epsilon^{\frac{r}{3}}\eta_4^{-2}, \tag{4.17}$$

可以找到$T_1 > 0$, 满足(4.17), 因此选取$T_1 = T_1(\epsilon) > 0$,

$$T_1 \overset{\text{def}}{=} 2C_5^2\epsilon^2\nu^{-1}Q(\epsilon),$$

其中

$$Q(\epsilon) = |\ln(2C_5^2\nu^{-2}\epsilon^{2+\frac{r}{3}-p}\eta_4^{-2}\eta_3^2)|. \tag{4.18}$$

为了保证(4, 18)成立, 应当满足下面要求

$$2C_5^2\epsilon^{2+\frac{r}{3}-p}\eta_3^2\eta_4^{-2} \le 1, \tag{4.19}$$

例如，只需选取$\eta_4 = -\ln\epsilon$,即可满足要求.
因此, $\exists\epsilon_3 > 0$, 对于$0 < \epsilon \le \epsilon_3$, 存在$T_1 = T_1(\epsilon) > 0$ 满足(4.19).
接下来, 断言$T_1 < T^\infty$. 对于$T_1 \le t < T^\infty$, 有

$$||A_\epsilon^{\frac{1}{2}}\omega||^2 \le e^{\frac{\nu C_5^{-2}\epsilon^{-2}}{2}T_1}||A_\epsilon^{\frac{1}{2}}\omega_0||^2 + \frac{2C_5^2\epsilon^2}{\nu^2}||(I-M)P_\epsilon f||_\infty^2$$
$$= \frac{4C_5^{-2}\epsilon^2}{\nu^2}\epsilon^{\frac{r}{3}}\eta_4^{-2}$$
$$= k_2^2\epsilon^{2+\frac{r}{3}}\eta_4^{-2},$$

其中$k_2^2 = 4C_5^{-2}\nu^{-2}$.
关于v 的估计, 我们限制t 在$[0, T_1]$上

$$\frac{\mathrm{dv}}{\mathrm{dt}} + \nu A_\epsilon v = MP_\epsilon f - MB_\epsilon(v,v) - MB_\epsilon(\omega,v) - B_\epsilon(v,\omega) - B_\epsilon(\omega,\omega), \tag{4.20}$$

(4.20) 式与$A_\epsilon v$ 作内积, 得到

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}||A_\epsilon^{\frac{1}{2}}v||^2+\nu||A_\epsilon v||^2 \le |(MP_\epsilon f, A_\epsilon v)|+|(MB_\epsilon(v,v),A_\epsilon v)|+|(MB_\epsilon(\omega,\omega),A_\epsilon\omega)|$$
$$\le |(MP_\epsilon f, A_\epsilon v)|+|b_\epsilon(v,v,A_\epsilon v)|+|_\epsilon(w,w,A_\epsilon v)|,$$

由于$b_\epsilon(v,\omega,A_\epsilon v)=b_\epsilon(\omega,v,A_\epsilon v)=0$ 具体结论详见 [1]
对上式利用Young不等式, 得到

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}||A_\epsilon^{\frac{1}{2}}v||^2+\nu||A_\epsilon v||^2 \le ||MP_\epsilon f||_\infty^2||A_\epsilon v||+C_1||v||^{\frac{1}{2}}||A_\epsilon^{\frac{1}{2}}v||||A_\epsilon v||^{\frac{3}{2}}$$
$$+C_2\epsilon^{\frac{1}{2}}||A_\epsilon^{\frac{1}{2}}\omega||^{\frac{3}{2}}||A_\epsilon\omega||^{\frac{1}{2}}||A_\epsilon v||$$
$$\le \frac{\nu}{2}||A_\epsilon v||^2+\frac{1}{2\nu}||MP_\epsilon f||_\infty^2||^2+C_1||v||^{\frac{1}{2}}||A_\epsilon^{\frac{1}{2}}v||||A_\epsilon v||^{\frac{3}{2}}$$
$$+C_2\epsilon^{\frac{1}{2}}||A_\epsilon^{\frac{1}{2}}\omega||^{\frac{3}{2}}||A_\epsilon\omega||^{\frac{1}{2}}||A_\epsilon v||,$$

即有

$$\frac{\mathrm{d}}{\mathrm{dt}}||A_\epsilon^{\frac{1}{2}}v||^2+\nu||A_\epsilon v||^2 \le \frac{1}{\nu}||MP_\epsilon f||_\infty^2$$
$$+2(\frac{\nu}{4}||A_\epsilon v||^2+\frac{27}{4\nu^3}C_1^4||v||^2||A_\epsilon^{\frac{1}{2}}v||^4+\frac{\nu}{4}||A_\epsilon v||^2+\frac{1}{\nu}C_2^2\epsilon||A_\epsilon^{\frac{1}{2}}\omega||^3||A_\epsilon\omega||)$$
$$\le \frac{1}{\nu}||MP_\epsilon f||_\infty^2+\frac{\nu}{2}||A_\epsilon v||^2+\frac{27}{2\nu^3}C_1^4||v||^2||A_\epsilon^{\frac{1}{2}}v||^4$$
$$+\frac{\nu}{2}||A_\epsilon v||^2+\frac{2}{\nu}C_2^2\epsilon||A_\epsilon^{\frac{1}{2}}\omega||^3||A_\epsilon\omega||,$$

接着得到

$$\frac{\mathrm{d}}{\mathrm{dt}}||A_\epsilon^{\frac{1}{2}}v||^2 \le \frac{1}{\nu}||MP_\epsilon f||_\infty^2+\frac{27}{2\nu^3}C_1^4||v||^2||A_\epsilon^{\frac{1}{2}}v||^4+\frac{2}{\nu}C_2^2\epsilon||A_\epsilon^{\frac{1}{2}}\omega||^3||A_\epsilon\omega||$$
$$=(\frac{27}{2\nu^3}C_1^4||v||^2||A_\epsilon^{\frac{1}{2}}v||^2)||A_\epsilon^{\frac{1}{2}}v||^2+\frac{1}{\nu}||MP_\epsilon f||_\infty^2+\frac{2}{\nu}C_2^2\epsilon||A_\epsilon^{\frac{1}{2}}\omega||^3||A_\epsilon\omega||,$$

对上式应用Gronwall不等式, 得

$$||A_\epsilon^{\frac{1}{2}}v||^2 \le e^{\int_0^t \frac{27}{2\nu^3}C_1^4||v||^2||A_\epsilon^{\frac{1}{2}}v||^2ds}[||A_\epsilon^{\frac{1}{2}}v_0||^2+\int_0^t\frac{1}{\nu}||MP_\epsilon f||_\infty^2+\frac{2}{\nu}C_2^2\epsilon||A_\epsilon^{\frac{1}{2}}\omega||^3||A_\epsilon\omega||ds]$$
$$\le e^{G(t)}(||A_\epsilon^{\frac{1}{2}}v_0||^2+H(t)), \tag{4.21}$$

其中

$$H(t)=\int_0^t\frac{1}{\nu}||MP_\epsilon f||_\infty^2+\frac{2}{\nu}C_2^2\epsilon||A_\epsilon^{\frac{1}{2}}\omega||^3||A_\epsilon\omega||ds,$$
$$G(t)=\int_0^t\frac{27}{2\nu^3}C_1^4||v||^2||A_\epsilon^{\frac{1}{2}}v||^2ds.$$

现在，估计$H(t)$, $0 \le t \le T_1$. 对(4.14) 式积分得

$$\frac{\nu}{2}\int_0^t ||A_\epsilon \omega||^2 ds \le \frac{t}{\nu}||(I-M)P_\epsilon f||_\infty^2 + ||A_\epsilon^{\frac{1}{2}} w_0||^2,$$

那么

$$\int_0^t ||A_\epsilon \omega||^2 ds \le \frac{2t}{\nu^2}||(I-M)P_\epsilon f||_\infty^2 + \frac{2}{\nu}||A_\epsilon^{\frac{1}{2}} w_0||^2,$$

(4.16)两边三次方，不等式放大为

$$\begin{aligned} ||A_\epsilon^{\frac{1}{2}}\omega||^6 &\le (e^{\frac{-\nu C_5^{-2}\epsilon^{-2}}{2}}t||A_\epsilon^{\frac{1}{2}}\omega_0||^2 + \frac{2C_5^2\epsilon^2}{\nu^2}||(I-M)P_\epsilon f||_\infty^2)^3 \\ &\le 4(e^{\frac{-3\nu C_5^{-2}\epsilon^{-2}}{2}}t||A_\epsilon^{\frac{1}{2}}\omega_0||^6 + \frac{8C_5^6\epsilon^6}{\nu^6}||(I-M)P_\epsilon f||_\infty^6), \end{aligned} \tag{4.22}$$

对(4.22)式积分得

$$\begin{aligned} \int_0^t ||A_\epsilon^{\frac{1}{2}}\omega||^6 ds &\le 4\int_0^t (e^{\frac{-3\nu C_5^{-2}\epsilon^{-2}}{2}}t||A_\epsilon^{\frac{1}{2}}\omega_0||^6 + \frac{8C_5^6\epsilon^6}{\nu^6}||(I-M)P_\epsilon f||_\infty^2)^6 ds \\ &\le 4(\frac{2C_5^2\epsilon^2}{3\nu}||A_\epsilon^{\frac{1}{2}}\omega_0||^6 + \frac{8C_5^6\epsilon^6}{\nu^6}t||(I-M)P_\epsilon f||_\infty^6), \end{aligned}$$

使用H?lder不等式得

$$\begin{aligned} \int_0^t ||A_\epsilon^{\frac{1}{2}}\omega||^3||A_\epsilon\omega||ds &\le (\int_0^t ||A_\epsilon\omega||ds)^{\frac{1}{2}}(\int_0^t ||A_\epsilon^{\frac{1}{2}}\omega||^6 ds)^{\frac{1}{2}} \\ &\le 2(\frac{2t}{\nu^2}||(I-M)P_\epsilon f||_\infty^2 + \frac{2}{\nu}||A_\epsilon^{\frac{1}{2}} w_0||^2)^{\frac{1}{2}} \\ &(\frac{2C_5^2\epsilon^2}{3\nu}||A_\epsilon^{\frac{1}{2}}\omega_0||^6 + \frac{8C_5^6\epsilon^6}{\nu^6}t||(I-M)P_\epsilon f||_\infty^6)^{\frac{1}{2}} \\ &\le \frac{4C_5\epsilon}{\nu}(\frac{t^{\frac{1}{2}}}{\nu^{\frac{1}{2}}}||(I-M)P_\epsilon f||_\infty^2 + ||A_\epsilon^{\frac{1}{2}} w_0||) \\ &(\frac{1}{\sqrt{3}}||A_\epsilon^{\frac{1}{2}}\omega_0||^3 + \frac{2C_5^2\epsilon^2}{\nu^{\frac{5}{2}}}t^{\frac{1}{2}}||(I-M)P_\epsilon f||_\infty^3), \end{aligned}$$

那么

$$\begin{aligned} \frac{2C_2^2\epsilon}{\nu}\int_0^t ||A_\epsilon^{\frac{1}{2}}\omega||^3||A_\epsilon\omega||ds &\le \frac{8C_2^2C_5}{\nu^{-2}}\epsilon^2(\frac{t^{\frac{1}{2}}}{\nu^{\frac{1}{2}}}||(I-M)P_\epsilon f||_\infty^2 + ||A_\epsilon^{\frac{1}{2}} w_0||) \\ &(\frac{1}{\sqrt{3}}||A_\epsilon^{\frac{1}{2}}\omega_0||^3 + \frac{2C_5^2\epsilon^2}{\nu^{\frac{5}{2}}}t^{\frac{1}{2}}||(I-M)P_\epsilon f||_\infty^3) \\ &\le D_2(t^{\frac{1}{2}}||(I-M)P_\epsilon f||_\infty^2 + ||A_\epsilon^{\frac{1}{2}} w_0||) \\ &(||A_\epsilon^{\frac{1}{2}}\omega_0||^3 + \epsilon^2 t^{\frac{1}{2}}||(I-M)P_\epsilon f||_\infty^3), \end{aligned}$$

其中 $D_2=\frac{8C_2^2C_5}{\nu^{-2}}\max(\frac{1}{\sqrt{3}},\frac{2C_5^2}{\nu^{\frac{5}{2}}})\max(1,\frac{1}{\nu^{\frac{1}{2}}})$.
最后得到

$$\begin{aligned}H(t)&\le \frac{1}{\nu}T_1\eta_2^{-2}+D_2\epsilon^2(T_1^{\frac{1}{2}}\epsilon^{\frac{r}{6}}\eta_4^{-1}+\epsilon^{\frac{p}{2}}\eta_3^{-1})(\epsilon^{\frac{3p}{2}}\eta_3^{-3}+T_1^{\frac{1}{2}}\epsilon^{2+\frac{r}{3}}\eta_4^{-3})\\&\le \frac{2C_5^2}{\nu^2}\epsilon^2Q(\epsilon)\eta_2^{-2}+D_2\eta_3^{-4}+\frac{2D_2C_5^2}{\nu}\epsilon^2Q(\epsilon)\eta_4^{-4}+\frac{3D_2}{4}\eta_3^{-4}\\&+\frac{D_2C_5^4}{\nu^2}\epsilon^2Q(\epsilon)^2\eta_4^{-4}+\frac{\sqrt{2}D_2C_5}{\nu^{\frac{1}{2}}}\epsilon^{\frac{3}{2}}Q(\epsilon)^{\frac{1}{2}}\eta_3^{-1}\eta_4^{-3}\\&\le E_1(\epsilon)+\frac{7}{4}D_2\eta_3^{-4},\end{aligned}$$

其中

$$E_1(\epsilon)=D_3(\epsilon^2Q\eta_2^{-2}+\epsilon^2Q\eta_4^{-4}+Q^{\frac{1}{2}}\epsilon^{\frac{3}{2}}\eta_3^{-1}\eta_4^{-3}+\epsilon^2Q\eta_4^{-4}),$$

且

$$D_3=\max(\frac{2C_5^2}{\nu^{-2}},\frac{2D_2C_5^2}{\nu},\frac{\sqrt{2}D_2C_5}{\nu^{\frac{1}{2}}},\frac{D_2C_5^4}{\nu^2}),$$

由假设条件$H1$, 不难验证当$\epsilon\to 0^+$时有

$$E_1(\epsilon)\to 0,$$

因此,

$$||A_\epsilon^{\frac{1}{2}}v(t)||^2\le e^{G(t)}(\eta_1^{-2}+E_1(\epsilon)+\frac{7}{4}D_2\eta_3^{-4}),\quad 0\le t\le T_1.$$

接下来,估计$G(t)$，并说明$G(t)$ 充分小.
(2.2)式与u作内积, 利用$b_\epsilon(u,u,u)=0$,和文献 [8]得到

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}||u||^2+\nu||\ A_\epsilon^{\frac{1}{2}}u||^2&\le|(P_\epsilon f,u)|\\&\le|(A_\epsilon^{-\frac{1}{2}}P_\epsilon f,A_\epsilon^{\frac{1}{2}}u)|\\&\le||A_\epsilon^{\frac{1}{2}}u||||A_\epsilon^{-\frac{1}{2}}P_\epsilon f||_\infty\\&\le\frac{\nu}{2}||A_\epsilon^{\frac{1}{2}}u||^2+\frac{1}{2\nu}||A_\epsilon^{-\frac{1}{2}}P_\epsilon f||_\infty^2,\end{aligned}$$

即有

$$\frac{\mathrm{d}}{\mathrm{dt}}||u||^2+\nu||A_\epsilon^{\frac{1}{2}}u_0||^2\le\frac{1}{\nu}||A_\epsilon^{-\frac{1}{2}}P_\epsilon f||_\infty^2,\tag{4.23}$$

积分得

$$\begin{aligned}||u||^2-||u_0||^2+\nu\int_0^t||A_\epsilon^{\frac{1}{2}}u_0||^2ds &\le \frac{t}{\nu}||A_\epsilon^{-\frac{1}{2}}P_\epsilon f||_\infty^2\\ &\le \frac{2t}{\nu}(||A_\epsilon^{-\frac{1}{2}}MP_\epsilon f||_\infty^2+||A_\epsilon^{-\frac{1}{2}}(I-M)P_\epsilon f||_\infty^2)\\ &\le \frac{2T_1}{\nu}(||A_\epsilon^{-\frac{1}{2}}MP_\epsilon f||_\infty^2+||A_\epsilon^{-\frac{1}{2}}(I-M)P_\epsilon f||_\infty^2)\\ &\le \frac{4C_5^2\epsilon^2Q}{\nu^2}(||A_\epsilon^{-\frac{1}{2}}MP_\epsilon f||_\infty^2+||A_\epsilon^{-\frac{1}{2}}(I-M)P_\epsilon f||_\infty^2),\end{aligned}$$

使用(3.3)与(3.4)得

$$\begin{aligned}||u||^2 &\le ||u_0||^2+\frac{4C_5^2\epsilon^2Q}{\nu^2\lambda_1}||MP_\epsilon f||_\infty^2+\frac{4C_5^4\epsilon^4Q}{\nu^2}||(I-M)P_\epsilon f||_\infty^2\\ &=||v_0||^2+||\omega_0||^2+\frac{4C_5^2\epsilon^2Q}{\nu^2\lambda_1}||MP_\epsilon f||_\infty^2+\frac{4C_5^4\epsilon^4Q}{\nu^2}||(I-M)P_\epsilon f||_\infty^2\\ &\le \frac{1}{\lambda_1}||A_\epsilon^{\frac{1}{2}}v_0||^2+C_5^2\epsilon^2||A_\epsilon^{\frac{1}{2}}\omega_0||^2+\frac{4C_5^2\epsilon^2Q}{\nu^2\lambda_1}||MP_\epsilon f||_\infty^2+\frac{4C_5^4\epsilon^4Q}{\nu^2}||(I-M)P_\epsilon f||_\infty^2,\end{aligned}$$

即有

$$\begin{aligned}||v||^2\le||u||^2 &\le \frac{1}{\lambda_1}||A_\epsilon^{\frac{1}{2}}v_0||^2+C_5^2\epsilon^2||A_\epsilon^{\frac{1}{2}}\omega_0||^2+\frac{4C_5^2\epsilon^2Q}{\nu^2\lambda_1}||MP_\epsilon f||_\infty^2+\frac{4C_5^4\epsilon^4Q}{\nu^2}||(I-M)P_\epsilon f||_\infty^2\\ &\le D_4(||A_\epsilon^{\frac{1}{2}}v_0||^2+\epsilon^2||A_\epsilon^{\frac{1}{2}}\omega_0||^2+\epsilon^2Q||MP_\epsilon f||_\infty^2+\epsilon^4Q||(I-M)P_\epsilon f||_\infty^2),\end{aligned}$$

其中

$$D_4=\max(\frac{1}{\lambda_1},C_5^2,\frac{4C_5^4}{\nu^2},\frac{4C_5^2}{\nu^2\lambda_1}).$$

利用下述结论

$$||A_\epsilon^{\frac{1}{2}}v(t)||^2\le||A_\epsilon^{\frac{1}{2}}u(t)||^2\le NR_0^2,$$

我们可以得到

$$\begin{aligned}G(t)&=\int_0^t\frac{27}{2\nu^3}C_1^4||v||^2||A_\epsilon^{\frac{1}{2}}v||^2ds\\ &\le\frac{27}{2\nu^3}C_1^4D_4T_1(\eta_1^{-2}+\epsilon^{2+p}\eta_3^{-2}+\epsilon^2Q\eta_2^{-4}+\epsilon^{4+\frac{r}{3}}Q\eta_4^{-2})NR_0^2\\ &\le\frac{27C_5^2C_1^4D_4}{\nu^4}\epsilon^2Q(\epsilon)(\eta_1^{-2}+\epsilon^{2+p}\eta_3^{-2}+\epsilon^2Q\eta_2^{-4}+\epsilon^{4+\frac{r}{3}}Q\eta_4^{-2})(\eta_1^{-2}+\epsilon^p\eta_3^{-2}+\eta_3^{-4}+2\eta_2^{-2}+2\epsilon^{\frac{r}{3}}\eta_4^{-2})\\ &\le E_2(\epsilon),\end{aligned}$$

其中

$$E_2(\epsilon)=D_5\epsilon^2Q(\epsilon)(\eta_1^{-2}+\epsilon^{2+p}\eta_3^{-2}+\epsilon^2Q\eta_2^{-4}+\epsilon^{4+\frac{r}{3}}Q\eta_4^{-2})(\eta_1^{-2}+\epsilon^p\eta_3^{-2}+\eta_3^{-4}+2\eta_2^{-2}+2\epsilon^{\frac{r}{3}}\eta_4^{-2}),$$

且

$$D_5 = \frac{27C_5^2C_1^4D_4}{\nu^4}.$$

类似地，利用假设条件$H1$,不难验证

$$E_2(\epsilon) \to 0 \quad as \quad \epsilon \to 0.$$

最后, 在$N = 1 + \max(4, \frac{7}{2}D_2)$下, 我们选取$\epsilon_4 > 0$, 使得

$$e^{E_2(\epsilon) \le 2}, \quad E_1(\epsilon) \le \eta_1^{-2}, \quad 2C_5^2\epsilon^{\frac{2}{3}} \le \nu^2, \quad 0 < \epsilon \le \epsilon_4$$

为了证明$T_1 < T^\infty$,, 我们使用反证法假设$T^\infty = \infty$, 那么显然有$T_1 < T^\infty = \infty$. 因此我们只需假设$T^\infty \le T_1 < \infty$, 一方面, 有

$$\begin{aligned}
||A_\epsilon^{\frac{1}{2}}\omega(T^\infty)||^2 &\le ||A_\epsilon^{\frac{1}{2}}\omega_0||^2 + \frac{1}{2}k_2^2\epsilon^2||(I-M)P_\epsilon f||_\infty^2 \\
&\le \epsilon^p\eta_3^{-2} + \frac{1}{2}k_2^2\epsilon^{2+\frac{r}{3}}\eta_4^{-2} \\
&\le \epsilon^p\eta_3^{-2} + \frac{1}{2}k_2^2\epsilon^{\frac{2}{3}+\frac{r}{3}}\eta_4^{-2} \\
&\le \epsilon^p\eta_3^{-2} + \epsilon^{\frac{r}{3}}\eta_4^{-2},
\end{aligned}$$

且

$$\begin{aligned}
||A_\epsilon^{\frac{1}{2}}v(T^\infty)||^2 &\le e^{E_2(\epsilon)}(\eta_1^{-2} + E_1(\epsilon) + \frac{7}{4}D_2\eta_3^{-4}) \\
&\le 4\eta_1^{-2} + \frac{7}{2}D_2\eta_3^{-4}.
\end{aligned}$$

最后, 我们得到

$$\begin{aligned}
||A_\epsilon^{\frac{1}{2}}u(T^\infty)||^2 &= 4\eta_1^{-2} + \frac{7}{2}D_2\eta_3^{-4} + \epsilon^p\eta_3^{-2} + \epsilon^{\frac{r}{3}}\eta_4^{-2} \\
&< (1+\max(4, \frac{7}{2}D_2))R_0^2 \\
&= NR_0^2.
\end{aligned} \tag{4.24}$$

另一方面, 如果$T^\infty < \infty$, 那么

$$||A_\epsilon^{\frac{1}{2}}u(T^\infty)||^2 = NR_0^2,$$

这与(4.23)式矛盾. 因此, $T_1 < T^\infty$.

令$k_1^2 = \frac{7}{2}D_2, k_2^2 = 4C_5^2\nu^{-2}, \epsilon_1 \overset{\text{def}}{=} \epsilon_4$, 我们有

$$||A_\epsilon^{\frac{1}{2}}v(T_1)||^2 \le 4\eta_1^{-2} + k_1^2\eta_3^{-4},$$

$$||A_\epsilon^{\frac{1}{2}}\omega(T_1)||^2 \le k_2^2\epsilon^{2+\frac{r}{3}}\eta_4^{-2}.$$

证毕.

引理 4.3 设条件$H1$和$H(a,b)$成立，其中a和b 充分大，那么存在$\epsilon_0>0$ 使得对每个$\epsilon, 0<\epsilon\leq\epsilon_0$，若初始数据满足假设条件(3.1)，则方程(2.2) 的解$u(t)$ 满足：当$0\leq t\leq 2T_0$ 时有$u(t)\in D(A_\epsilon^{\frac{1}{2}})$ ，且当$T_0\leq t\leq 2T_0$时有

$$\begin{cases} ||A_\epsilon^{\frac{1}{2}}v(t)||^2\leq \frac{1}{2}(4\eta_1^{-2}+k_1^2\eta_3^{-4}), \\ ||A_\epsilon^{\frac{1}{2}}w(t)||^2\leq k_2^2\epsilon^{2+\frac{r}{3}}\eta_4^{-2}. \end{cases} \tag{4.25}$$

证明. 我们在引理4.2的基础上证明引理4.3. 定义

$$R_0^2=1+(\eta^{-2}+2\eta_2^{-2}+d_1)[1+e^{2D_{19}\eta^{-4}}e^{4D_{19}\eta_2^{-4}}]+2\epsilon^{\frac{r}{3}}\eta_4^{-2}.$$

由于

$$\begin{aligned} ||A_\epsilon^{\frac{1}{2}}u_0||^2+||P_\epsilon f||_\infty^2 &\leq ||A_\epsilon^{\frac{1}{2}}v_0||^2+||A_\epsilon^{\frac{1}{2}}\omega_0||^2+2||MP_\epsilon f||_\infty^2+2||(I-M)P_\epsilon f||_\infty^2 \\ &\leq 4\eta_1^{-2}+k_1^2\eta_3^{-4}+k_2^2\epsilon^{2+\frac{r}{3}}\eta_4^{-2}+2\eta_2^{-2}+2\epsilon^{\frac{r}{3}}\eta_4^{-2} \\ &\leq R_0^2, \end{aligned}$$

取定$N=\max(1,D_{21},D_{22})>1$, 应用Lemma 4.1, 存在$T^N>0$, 使得

$$||A_\epsilon^{\frac{1}{2}}u(t)||^2\leq NR_0^2 \qquad t\in[0,T^N].$$

记$[0,T^\infty)$ 表示使得上面不等式成立的最大时间区间. 若$T^\infty<\infty$, 则有

$$||A_\epsilon^{\frac{1}{2}}u(t)||^2=NR_0^2. \tag{4.26}$$

(4.11) 式与$A_\epsilon\omega$, 得到(4.13) 式和D_1. 对于$0\leq t<T^\infty$ 有

$$D_1^2\epsilon^{\frac{5}{4}}||A_\epsilon^{\frac{1}{2}}u||^2\leq D_1^2N\epsilon^{\frac{5}{4}}R_0^2\to 0,\quad as\quad \epsilon\to 0$$

假设$a\geq 2D_{19}, b\geq 4D_{19}$, 从假设条件$H1$ 和$H(a,b)$得到：当$\epsilon\to 0$式, 下式趋于0

$$\epsilon^{\frac{5}{4}}R_0^2=\epsilon^{\frac{5}{4}}(1+\eta^{-2}+2\eta_2^{-2}+d_1+\eta^{-2}e^{2D_19\eta^{-4}}e^{4D_{19}\eta_2^{-4}}+2\eta_2^{-2}e^{2D_{19}\eta^{-4}}e^{4D_{19}\eta_2^{-4}}+d_1e^{2D_{19}\eta^{-4}}e^{4D_{19}\eta_2^{-4}}+2\epsilon^{\frac{r}{3}}\eta_4^{-2}).$$

那么, 存在$\epsilon_5>0$, 使得

$$D_1N^{\frac{1}{2}}\epsilon^{\frac{5}{8}}R_0\leq\frac{\nu}{2},\quad 0<\epsilon\leq\epsilon_5$$

从(4.16)式得

$$\begin{aligned} ||A_\epsilon^{\frac{1}{2}}\omega(t)||^2 &\leq e^{-\frac{\nu C_5^{-2}\epsilon^{-2}}{2}t}||A_\epsilon^{\frac{1}{2}}\omega_0||^2+\frac{2C_5^2\epsilon^2}{\nu^2}||(I-M)P_\epsilon f||_\infty^2 \\ &\leq \frac{3}{2}k_2^2\epsilon^{2+\frac{r}{3}}\eta_4^{-2}, \end{aligned} \tag{4.27}$$

对上式从0到t积分得

$$\begin{aligned}\int_0^t ||A_\epsilon^{\frac{1}{2}}\omega||^2 ds &\le \frac{2}{\nu}||A_\epsilon^{\frac{1}{2}}\omega_0||^2 + \frac{2t}{\nu^2}||(I-M)P_\epsilon f||_\infty^2 \\ &\le \max(\frac{2k_2^2}{\nu}, \frac{2}{\nu^2})[\epsilon^2+t]\epsilon^{\frac{r}{3}}\eta_4^{-2} \\ &\le D_9^2[\epsilon^2+t]\epsilon^{\frac{r}{3}}\eta_4^{-2},\end{aligned}$$

其中$0 \le t < 1$.
对上式从$t-1$到t积分得

$$\begin{aligned}\int_{t-1}^t ||A_\epsilon^{\frac{1}{2}}\omega||^2 ds &\le \frac{2}{\nu}||A_\epsilon^{\frac{1}{2}}\omega(t-1)||^2 + \frac{2}{\nu^2}||(I-M)P_\epsilon f||_\infty^2 \\ &\le \max(\frac{3}{\nu}k_2^2, \frac{2}{\nu^2})\epsilon^{\frac{r}{3}}\eta_4^{-2} \\ &\le D_{10}^2\epsilon^{\frac{r}{3}}\eta_4^{-2},\end{aligned}$$

其中$1 \le t < T^\infty$.
同样，积分得

$$\begin{aligned}\int_0^t ||A_\epsilon^{\frac{1}{2}}\omega(s)||^6 ds &\le 4\int_0^t (e^{-\frac{3\nu C_5^{-2}\epsilon^2}{2}}k_2^2\epsilon^{6+r}\eta_4^{-6} + \frac{8C_5^6\epsilon^2}{\nu^6}\epsilon^r\eta_4^{-6}ds) \\ &\le 4\int_0^t (e^{-\frac{3\nu C_5^{-2}\epsilon^2}{2}}k_2^2 + \frac{1}{8}k_2^6)ds \quad \epsilon^{6+r}\eta_4^{-6} \\ &\le 4k_2^6\max(\frac{2C_5^2}{3\nu}, \frac{1}{8})[\epsilon^2+t]\epsilon^{6+r}\eta_4^{-6} \\ &\le D_{11}^2[\epsilon^2+t]\epsilon^{6+r}\eta_4^{-6},\end{aligned}$$

其中$0 \le t < 1$.
由$0 < \epsilon \le 1$有, 类似得到

$$\int_{t-1}^t ||A_\epsilon^{\frac{1}{2}}\omega(s)||^6 ds \le 2D_{11}^2\epsilon^{6+r}\eta_4^{-6},$$

其中$1 \le t < T^\infty$.
使用H?lder不等式得到

$$\begin{aligned}\int_0^t ||A_\epsilon^{\frac{1}{2}}\omega||^3||A_\epsilon\omega|| &\le (\int_0^t ||A_\epsilon^{\frac{1}{2}}\omega||^6)^{\frac{1}{2}}(\int_0^t ||A_\epsilon\omega||^2)^{\frac{1}{2}} \\ &\le D_9[\epsilon^2+t]^{\frac{1}{2}}\epsilon^{\frac{r}{6}}\eta_4^{-1}D_{11}[\epsilon^2+t]^{\frac{1}{2}}\epsilon^{3+\frac{r}{2}}\eta_4^{-3} \qquad (4.28) \\ &= D_9D_{11}[\epsilon^2+t]\epsilon^{3+\frac{2r}{3}}\eta_4^{-4},\end{aligned}$$

其中$0 \le t < 1$.
同样使用H?lder不等式得到

$$\begin{aligned}\int_{t-1}^{t} ||A_\epsilon^{\frac{1}{2}}\omega||^3||A_\epsilon\omega|| &\le (\int_{t-1}^{t} ||A_\epsilon^{\frac{1}{2}}\omega||^6)^{\frac{1}{2}}(\int_{t-1}^{t} ||A_\epsilon\omega||^2)^{\frac{1}{2}} \\ &\le D_{10}\epsilon^{\frac{r}{6}}\eta_4^{-1}\sqrt{2}D_{11}\epsilon^{3+\frac{r}{2}}\eta_4^{-3} \\ &\le 2D_{10}D_{11}\epsilon^{3+\frac{2r}{3}}\eta_4^{-4},\end{aligned} \tag{4.29}$$

其中$1 \le t < T^\infty$.
接下来, (4.20) 式与v 作内积, 结合$b_\epsilon(v, v, v) = 0$ 、(3.3) 式和(3.4)式得到

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}||v||^2 + \lambda_1\nu||v||^2 &\le \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}||v||^2 + \nu||A_\epsilon v||^2 \\ &\le ||MP\epsilon f||_\infty||v|| + C_2\epsilon^{\frac{1}{2}}||A_\epsilon^{\frac{1}{2}}\omega||^{\frac{3}{2}}||A_\epsilon\omega||^{\frac{1}{2}}||v|| \\ &\le \frac{1}{\lambda_1^{\frac{1}{2}}}(||MP\epsilon f||_\infty + C_2\epsilon^{\frac{1}{2}}||A_\epsilon^{\frac{1}{2}}\omega||^{\frac{3}{2}}||A_\epsilon\omega||^{\frac{1}{2}}),\end{aligned}$$

利用Young不等式得
$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}||v||^2 + \nu||A_\epsilon^{\frac{1}{2}}v||^2 \le \frac{\nu}{4}||A_\epsilon^{\frac{1}{2}}v||^2 + \frac{1}{\nu\lambda_1}||MP\epsilon f||_\infty^2 + \frac{\nu}{4}||A_\epsilon^{\frac{1}{2}}v||^2 + \frac{1}{\nu\lambda_1}C_2^2\epsilon||A_\epsilon^{\frac{1}{2}}\omega||^3||A_\epsilon\omega||$,
不难证得下述结论

$$\frac{\mathrm{d}}{\mathrm{dt}}||v||^2 + \nu||A_\epsilon^{\frac{1}{2}}v||^2 \le \frac{2}{\nu\lambda_1}(||MP\epsilon f||_\infty^2 + C_2^2\epsilon||A_\epsilon^{\frac{1}{2}}\omega||^3||A_\epsilon\omega||),$$

$$\frac{\mathrm{d}}{\mathrm{dt}}||v||^2 + \lambda_1\nu||A_\epsilon^{\frac{1}{2}}v||^2 \le \frac{2}{\nu\lambda_1}(||MP\epsilon f||_\infty^2 + C_2^2\epsilon||A_\epsilon^{\frac{1}{2}}\omega||^3||A_\epsilon\omega||),$$

使用Gronwall不等式得

$$\begin{aligned}||v||^2 &\le e^{-\lambda_1 t}(||v_0||^2 + \int_0^t \frac{2}{\nu\lambda_1}(||MP\epsilon f||_\infty^2 + C_2^2\epsilon||A_\epsilon^{\frac{1}{2}}\omega||^3||A_\epsilon\omega||))ds \\ &\le e^{-\lambda_1 t}||v_0||^2 + \frac{2}{\nu\lambda_1}||MP\epsilon f||_\infty^2 te^{-\lambda_1 t} + \frac{2}{\nu\lambda_1}C_2^2\epsilon e^{-\lambda_1 t}\int_0^t ||A_\epsilon^{\frac{1}{2}}\omega||^3||A_\epsilon\omega||ds \\ &\le e^{-\lambda_1 t}||v_0||^2 + \frac{2}{(\nu\lambda_1)^2}||MP\epsilon f||_\infty^2 + \frac{2}{\nu\lambda_1}C_2^2\epsilon\int_0^t ||A_\epsilon^{\frac{1}{2}}\omega||^3||A_\epsilon\omega||ds \\ &\le e^{-\lambda_1 t}\frac{1}{\lambda_1}||A_\epsilon^{\frac{1}{2}}v_0||^2 + \frac{2}{(\nu\lambda_1)^2}||MP\epsilon f||_\infty^2 + \frac{2}{\nu\lambda_1}C_2^2\epsilon\int_0^t ||A_\epsilon^{\frac{1}{2}}\omega||^3||A_\epsilon\omega||ds \\ &\le e^{-\lambda_1 t}\frac{1}{\lambda_1}(4\eta_1^{-2} + k_1^2\eta_3^{-4}) + \frac{2}{(\nu\lambda_1)^2}\eta_2^{-2} + \frac{2}{(\nu\lambda_1)^2}C_2^2\epsilon D_9 D_{11}[\epsilon^2 + t]\epsilon^{3+\frac{2r}{3}}\eta_4^{-4} \\ &\le \max(\frac{1}{\lambda_1}, \frac{2}{(\nu\lambda_1)^2}, \frac{2}{(\nu\lambda_1)^2}C_2^2 D_9 D_{11})(e^{-\nu\lambda_1 t}(4\eta_1^{-2} + k_1^2\eta_3^{-4}) + \eta_2^{-2} + [\epsilon^2 + t]\epsilon^{4+\frac{2r}{3}}\eta_4^{-4}) \\ &= D_{12}\gamma(\epsilon, t),\end{aligned}$$

其中

$$D_{12} \stackrel{\text{def}}{=} \max(\frac{1}{\lambda_1}, \frac{2}{(\nu\lambda_1)^2}, \frac{2}{(\nu\lambda_1)^2}C_2^2D_9D_{11}),$$

$$\gamma(\epsilon,t) \stackrel{\text{def}}{=} e^{-\nu\lambda_1 t}\eta^{-2} + \eta_2^{-2} + [\epsilon^2+t]\epsilon^{4+\frac{2r}{3}}\eta_4^{-4}.$$

类似地, 得到

$$||v||^2 - ||v_0||^2 - \nu\int_0^t ||A_\epsilon^{\frac{1}{2}}v||^2 ds \leq \frac{2}{\nu\lambda_1}(||MP_\epsilon f||_\infty^2 t + C_2^2\epsilon\int_0^t ||A_\epsilon^{\frac{1}{2}}\omega||^3||A_\epsilon\omega||ds),$$

由假设条件(3.1)和(4.28)

$$\begin{aligned}\int_0^t ||A_\epsilon^{\frac{1}{2}}v||^2 ds &\leq \frac{2t}{\lambda_1\nu^2}||MP_\epsilon f||_\infty^2 + \frac{1}{\nu}||v_0||^2 + \frac{2}{\lambda_1\nu^2}C_2^2\epsilon\int_0^t ||A_\epsilon^{\frac{1}{2}}\omega||^3||A_\epsilon\omega||ds\\ &\leq \frac{2t}{\lambda_1\nu^2}\eta_2^{-2} + \frac{1}{\nu\lambda_1}(4\eta_1^{-2}+k_1^2\eta_3^{-4}) + \frac{2}{\lambda_1\nu^2}C_2^2\epsilon D_9D_{11}[\epsilon^2+t]\epsilon^{3+\frac{2r}{3}}\eta_4^{-4}\\ &\leq \max(\frac{e^{\nu\lambda_1}}{\nu\lambda_1}, \frac{2}{\lambda_1\nu^2}, \frac{2}{\lambda_1\nu^2}C_2^2D_9D_{11})(e^{-\nu\lambda_1}(4\eta_1^{-2}+k_1^2\eta_3^{-4}) + \eta_2^{-2} + [\epsilon^2+t]\epsilon^{3+\frac{2r}{3}}\eta_4^{-4})\\ &\leq D_{13}\gamma(\epsilon,t),\end{aligned}$$

其中

$$D_{13} \stackrel{\text{def}}{=} \max(\frac{e^{\nu\lambda_1}}{\nu\lambda_1}, \frac{2}{\lambda_1\nu^2}, \frac{2}{\lambda_1\nu^2}C_2^2D_9D_{11}).$$

类似地

$$\begin{aligned}\int_{t-1}^t ||A_\epsilon^{\frac{1}{2}}v||^2 ds &\leq \frac{2}{\lambda_1\nu^2}||MP_\epsilon f||_\infty^2 + \frac{1}{\nu}||v(t-1)||^2 + \frac{2}{\lambda_1\nu^2}C_2^2\epsilon\int_{t-1}^t ||A_\epsilon^{\frac{1}{2}}\omega||^3||A_\epsilon\omega||ds\\ &\leq \frac{1}{\nu}D_{12}\gamma(\epsilon,t) + \frac{2}{\lambda_1\nu^2}\eta_2^{-2} + \frac{2}{\lambda_1\nu^2}C_2^2\epsilon 2D_9D_{11}[\epsilon^2+t]\epsilon^{3+\frac{2r}{3}}\eta_4^{-4}\\ &\leq \frac{1}{\nu}(D_{12}\gamma(\epsilon,t) + \frac{2}{\lambda_1\nu}\eta_2^{-2} + \frac{2}{\lambda_1\nu}C_2^2 2D_{10}D_{11}\epsilon^{3+\frac{2r}{3}}\eta_4^{-4})\\ &\leq \frac{1}{\nu}\max(D_{12}, \frac{2}{\lambda_1\nu} + D_{12}, 2D_{12} + \frac{4}{\lambda_1\nu}C_2^2 10D_{11})\gamma(\epsilon,t)\\ &\leq D_{14}\gamma(\epsilon,t),\end{aligned}$$

利用以上估计，有

$$\begin{aligned}\int_0^t ||v||^2||A_\epsilon^{\frac{1}{2}}v||^2 ds &\leq \sup_{0\leq s\leq t}||v||^2\int_0^t ||A_\epsilon^{\frac{1}{2}}v||^2 ds\\ &\leq D_{12}D_{13}\gamma^2(\epsilon,t)\\ &\leq e^{\nu\lambda_1}D_{12}D_{13}\gamma^2(\epsilon,t).\end{aligned}$$

类似地, 得到

$$\int_{t-1}^{t} ||v||^2||A_\epsilon^{\frac{1}{2}}v||^2 ds \le e^{\nu\lambda_1} D_{12}D_{14}\gamma^2(\epsilon,t).$$

从(4.21)式知

$$||A_\epsilon^{\frac{1}{2}}v||^2 \le e^{G(t)}(||A_\epsilon^{\frac{1}{2}}v_0||^2 + H(t)),$$

其中

$$\begin{aligned} G(t) &= \int_0^t \frac{27}{2\nu^3} C_1^4 ||A_\epsilon^{\frac{1}{2}}v||^2||v||^2 ds \\ &= \frac{27}{2\nu^3} C_1^4 e^{\nu\lambda_1} D_{12}D_{13}\gamma^2(\epsilon,t) \\ &= D_{17}\gamma^2(\epsilon,t), \end{aligned}$$

且

$$\begin{aligned} H(t) &= \frac{1}{\nu}\int_0^t ||MP_\epsilon f||_\infty^2 + \frac{2C_2^2\epsilon}{\nu}\int_0^t ||A_\epsilon^{\frac{1}{2}}\omega||^3||A_\epsilon\omega||^2 ds \\ &\le \frac{1}{\nu}\eta_2^{-2} + \frac{2C_2^2 D_9 D_{11}}{\nu}[\epsilon^2+t]\epsilon^{4+\frac{2r}{3}}\eta_4^{-4}. \end{aligned}$$

结合上述估计，得到

$$\begin{aligned} ||A_\epsilon^{\frac{1}{2}}v(t)||^2 &\le e^{D_{17}}\gamma^2(\epsilon,t)(\frac{1}{\nu}\eta_2^{-2} + \frac{2C_2^2 D_9 D_{11}}{\nu}[\epsilon^2+t]\epsilon^{4+\frac{2r}{3}}\eta_4^{-4} + 4\eta_1^{-2} + k_1^2\eta_3^{-4}) \\ &\le \max(e^{\nu\lambda_1}, \frac{1}{\nu}, \frac{2C_2^2 D_9 D_{11}}{\nu}) \\ &= D_{16}\gamma(\epsilon,t)e^{D_{17}}\gamma^2(\epsilon,t) \end{aligned},$$

注意到

$$\frac{\mathrm{d}}{\mathrm{dt}}||A_\epsilon^{\frac{1}{2}}v||^2 \le (\frac{27}{2\nu^3}C_1^4||v||^2||A_\epsilon^{\frac{1}{2}}v||^2)||A_\epsilon^{\frac{1}{2}}v||^2 + \frac{1}{\nu}||MP_\epsilon f||_\infty^2 + \frac{2C_2^2\epsilon}{\nu}||A_\epsilon^{\frac{1}{2}}\omega||^3||A_\epsilon\omega||^2,$$

对上式在$t\in[1,\infty)$上使用一致Gronwall不等式，得到

$$\begin{aligned} ||A_\epsilon^{\frac{1}{2}}v||^2 &\le [\int_{t-1}^t ||A_\epsilon^{\frac{1}{2}}v||^2 ds + \int_{t-1}^t (\frac{1}{\nu}||MP_\epsilon f||_\infty^2 + \frac{2C_2^2\epsilon}{\nu}||A_\epsilon^{\frac{1}{2}}\omega||^3||A_\epsilon\omega||^2)ds]e^{\frac{27}{2\nu^3}C_1^4 e^{\nu\lambda_1}D_{12}D_{14}\gamma^2(\epsilon,t)} \\ &\le [D_{14}\nu^2\gamma(\epsilon,t) + \frac{1}{\nu}\eta_2^{-2} + \frac{4C_2^2 D_{10}D_{11}\epsilon}{\nu}\epsilon^{3+\frac{2r}{3}}\eta_4^{-4}]e^{\frac{27}{2\nu^3}C_1^4 e^{\nu\lambda_1}D_{12}D_{14}\gamma^2(\epsilon,t)} \\ &\le [D_{14}\nu^2\gamma(\epsilon,t) + \frac{1}{\nu}\eta_2^{-2} + \frac{4C_2^2 D_{10}D_{11}}{\nu}\epsilon^{4+\frac{2r}{3}}\eta_4^{-4}]e^{\frac{27}{2\nu^3}C_1^4 e^{\nu\lambda_1}D_{12}D_{14}\gamma^2(\epsilon,t)} \\ &\le (D_{14} + \max(\frac{1}{\nu}, \frac{4C_2^2 D_{10}D_{11}}{\nu}))\gamma(\epsilon,t)e^{\frac{27}{2\nu^3}C_1^4 e^{\nu\lambda_1}D_{12}D_{14}\gamma^2(\epsilon,t)}, \end{aligned}$$

那么我们得到

$$\begin{cases} ||A_\epsilon^{\frac{1}{2}}v(t)||^2 \le D_{16}\gamma(\epsilon,t)e^{D_{17}\gamma^2(\epsilon,t)} \\ ||A_\epsilon^{\frac{1}{2}}v(t)||^2 \le (D_{14}+\max(\frac{1}{\nu},\frac{4C_2^2D_{10}D_{11}}{\nu}))\gamma(\epsilon,t)e^{\frac{27}{2\nu^3}}C_1^4e^{\nu\lambda_1}D_{12}D_{14}\gamma^2(\epsilon,t), \end{cases}$$

且

$$||A_\epsilon^{\frac{1}{2}}v(t)||^2 \le D_{18}\gamma(\epsilon,t)e^{D_{19}\gamma^2(\epsilon,t)},$$

其中$D_{18}=\max(D_{16},D_{14}+\max(\frac{1}{\nu},\frac{4C_2^2D_{10}D_{11}}{\nu}))$, $\quad D_{19}=\frac{27C_1^4}{2\nu^3}e^{\nu\lambda_1}\max(D_{13},D_{14})D_{12}$.
我们想要找到$T_0=T_0(\epsilon)$. 因为$\eta\to\infty$，$\epsilon\to 0^+$, 可以假设

$$2D_{19}\eta^{-4}>1,$$

要求

$$2D_{19}e^{-2\nu\lambda_1 t}\eta^{-4}\le 1.$$

令

$$T_0 \overset{\text{def}}{=} \frac{1}{2\nu\lambda_1}\ln(2D_{19}\eta^{-4})>0,$$

且

$$E_3(\epsilon) \overset{\text{def}}{=} (\epsilon^2+2T_0(\epsilon))\epsilon^{4+\frac{2r}{3}}\eta_4^{-4},$$

不难估计$E_3(\epsilon)$ 和$\frac{3}{2}k_2^2\epsilon^{2+\frac{r}{3}}\eta_4^{-2}$

$$\begin{aligned} E_3(\epsilon) &= (\epsilon^2+2T_0(\epsilon))\epsilon^{4+\frac{2r}{3}}\eta_4^{-4} \\ &= (1+\frac{1}{\nu\lambda_1}\ln(2D_{19})\eta^{-4})\epsilon^{4+\frac{2r}{3}}\eta_4^{-4} \\ &= (1+\frac{1}{\nu\lambda_1}(\ln(2D_{19})+\ln\eta^{-4}))\epsilon^{4+\frac{2r}{3}}\eta_4^{-4} \\ &= (1+\frac{1}{\nu\lambda_1}\max(\ln(2D_{19},1)))(1+\ln\eta^{-4})\epsilon^{4+\frac{2r}{3}}\eta_4^{-4} \\ &\le D_{20}, \end{aligned}$$

且

$$\begin{aligned} \frac{3}{2}k_2^2\epsilon^{2+\frac{r}{3}}\eta_4^{-2} &\le \frac{3}{2}k_2^2(\epsilon^{4+\frac{2r}{3}}\eta_4^{-4})^{\frac{1}{2}} \\ &\le \frac{3}{2}k_2^2(1+\ln\eta^{-4})^{\frac{1}{2}}(\epsilon^{4+\frac{2r}{3}}\eta_4^{-4})^{\frac{1}{2}} \\ &\le \frac{3}{2}k_2^2(1+\ln\eta^{-4})(\epsilon^{4+\frac{2r}{3}}\eta_4^{-4})^{\frac{1}{2}} \\ &\le D_{21}, \end{aligned}$$

而且有

$$\begin{aligned}||A_\epsilon^{\frac{1}{2}}v(t)||^2 &\leq D_{18}\gamma(\epsilon,t)e^{D_{19}\gamma^2(\epsilon,t)} \\ &\leq D_{18}(e^{-\nu\lambda_1 t}\eta^{-2}+\eta_2^{-2}+[\epsilon+t]\epsilon^{4+\frac{2r}{3}}\eta_4^{-4})e^{D_{19}(2e^{-2\nu\lambda_1}\eta^{-4}+4\eta_2^{-2}+4D_{19}^2)} \\ &\leq D_{18}e^{4D_{19}D_{20}^2}(e^{-\nu\lambda_1 t}\eta^{-2}+\eta_2^{-2}+D_{20})e^(4D_{19}\eta_2^{-4}+1) \\ &\leq D_{22}(e^{-\nu\lambda_1 t}\eta^{-2}+\eta_2^{-2}+D_{20})e^(4D_{19}\eta_2^{-4}+1)\end{aligned},$$

因此,

$$\begin{aligned}||A_\epsilon^{\frac{1}{2}}u(t)||^2 &= ||A_\epsilon^{\frac{1}{2}}v(t)||^2+||A_\epsilon^{\frac{1}{2}}\omega(t)||^2 \\ &\leq D_{22}(e^{-\nu\lambda_1 t}\eta^{-2}+\eta_2^{-2}+D_{20})e^{2D_{19}\eta^{-4}}e^{4D_{19}\eta_2^{-4}}+D_{21}e^0 \\ &\leq \max(1,D_{21},D_{22})((\eta^{-2}+\eta_2^{-2}+D_{20})e^{2D_{19}\eta^{-4}}e^{4D_{19}\eta_2^{-4}}) \\ &< NR_0^2.\end{aligned} \tag{4.30}$$

接下来, 我们使用反证法证明$2T_0\leq T^\infty$. 如果$T^\infty=\infty$, 那么$2T_0\leq T^\infty$.
因此我们假设$T^\infty<2T_0<\infty$,
从(4.30) 式知

$$||A_\epsilon^{\frac{1}{2}}u(T^\infty)||^2<NR_0^2,$$

这与(4.26)式矛盾. 因此$2T_0\leq T^\infty$.
最后, 我们来证明(4.25)式. 限制t 在$[T_0,2T_0]$上
从(4.27) 式得

$$||A_\epsilon^{\frac{1}{2}}\omega(t)||^2\leq[k_2^2e^{-\frac{2\nu C_5^{-2}\epsilon^{-2}}{2}T_0}+\frac{1}{2}k_2^2]\epsilon^{2+\frac{r}{3}\eta_4^{-2}},$$

只要选取$T_0\geq\frac{2\ln 2C_5^2\epsilon^2}{\nu}$, 那么$\exists\epsilon_8, 0<\epsilon\leq\epsilon_8$, 使得

$$e^{-\frac{\nu C_5^{-2}\epsilon^{-2}}{2}T_0}\leq\frac{1}{2},\quad 0<\epsilon\leq\epsilon_8.$$

因此,

$$\begin{aligned}||A_\epsilon^{\frac{1}{2}}\omega(t)||^2 &\leq [k_2^2e^{-\frac{\nu C_5^{-2}\epsilon^{-2}}{2}T_0}+\frac{1}{2}k_2^2]\epsilon^{2+\frac{r}{3}\eta_4^{-2}} \\ &\leq [\frac{1}{2}k_2^2+\frac{1}{2}k_2^2]\epsilon^{2+\frac{r}{3}\eta_4^{-2}} \\ &= k_2^2\epsilon^{2+\frac{r}{3}\eta_4^{-2}},\end{aligned}$$

由已知估计式可得

$$\begin{aligned}||A_\epsilon^{\frac{1}{2}}v(t)||^2 &\leq D_{22}(e^{-\nu\lambda_1 t}\eta^{-2}+\eta_2^{-2}+D_{20})e^{4D_{19}\eta_2^{-2}+1}\\ &\leq D_{22}(\eta_2^{-2}+\frac{1}{\sqrt{2D_{19}}}+D_{20})e^{4D_{19}\eta_2^{-2}+1}\\ &\leq D_{22}(\eta_2^{-2}+D_{24})e^{4D_{19}\eta_2^{-2}+1}\\ &=\Gamma(\eta_2^{-2}),\end{aligned}$$

那么有

$$\Gamma(\eta_2^{-2})=D_{22}\eta_2^{-2}e^{4D_{19}\eta_2^{-4}+1}+\frac{D_{22}}{\sqrt{2D_{19}}}e^{4D_{19}\eta_2^{-4}+1}+D_{20}D_{22}e^{4D_{19}\eta_2^{-4}+1}. \tag{4.31}$$

由假设条件$H(a,b)$, 知$\exists\epsilon_{10}$, f $u0<\epsilon\leq\epsilon_{10}$

$$\begin{aligned}D_{22}\eta_2^{-2}e^{4D_{19}\eta_2^{-4}+1}&=D_{22}\eta_2^{-2}e^{4D_{19}\eta_2^{-4}}e\\ &\leq\frac{1}{6}(4\eta_1^{-2}+k_1^2\eta_3^{-4}).\end{aligned}$$

类似地, $\exists\epsilon_{11}$, 对于$0<\epsilon\leq\epsilon_{11}$,

$$\frac{D_{22}}{\sqrt{2D_{19}}}e^{4D_{19}\eta_2^{-4}+1}\leq\frac{1}{6}(4\eta_1^{-2}+k_1^2\eta_3^{-4}).$$

类似地, $\exists\epsilon_{12}$, 对于$0<\epsilon\leq\epsilon_{12}$,

$$D_{20}D_{22}e^{4D_{19}\eta_2^{-4}+1}\leq\frac{1}{6}(4\eta_1^{-2}+k_1^2\eta_3^{-4}).$$

取$\epsilon_{13}=\min(\epsilon_{10},\epsilon_{11},\epsilon_{12})$, 使得

$$\Gamma(\eta_2^{-2})\leq\frac{1}{2}(4\eta_1^{-2}+k_1^2\eta_3^{-4}),\quad 0<\epsilon\leq\epsilon_{13}.$$

对于$0<\epsilon\leq\epsilon_{13}$,有

$$||A_\epsilon^{\frac{1}{2}}v(t)||^2\leq\frac{1}{2}(4\eta_1^{-2}+k_1^2\eta_3^{-4}),\quad T_0\leq t\leq 2T_0.$$

取$\epsilon_0=\epsilon_{13}$. 证毕.

定理 4.2 *(H^1正则性)*

设$\eta_i, i=1,2,3,4,r$,和p 满足条件$H1$ 和$H(a,b)$, 其中a和b充分大，那么存在$\epsilon_0>0, k_2>0$, 和一个连续函数$\Gamma\in C([0,\infty),R)$, 且对$\epsilon, 0<\epsilon\leq\epsilon_0$, 存在$\hat{T}_1=\hat{T}_1(\epsilon)>0$ 使得对$\epsilon, 0<\epsilon\leq\epsilon_0$, 当$u_0\in D(A_\epsilon^{\frac{1}{2}})$且$f\in L^\infty([0,\infty),L^2(Q_3))$ 满足假设条件(3.1).那么(2.5)存在唯一解u满足$C([0,\infty),V_\epsilon^1)\cap L^\infty([0,\infty),V_\epsilon^1)$, 即有

$$||A_\epsilon^{\frac{1}{2}}u(t)||^2\leq K_1^2,\quad t\geq 0,$$

其中K_1 依赖ν, λ_1, 和$\eta_i, i = 1, 2, 3, 4$, 但不依赖$t \geq 0$. 且v 和ω 满足

$$||A_\epsilon^{\frac{1}{2}} v(t)||^2 \leq \Gamma(\eta_2^{-2}), \quad t \geq \hat{T}_1,$$

其中Γ由(4.31)式给出，且

$$||A_\epsilon^{\frac{1}{2}} \omega(t)||^2 \leq k_2^2 \epsilon^{2+\frac{r}{3}} \eta_4^{-2}, \quad t \geq \hat{T}_1.$$

证明. 先利用引理(4.2), 再利用引理(4.3), 最后对引理(4.3)利用数学归纳法, 不难证得定理成立结论成立.

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