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关于第三类退化的Poly-Cauchy多项式的组合恒等式

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摘 要

本文利用发生函数和Riordan阵研究了第三类退化的Poly-Cauchy多项式相关的恒等式。首先，运用发生函数方法给出第三类退化的Poly-Cauchy多项式的性质，从而得到了关于第三类退化的Poly-Cauchy多项式的一些组合恒等式。其次，应用Riordan阵法，建立了第三类退化的Poly-Cauchy多项式与两类Stirling数、Lab数、Bell数之间的一些关系式。

关键词

Cauchy多项式，发生函数，Riordan阵，第三类退化的Poly-Cauchy多项式，Stirling数，Lab数

Some Identities on Degenerate Poly-Cauchy Polynomials of the Third Kind

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Abstract

In this paper, using generating functions and Riordan arrays, we establish some identities involving the degenerate Poly-Cauchy polynomials of the third kind. Using the generating functions, we explore some properties of the degenerate Poly-Cauchy polynomials of the third kind, and obtain some combinatorial identities involving the degenerate Poly-Cauchy polynomials of the third kind. In addition, using Riordan arrays, we give some interesting relations involving degenerate Poly-Cauchy polynomials of the third kind with the Stirling numbers of both kinds, the

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Lab numbers and the Bell numbers.

Keywords

Cauchy Polynomials, Generating Functions, Riordan Matrices, The Degenerate Poly-Cauchy Polynomials of Third Kind, Stirling Numbers, Lab Numbers

1. 预备知识

在文献[1]中，Cauchy 多项式 $C_n(x)$ 的发生函数定义为：

$$\sum_{n\geq 0} C_n(x)\frac{t^n}{n!}=\frac{t}{\ln(1+t)}(1+t)^x. \tag{1}$$

其中，当 $x=0$ 时，称 $C_n(0)=C_n$ 为 Cauchy 数。

首先，我们引入第三类退化的 Poly-Cauchy 多项式的定义，其发生函数的定义为(文献[2])：

$$\sum_{n\geq 0} C_{n,\lambda,3}(x)\frac{t^n}{n!}=\frac{\lambda\left(\left(1+\lambda\ln(1+t)\right)^{\frac{1}{\lambda}}-1\right)}{\ln\left(1+\lambda\ln(1+t)\right)}\left(1+\lambda\ln(1+t)\right)^{\frac{x}{\lambda}}. \tag{2}$$

其次，我们给出本文中用到的几种组合序列的发生函数的定义：即第一类 Stirling 数 $s(n,k)$；第一类无符号 Stirling 数 $|s(n,k)|$；第二类 Stirling 数 $S(n,k)$；Lab 数 $L(n,k)$；第一类 Bell 数 $B(n,k)$ 以及第二类 Bell 数 $\beta(n,k)$ 的发生函数定义如下(文献[3])：

$$\sum_{n\geq k} s(n,k)\frac{t^n}{n!}=\frac{1}{k!}\ln^k(1+t), \tag{3}$$

$$\sum_{n\geq k} |s(n,k)|\frac{t^n}{n!}=\frac{1}{k!}\left(\ln\frac{1}{1-t}\right)^k, \tag{4}$$

$$\sum_{n\geq k} S(n,k)\frac{t^n}{n!}=\frac{1}{k!}\left(\mathrm{e}^t-1\right)^k, \tag{5}$$

$$\sum_{n\geq k} L(n,k)\frac{t^n}{n!}=\frac{1}{k!}\left(\frac{-t}{1+t}\right)^k, \tag{6}$$

$$\sum_{n\geq k} B(n,k)\frac{t^n}{n!}=\frac{1}{k!}\left(\exp\left(\mathrm{e}^t-1\right)-1\right)^k, \tag{7}$$

$$\sum_{n\geq k} \beta(n,k)\frac{t^n}{n!}=\frac{1}{k!}\ln^k\left(1+\ln(1+t)\right). \tag{8}$$

下面引进一个引理：

引理 1 令 $D=\left(g(t),f(t)\right)=\left\{d_{n,k}\right\}_{n,k\in N}$ 为一个 Riordan 阵，令 $h(t)=\sum_{k\geq 0}h_k t^k$ 为序列 $\left\{h_n\right\}_{n\in N}$ 的发生函数，则有(文献[3])

$$\sum_{k=0}^{n} d_{n,k} h_k = \left[t^n\right] g(t) h\left(f(t)\right) = \left[t^n\right] g(t) h(y) \left(y = f(t)\right). \tag{9}$$

2. 第三类退化的 Poly-Cauchy 多项式与一些组合数间的关系

定理 1 设 n 为非负整数，则有

$$C_{n,\lambda,3}(x+y) = \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}\left(\frac{y}{\lambda}\right)_j \lambda^j s(k,j) C_{n-k,\lambda,3}(x). \tag{10}$$

证明： 由第三类退化的 Poly-Cauchy 多项式的发生函数(2)，可得

$$\begin{aligned}
\sum_{n\ge 0} C_{n,\lambda,3}(x+y)\frac{t^n}{n!} &= \frac{\lambda\left(\left(1+\lambda\ln(1+t)\right)^{\frac{1}{\lambda}}-1\right)}{\ln\left(1+\lambda\ln(1+t)\right)}\left(1+\lambda\ln(1+t)\right)^{\frac{x+y}{\lambda}} \\
&= \frac{\lambda\left(\left(1+\lambda\ln(1+t)\right)^{\frac{1}{\lambda}}-1\right)}{\ln\left(1+\lambda\ln(1+t)\right)}\left(1+\lambda\ln(1+t)\right)^{\frac{x}{\lambda}}\left(1+\lambda\ln(1+t)\right)^{\frac{y}{\lambda}} \\
&= \left(\sum_{n\ge 0} C_{n,\lambda,3}(x)\frac{t^n}{n!}\right)\left(\sum_{j\ge 0}\left(\frac{y}{\lambda}\right)_j \frac{\lambda^j \ln^j(1+t)}{j!}\right) \\
&= \left(\sum_{n\ge 0} C_{n,\lambda,3}(x)\frac{t^n}{n!}\right)\left(\sum_{j\ge 0}\left(\frac{y}{\lambda}\right)_j \lambda^j \sum_{n\ge j} s(n,j)\frac{t^n}{n!}\right) \\
&= \left(\sum_{n\ge 0} C_{n,\lambda,3}(x)\frac{t^n}{n!}\right)\left(\sum_{n\ge 0}\sum_{j=0}^{n}\left(\frac{y}{\lambda}\right)_j \lambda^j s(n,j)\frac{t^n}{n!}\right) \\
&= \sum_{n\ge 0}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}\left(\frac{y}{\lambda}\right)_j \lambda^j s(k,j) C_{n-k,\lambda,3}(x)\frac{t^n}{n!}
\end{aligned}$$

比较等式两边 $\frac{t^n}{n!}$ 的系数，可完成定理的证明。

定理 2 设 n 为非负整数，则有

$$C_{n,\lambda,3}(x+1) + C_{n,\lambda,3}(x) = \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{2}{\lambda}\right)_{j+1}\frac{\lambda^{k+1}}{j+1} C_{k-j}\left(\frac{x}{\lambda}\right) s(n,k). \tag{11}$$

证明： 由式(2)，可得

$$\begin{aligned}
&\sum_{n\ge 0} C_{n,\lambda,3}(x+1) + C_{n,\lambda,3}(x)\frac{t^n}{n!} \\
&= \frac{\lambda\left(\left(1+\lambda\ln(1+t)\right)^{\frac{1}{\lambda}}-1\right)}{\ln\left(1+\lambda\ln(1+t)\right)}\left(1+\lambda\ln(1+t)\right)^{\frac{x+1}{\lambda}} + \frac{\lambda\left(\left(1+\lambda\ln(1+t)\right)^{\frac{1}{\lambda}}-1\right)}{\ln\left(1+\lambda\ln(1+t)\right)}\left(1+\lambda\ln(1+t)\right)^{\frac{x}{\lambda}} \\
&= \frac{\lambda\left(\left(1+\lambda\ln(1+t)\right)^{\frac{1}{\lambda}}-1\right)}{\ln\left(1+\lambda\ln(1+t)\right)}\left(1+\lambda\ln(1+t)\right)^{\frac{x}{\lambda}}\left(\left(1+\lambda\ln(1+t)\right)^{\frac{1}{\lambda}}+1\right) \\
&= \frac{\lambda\left(\left(1+\lambda\ln(1+t)\right)^{\frac{2}{\lambda}}-1\right)}{\ln\left(1+\lambda\ln(1+t)\right)}\left(1+\lambda\ln(1+t)\right)^{\frac{x}{\lambda}}
\end{aligned}$$

$$=\lambda\left(\sum_{k\ge1}\left(\frac{2}{\lambda}\right)_k\frac{\lambda^k\ln^k(1+t)}{k!}\right)\frac{1}{\ln(1+\lambda\ln(1+t))}(1+\lambda\ln(1+t))^{\frac{x}{\lambda}}$$

$$=\lambda\left(\sum_{k\ge1}\left(\frac{2}{\lambda}\right)_k\frac{\lambda^k\ln^{k-1}(1+t)}{k!}\right)\frac{\lambda\ln(1+t)}{\ln(1+\lambda\ln(1+t))}(1+\lambda\ln(1+t))^{\frac{x}{\lambda}}$$

$$=\lambda\left(\sum_{k\ge0}\left(\frac{2}{\lambda}\right)_{k+1}\frac{1}{k+1}\frac{\lambda^k\ln^k(1+t)}{k!}\right)\left(\sum_{k\ge0}C_k\left(\frac{x}{\lambda}\right)\frac{\lambda^k\ln^k(1+t)}{k!}\right)$$

$$=\lambda\sum_{k\ge0}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{2}{\lambda}\right)_{j+1}\frac{\lambda^k}{j+1}C_{k-l}\left(\frac{x}{\lambda}\right)\frac{\ln^k(1+t)}{k!}$$

$$=\sum_{k\ge0}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{2}{\lambda}\right)_{j+1}\frac{\lambda^{k+1}}{j+1}C_{k-l}\left(\frac{x}{\lambda}\right)\sum_{n\ge k}s(n,k)\frac{t^n}{n!}$$

$$=\sum_{n\ge0}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}s(n,k)\left(\frac{2}{\lambda}\right)_{j+1}\frac{\lambda^{k+1}}{j+1}C_{k-l}\left(\frac{x}{\lambda}\right)\frac{t^n}{n!}$$

比较等式两边$\frac{t^n}{n!}$的系数，可得定理的证明。

在定理 2 中，令 $x=0$，可得如下推论。

推论 2 设 n 为非负整数，则有

$$C_{n,\lambda,3}(1)+C_{n,\lambda,3}(0)=\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{2}{\lambda}\right)_{j+1}\frac{\lambda^{k+1}}{j+1}C_{k-j}s(n,k).\tag{12}$$

定理 3 设 n 为非负整数，则有

$$C_{n,\lambda,3}(x-1)=\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\left\langle\frac{1}{\lambda}\right\rangle_{j+1}\frac{\lambda^{k+1}}{j+1}C_{k-j}\left(\frac{x}{\lambda}\right)s(n,k).\tag{13}$$

证明：由式(2)，可得

$$\sum_{n\ge0}C_{n,\lambda,3}(x-1)\frac{t^n}{n!}=\frac{\lambda\left((1+\lambda\ln(1+t))^{\frac{1}{\lambda}}-1\right)}{\ln(1+\lambda\ln(1+t))}(1+\lambda\ln(1+t))^{\frac{x-1}{\lambda}}$$

$$=\frac{\lambda\left((1+\lambda\ln(1+t))^{\frac{1}{\lambda}}-1\right)}{\ln(1+\lambda\ln(1+t))}\frac{(1+\lambda\ln(1+t))^{\frac{x}{\lambda}}}{(1+\lambda\ln(1+t))^{\frac{1}{\lambda}}}$$

$$=\frac{\lambda\left(1-(1+\lambda\ln(1+t))^{-\frac{1}{\lambda}}\right)}{\ln(1+\lambda\ln(1+t))}(1+\lambda\ln(1+t))^{\frac{x}{\lambda}}$$

$$=\frac{-\lambda\left((1+\lambda\ln(1+t))^{-\frac{1}{\lambda}}-1\right)}{\ln(1+\lambda\ln(1+t))}(1+\lambda\ln(1+t))^{\frac{x}{\lambda}}$$

$$=\frac{-\lambda\sum_{k\ge1}\left(-\frac{1}{\lambda}\right)_k\frac{\lambda^k\ln^k(1+t)}{k!}}{\ln(1+\lambda\ln(1+t))}(1+\lambda\ln(1+t))^{\frac{x}{\lambda}}$$

$$=-\lambda\sum_{k\geq 0}(-1)^{k+1}\left\langle\frac{1}{\lambda}\right\rangle_{k+1}\frac{\lambda^k\ln^k(1+t)}{(k+1)!}\frac{\lambda\ln(1+t)}{\ln(1+\lambda\ln(1+t))}(1+\lambda\ln(1+t))^{\frac{x}{\lambda}}$$

$$=\left(-\lambda\sum_{k\geq 0}(-1)^{k+1}\left\langle\frac{1}{\lambda}\right\rangle_{k+1}\frac{\lambda^k}{k+1}\frac{\ln^k(1+t)}{k!}\right)\left(\sum_{k\geq 0}\lambda^k C_k\left(\frac{x}{\lambda}\right)\frac{\ln^k(1+t)}{k!}\right)$$

$$=-\lambda\sum_{k\geq 0}\sum_{j=0}^{k}\binom{k}{j}(-1)^{j+1}\left\langle\frac{1}{\lambda}\right\rangle_{j+1}\frac{\lambda^k}{j+1}C_{k-j}\left(\frac{x}{\lambda}\right)\frac{\ln^k(1+t)}{k!}$$

$$=\sum_{k\geq 0}\sum_{j=0}^{k}\binom{k}{j}(-1)^{j}\left\langle\frac{1}{\lambda}\right\rangle_{j+1}\frac{\lambda^{k+1}}{j+1}C_{k-j}\left(\frac{x}{\lambda}\right)\sum_{n\geq k}s(n,k)\frac{t^n}{n!}$$

$$=\sum_{n\geq 0}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}(-1)^{j}\left\langle\frac{1}{\lambda}\right\rangle_{j+1}\frac{\lambda^{k+1}}{j+1}C_{k-j}\left(\frac{x}{\lambda}\right)s(n,k)\frac{t^n}{n!}$$

比较等式两边$\frac{t^n}{n!}$的系数，可以得到结论(13)。

在定理 3 中，令 $x=0$，即可得如下推论。

推论 3 设 n 为非负整数，则有

$$C_{n,\lambda,3}(-1)=\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}(-1)^{j}\left\langle\frac{1}{\lambda}\right\rangle_{j+1}\frac{\lambda^{k+1}}{j+1}C_{k-j}s(n,k). \tag{14}$$

定理 4 设 n 为非负整数，则有

$$(-1)^n C_{n,\lambda,3}(x)=\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k}\left(\frac{1}{\lambda}\right)_{j+1}\frac{\lambda^{k+1}}{j+1}C_{k-j}\left(\frac{x}{\lambda}\right)|s(n,k)|. \tag{15}$$

证明：由式(2)，可得

$$\sum_{n\geq 0}(-1)^n C_{n,\lambda,3}(x)\frac{t^n}{n!}=\frac{\lambda\left((1+\lambda\ln(1-t))^{\frac{1}{\lambda}}-1\right)}{\ln(1+\lambda\ln(1-t))}(1+\lambda\ln(1-t))^{\frac{x}{\lambda}}$$

$$=\frac{\lambda\left(\left(1-\lambda\ln\left(\frac{1}{1-t}\right)\right)^{\frac{1}{\lambda}}-1\right)}{\ln\left(1-\lambda\ln\left(\frac{1}{1-t}\right)\right)}\left(1-\lambda\ln\left(\frac{1}{1-t}\right)\right)^{\frac{x}{\lambda}}$$

$$=\lambda\frac{\sum_{k\geq 1}(-\lambda)^k\left(\frac{1}{\lambda}\right)_k\frac{1}{k!}\left(\ln\left(\frac{1}{1-t}\right)\right)^k}{\ln\left(1-\lambda\ln\left(\frac{1}{1-t}\right)\right)}\left(1-\lambda\ln\left(\frac{1}{1-t}\right)\right)^{\frac{x}{\lambda}}$$

$$=\lambda\sum_{k\geq 0}(-\lambda)^k\left(\frac{1}{\lambda}\right)_{k+1}\frac{1}{(k+1)!}\left(\ln\left(\frac{1}{1-t}\right)\right)^k\frac{-\lambda\ln\left(\frac{1}{1-t}\right)}{\ln\left(1-\lambda\ln\left(\frac{1}{1-t}\right)\right)}\left(1-\lambda\ln\left(\frac{1}{1-t}\right)\right)^{\frac{x}{\lambda}}$$

$$=\lambda\left(\sum_{k\geq 0}(-\lambda)^k\left(\frac{1}{\lambda}\right)_{k+1}\frac{1}{(k+1)!}\left(\ln\left(\frac{1}{1-t}\right)\right)^k\right)\left(\sum_{k\geq 0}(-\lambda)^k C_k\left(\frac{x}{\lambda}\right)\frac{1}{k!}\left(\ln\left(\frac{1}{1-t}\right)\right)^k\right)$$

$$=\lambda\sum_{k\geq 0}\sum_{j=0}^{k}\binom{k}{j}(-\lambda)^k\left(\frac{1}{\lambda}\right)_{j+1}\frac{1}{j+1}C_{k-j}\left(\frac{x}{\lambda}\right)\frac{1}{k!}\left(\ln\left(\frac{1}{1-t}\right)\right)^k$$

$$=\lambda\sum_{k\geq 0}\sum_{j=0}^{k}\binom{k}{j}(-\lambda)^k\left(\frac{1}{\lambda}\right)_{j+1}\frac{1}{j+1}C_{k-j}\left(\frac{x}{\lambda}\right)\sum_{n\geq k}\left|s(n,k)\right|\frac{t^n}{n!}$$

$$=\sum_{n\geq 0}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}(-1)^k\left(\frac{1}{\lambda}\right)_{j+1}\frac{\lambda^{k+1}}{j+1}C_{k-j}\left(\frac{x}{\lambda}\right)\left|s(n,k)\right|\frac{t^n}{n!}$$

比较等式两边$\dfrac{t^n}{n!}$的系数，可完成定理的证明。

定理 5 设 n 为非负整数，则有

$$\sum_{k=0}^{n}S(n,k)C_{k,\lambda,3}(x)=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{\lambda}\right)_{k+1}\frac{\lambda^{n+1}}{k+1}C_{n-k}\left(\frac{x}{\lambda}\right). \tag{16}$$

证明： 因为$\left\{\frac{k!}{n!}S(n,k)\right\}=\left(1,e^t-1\right)$，再由引理 1 和式(2)，则有

$$\sum_{k=0}^{n}S(n,k)C_{k,\lambda,3}(x)=n!\sum_{k=0}^{n}\frac{k!}{n!}S(n,k)\frac{C_{k,\lambda,3}(x)}{k!}$$

$$=n!\left[t^n\right]\frac{\lambda\left(\left(1+\lambda\ln(1+y)\right)^{\frac{1}{\lambda}}-1\right)}{\ln\left(1+\lambda\ln(1+y)\right)}\left(1+\lambda\ln(1+y)\right)^{\frac{xz}{\lambda}}\left(y=e^t-1\right)$$

$$=n!\left[t^n\right]\frac{\lambda\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}{\ln(1+\lambda t)}(1+\lambda t)^{\frac{xz}{\lambda}}$$

$$=n!\left[t^n\right]\sum_{n\geq 1}\left(\frac{1}{\lambda}\right)_n\frac{(\lambda t)^n}{n!}\frac{\lambda}{\ln(1+\lambda t)}(1+\lambda t)^{\frac{xz}{\lambda}}$$

$$=n!\left[t^n\right]\sum_{n\geq 0}\left(\frac{1}{\lambda}\right)_{n+1}\frac{\lambda^{n+1}t^n}{(n+1)!}\frac{\lambda t}{\ln(1+\lambda t)}(1+\lambda t)^{\frac{xz}{\lambda}}$$

$$=n!\left[t^n\right]\left(\sum_{n\geq 0}\left(\frac{1}{\lambda}\right)_{n+1}\frac{\lambda^{n+1}}{n+1}\frac{t^n}{n!}\right)\left(\sum_{n\geq 0}\lambda^n C_n\left(\frac{x}{\lambda}\right)\frac{t^n}{n!}\right)$$

$$=n!\left[t^n\right]\sum_{n\geq 0}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{\lambda}\right)_{k+1}\frac{\lambda^{n+1}}{k+1}C_{n-k}\left(\frac{x}{\lambda}\right)\frac{t^n}{n!}$$

$$=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{\lambda}\right)_{k+1}\frac{\lambda^{n+1}}{k+1}C_{n-k}\left(\frac{x}{\lambda}\right)$$

下面引进序列$\{f_n\}_{n\in N}$和$\{g_n\}_{n\in N}$的反演公式

$$f_n=\sum_{k=0}^{n}S(n,k)g_k\Leftrightarrow g_n=\sum_{k=0}^{n}s(n,k)f_k. \tag{17}$$

由式(17)和定理 5 立即可得如下等式。

定理 6 设 n 为非负整数，则有

$$\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{1}{\lambda}\right)_{j+1}\frac{\lambda^{k+1}}{j+1}C_{k-j}\left(\frac{x}{\lambda}\right)s(n,k)=C_{n,\lambda,3}(x). \tag{18}$$

定理 7 设 n 为非负整数，则有

$$\sum_{k=0}^{n}\sum_{j=0}^{k}\lambda^{-k}S(n,k)S(k,j)C_{j,\lambda,3}(x)=\lambda^{-(n-1)}\frac{(x+1)^{n+1}-x^{n+1}}{n+1}. \tag{19}$$

证明：由定理 5 的证明可知

$$\sum_{k=0}^{n}S(n,k)C_{k,\lambda,3}(x)=\frac{\lambda\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}{\ln(1+\lambda t)}(1+\lambda t)^{\frac{x}{\lambda}}. \tag{20}$$

因为 $\left\{\frac{k!}{n!}S(n,k)\right\}=\left(1,\mathrm{e}^{t}-1\right)$，再由式(9)和式(20)，可得

$$\begin{aligned}
&\sum_{k=0}^{n}\sum_{j=0}^{k}\lambda^{-k}S(n,k)S(k,j)C_{j,\lambda,3}(x)\\
&=n!\sum_{k=0}^{n}\frac{k!}{n!}\lambda^{-k}S(n,k)\frac{\sum_{j=0}^{k}S(k,j)C_{j,\lambda,3}(x)}{k!}\\
&=n!\left[t^{n}\right]\frac{\lambda\left((1+\lambda y)^{\frac{1}{\lambda}}-1\right)}{\ln(1+\lambda y)}(1+\lambda y)^{\frac{x}{\lambda}}\left(y=\frac{\mathrm{e}^{t}-1}{\lambda}\right)\\
&=n!\left[t^{n}\right]\frac{\lambda\left(\left(1+\lambda\frac{\mathrm{e}^{t}-1}{\lambda}\right)^{\frac{1}{\lambda}}-1\right)}{\ln\left(1+\lambda\frac{\mathrm{e}^{t}-1}{\lambda}\right)}\left(1+\lambda\frac{\mathrm{e}^{t}-1}{\lambda}\right)^{\frac{x}{\lambda}}\\
&=n!\left[t^{n}\right]\frac{\lambda\left(\mathrm{e}^{\frac{t}{\lambda}}-1\right)}{t}\mathrm{e}^{\frac{xt}{\lambda}}=n!\left[t^{n+1}\right]\lambda\left(\mathrm{e}^{\frac{t}{\lambda}}-1\right)\mathrm{e}^{\frac{xt}{\lambda}}\\
&=\lambda^{-(n-1)}\frac{(x+1)^{n+1}-x^{n+1}}{n+1}
\end{aligned}$$

由式(17)和定理 7 立即可得如下结论。

定理 8 设 n 为非负整数，则有

$$\sum_{k=0}^{n}\sum_{j=0}^{k}\lambda^{-n+k+1}\frac{(x+1)^{n+1}-x^{n+1}}{n+1}s(n,k)s(k,j)=C_{n,\lambda,3}(x). \tag{21}$$

定理 9 设 n 为非负整数，则有

$$\sum_{k=0}^{n}\sum_{j=0}^{k}S(n,k)L(k,j)C_{j,\lambda,3}(x)=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{\lambda}\right)_{k+1}(-1)^{n}\frac{\lambda^{n+1}}{k+1}C_{n-k}\left(\frac{x}{\lambda}\right). \tag{22}$$

证明：因为 $\left\{\frac{k!}{n!}L(n,k)\right\}=\left(1,\frac{-t}{1+t}\right)$，再由式(9)和式(2)，可得

$$\sum_{k=0}^{n}L(n,k)C_{k,\lambda,3}(x)=n!\left[t^{n}\right]\frac{\lambda\left((1-\lambda\ln(1+t))^{\frac{1}{\lambda}}-1\right)}{\ln(1-\lambda\ln(1+t))}(1-\lambda\ln(1+t))^{\frac{x}{\lambda}}, \tag{23}$$

又由式(5)，可得

$$\sum_{k=0}^{n}\sum_{j=0}^{k}S(n,k)L(k,j)C_{j,\lambda,3}(x)=n!\sum_{k=0}^{n}\frac{\frac{k!}{n!}\sum_{j=0}^{k}L(k,j)C_{j,\lambda,3}(x)}{k!}S(n,k)$$

$$=n!\left[t^n\right]\frac{\lambda\left(\left(1-\lambda\ln(1+y)\right)^{\frac{1}{\lambda}}-1\right)}{\ln\left(1-\lambda\ln(1+y)\right)}\left(1-\lambda\ln(1+y)\right)^{\frac{xz}{\lambda}}\left(y=\mathrm{e}^t-1\right)$$

$$=n!\left[t^n\right]\frac{\lambda\left(\left(1-\lambda\ln(1+\mathrm{e}^t-1)\right)^{\frac{1}{\lambda}}-1\right)}{\ln\left(1-\lambda\ln(1+\mathrm{e}^t-1)\right)}\left(1-\lambda\ln(1+\mathrm{e}^t-1)\right)^{\frac{xz}{\lambda}}$$

$$=n!\left[t^n\right]\frac{\lambda\left((1-\lambda t)^{\frac{1}{\lambda}}-1\right)}{\ln(1-\lambda t)}(1-\lambda t)^{\frac{xz}{\lambda}}$$

$$=n!\left[t^n\right]\sum_{n\ge1}\left(\frac{1}{\lambda}\right)_n\frac{(-\lambda t)^n}{n!}\frac{\lambda}{\ln(1-\lambda t)}(1-\lambda t)^{\frac{xz}{\lambda}}$$

$$=n!\left[t^n\right]\sum_{n\ge0}\left(\frac{1}{\lambda}\right)_{n+1}\frac{\lambda^{n+1}(-t)^n}{(n+1)!}\frac{-\lambda t}{\ln(1-\lambda t)}(1-\lambda t)^{\frac{xz}{\lambda}}$$

$$=n!\left[t^n\right]\left(\sum_{n\ge0}(-1)^n\left(\frac{1}{\lambda}\right)_{n+1}\frac{\lambda^{n+1}}{n+1}\frac{t^n}{n!}\right)\left(\sum_{n\ge0}(-\lambda)^n C_n\left(\frac{x}{\lambda}\right)\frac{t^n}{n!}\right)$$

$$=n!\left[t^n\right]\left(\frac{1}{\lambda}\right)_{k+1}\frac{(-1)^n\lambda^{n+1}}{k+1}C_{n-k}\left(\frac{x}{\lambda}\right)\frac{t^n}{n!}$$

$$=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{\lambda}\right)_{k+1}(-1)^n\frac{\lambda^{n+1}}{k+1}C_{n-k}\left(\frac{x}{\lambda}\right)$$

由式(17)和定理 9，可得如下结论。

定理 10 设 n 为非负整数，则有

$$\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{1}{\lambda}\right)_{j+1}\frac{(-\lambda)^{k+1}}{j+1}C_{k-j}\left(\frac{x}{\lambda}\right)s(n,k)=\sum_{k=0}^{n}L(n,k)C_{k,\lambda,3}(x).\tag{24}$$

定理 11 设 n 为非负整数，则有

$$\begin{aligned}&\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{1}{\lambda}\right)_{j+1}\frac{\lambda^{k+1}}{j+1}C_{k-j}\left(\frac{x}{\lambda}\right)\left|s(n,k)\right|\\&=\sum_{k=0}^{n}\sum_{j=0}^{k}\left|s(n,k)\right|S(k,j)C_{j,\lambda,3}(x)=\sum_{k=0}^{n}(-1)^n L(n,k)C_{k,\lambda,3}(x)\end{aligned}\tag{25}$$

证明：由式(6)有

$$\sum_{n\ge k}L(n,k)\frac{(-t)^n}{n!}=\frac{1}{k!}\left(\frac{t}{1-t}\right)^k,\tag{26}$$

再由式(9)有

$$\sum_{k=0}^{n}(-1)^{n}L(n,k)C_{k,\lambda,3}(x)=n![t^{n}]\frac{\lambda\left(\left(1-\lambda\ln(1-t)\right)^{\frac{1}{\lambda}}-1\right)}{\ln\left(1-\lambda\ln(1-t)\right)}\left(1-\lambda\ln(1-t)\right)^{\frac{x}{\lambda}}$$

$$=n![t^{n}]\frac{\lambda\left(\left(1+\lambda\ln\left(\frac{1}{1-t}\right)\right)^{\frac{1}{\lambda}}-1\right)}{\ln\left(1+\lambda\ln\left(\frac{1}{1-t}\right)\right)}\left(1+\lambda\ln\left(\frac{1}{1-t}\right)\right)^{\frac{x}{\lambda}}$$

$$=n![t^{n}]\lambda\frac{\sum_{k\geq 1}\lambda^{k}\left(\frac{1}{\lambda}\right)_{k}\frac{1}{k!}\left(\ln\left(\frac{1}{1-t}\right)\right)^{k}}{\ln\left(1+\lambda\ln\left(\frac{1}{1-t}\right)\right)}\left(1+\lambda\ln\left(\frac{1}{1-t}\right)\right)^{\frac{x}{\lambda}}$$

$$=n![t^{n}]\lambda\sum_{k\geq 0}\lambda^{k}\left(\frac{1}{\lambda}\right)_{k+1}\frac{1}{(k+1)!}\left(\ln\left(\frac{1}{1-t}\right)\right)^{k}\frac{\lambda\ln\left(\frac{1}{1-t}\right)}{\ln\left(1+\lambda\ln\left(\frac{1}{1-t}\right)\right)}\left(1+\lambda\ln\left(\frac{1}{1-t}\right)\right)^{\frac{x}{\lambda}}$$

$$=n![t^{n}]\lambda\left(\sum_{k\geq 0}\lambda^{k}\left(\frac{1}{\lambda}\right)_{k+1}\frac{1}{(k+1)!}\left(\ln\left(\frac{1}{1-t}\right)\right)^{k}\right)\left(\sum_{k\geq 0}\lambda^{k}C_{k}\left(\frac{x}{\lambda}\right)\frac{1}{k!}\left(\ln\left(\frac{1}{1-t}\right)\right)^{k}\right)$$

$$=n![t^{n}]\lambda\sum_{k\geq 0}\sum_{j=0}^{k}\binom{k}{j}\lambda^{k}\left(\frac{1}{\lambda}\right)_{j+1}\frac{1}{j+1}C_{k-j}\left(\frac{x}{\lambda}\right)\frac{1}{k!}\left(\ln\left(\frac{1}{1-t}\right)\right)^{k}$$

$$=n![t^{n}]\lambda\sum_{k\geq 0}\sum_{j=0}^{k}\binom{k}{j}\lambda^{k}\left(\frac{1}{\lambda}\right)_{j+1}\frac{1}{j+1}C_{k-j}\left(\frac{x}{\lambda}\right)\sum_{n\geq k}\left|s(n,k)\right|\frac{t^{n}}{n!}$$

$$=\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{1}{\lambda}\right)_{j+1}\frac{\lambda^{k+1}}{j+1}C_{k-j}\left(\frac{x}{\lambda}\right)\left|s(n,k)\right|$$

而由式(14)和式(9)，还可以得到

$$\sum_{k=0}^{n}\sum_{j=0}^{k}\left|s(n,k)\right|S(k,j)C_{j,\lambda,3}(x)=n!\sum_{k=0}^{n}\frac{k!}{n!}\left|s(n,k)\right|\frac{\sum_{j=0}^{k}S(k,j)C_{j,\lambda,3}(x)}{k!}$$

$$=n![t^{n}]\frac{\lambda\left((1+\lambda y)^{\frac{1}{\lambda}}-1\right)}{\ln(1+\lambda y)}(1+\lambda y)^{\frac{x}{\lambda}}\left(y=\ln\left(\frac{1}{1-t}\right)\right)$$

$$=n![t^{n}]\frac{\lambda\left(\left(1+\lambda\ln\left(\frac{1}{1-t}\right)\right)^{\frac{1}{\lambda}}-1\right)}{\ln\left(1+\lambda\ln\left(\frac{1}{1-t}\right)\right)}\left(1+\lambda\ln\left(\frac{1}{1-t}\right)\right)^{\frac{x}{\lambda}}$$

$$=n![t^{n}]\frac{\lambda\left(\left(1-\lambda\ln(1-t)\right)^{\frac{1}{\lambda}}-1\right)}{\ln\left(1-\lambda\ln(1-t)\right)}\left(1-\lambda\ln(1-t)\right)^{\frac{x}{\lambda}}$$

$$=\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{1}{\lambda}\right)_{j+1}\frac{\lambda^{k+1}}{j+1}C_{k-j}\left(\frac{x}{\lambda}\right)\left|s(n,k)\right|$$

定理 12 设 n 为非负整数，则有

$$\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}S(n,k)\left(\frac{1}{\lambda}\right)_{j+1}\frac{\lambda^{k+1}}{j+1}C_{k-j}\left(\frac{x}{\lambda}\right)=\sum_{k=0}^{n}B(n,k)C_{k,\lambda,3}(x). \tag{27}$$

证明： 因为 $\left\{\frac{k!}{n!}B(n,k)\right\}=\left(1,\exp\left(\mathrm{e}^{t}-1\right)-1\right)$，再由式(9)和式(2)，可得

$$\begin{aligned}
&\sum_{k=0}^{n}B(n,k)C_{k,\lambda,3}(x)=n!\sum_{k=0}^{n}\frac{k!}{n!}B(n,k)\frac{C_{k,\lambda,3}(x)}{k!}\\
&=n!\left[t^{n}\right]\frac{\lambda\left(\left(1+\lambda\ln(1+y)\right)^{\frac{1}{\lambda}}-1\right)}{\ln\left(1+\lambda\ln(1+y)\right)}\left(1+\lambda\ln(1+y)\right)^{\frac{x}{\lambda}}\left(y=\exp\left(\mathrm{e}^{t}-1\right)-1\right)\\
&=n!\left[t^{n}\right]\frac{\lambda\left(\left(1+\lambda\ln\left(1+\exp\left(\mathrm{e}^{t}-1\right)-1\right)\right)^{\frac{1}{\lambda}}-1\right)}{\ln\left(1+\lambda\ln\left(1+\exp\left(\mathrm{e}^{t}-1\right)-1\right)\right)}\left(1+\lambda\ln\left(1+\exp\left(\mathrm{e}^{t}-1\right)-1\right)\right)^{\frac{x}{\lambda}}\\
&=n!\left[t^{n}\right]\frac{\lambda\left(\left(1+\lambda\left(\mathrm{e}^{t}-1\right)\right)^{\frac{1}{\lambda}}-1\right)}{\ln\left(1+\lambda\left(\mathrm{e}^{t}-1\right)\right)}\left(1+\lambda\left(\mathrm{e}^{t}-1\right)\right)^{\frac{x}{\lambda}}\\
&=n!\left[t^{n}\right]\lambda\sum_{k\ge1}\left(\frac{1}{\lambda}\right)_{k}\frac{\lambda^{k}\left(\mathrm{e}^{t}-1\right)^{k}}{k!}\frac{1}{\ln\left(1+\lambda\left(\mathrm{e}^{t}-1\right)\right)}\left(1+\lambda\left(\mathrm{e}^{t}-1\right)\right)^{\frac{x}{\lambda}}\\
&=n!\left[t^{n}\right]\lambda\sum_{k\ge1}\left(\frac{1}{\lambda}\right)_{k}\frac{\lambda^{k}\left(\mathrm{e}^{t}-1\right)^{k-1}}{k!}\frac{\lambda\left(\mathrm{e}^{t}-1\right)}{\ln\left(1+\lambda\left(\mathrm{e}^{t}-1\right)\right)}\left(1+\lambda\left(\mathrm{e}^{t}-1\right)\right)^{\frac{x}{\lambda}}\\
&=n!\left[t^{n}\right]\lambda\left(\sum_{k\ge0}\left(\frac{1}{\lambda}\right)_{k+1}\frac{\lambda^{k}\left(\mathrm{e}^{t}-1\right)^{k-1}}{(k+1)!}\right)\left(\sum_{k\ge0}C_{k}\left(\frac{x}{\lambda}\right)\frac{\lambda^{k}\left(\mathrm{e}^{t}-1\right)^{k}}{k!}\right)\\
&=n!\left[t^{n}\right]\sum_{k\ge0}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{1}{\lambda}\right)_{j+1}\frac{\lambda^{k+1}}{j+1}C_{k-j}\left(\frac{x}{\lambda}\right)\frac{\left(\mathrm{e}^{t}-1\right)^{k}}{k!}\\
&=n!\left[t^{n}\right]\sum_{k\ge0}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{1}{\lambda}\right)_{j+1}\frac{\lambda^{k+1}}{j+1}C_{k-j}\left(\frac{x}{\lambda}\right)\sum_{n\ge k}S(n,k)\frac{t^{n}}{n!}\\
&=n!\left[t^{n}\right]\sum_{n\ge0}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}S(n,k)\left(\frac{1}{\lambda}\right)_{j+1}\frac{\lambda^{k+1}}{j+1}C_{k-j}\left(\frac{x}{\lambda}\right)\frac{t^{n}}{n!}\\
&=\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}S(n,k)\left(\frac{1}{\lambda}\right)_{j+1}\frac{\lambda^{k+1}}{j+1}C_{k-j}\left(\frac{x}{\lambda}\right)
\end{aligned}$$

定理 13 设 n 为非负整数，则有

$$\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{1}{\lambda}\right)_{j+1}\frac{\lambda^{k+1}}{j+1}C_{k-j}\left(\frac{x}{\lambda}\right)\beta(n,k)=\sum_{k=0}^{n}C_{k,\lambda,3}(x)s(n,k). \tag{28}$$

证明： 由第三类退化的 Poly-Cauchy 多项式的发生函数(2)有

$$\sum_{k\ge0}C_{k,\lambda,3}(x)\frac{\ln^k(1+t)}{k!}=\sum_{k\ge0}C_{k,\lambda,3}(x)\sum_{n\ge k}s(n,k)\frac{t^n}{n!}=\sum_{n\ge0}\sum_{k=0}^{n}C_{k,\lambda,3}(x)s(n,k)\frac{t^n}{n!}$$
$$=\frac{\lambda\left(\left(1+\lambda\ln(1+\ln(1+t))\right)^{\frac{1}{\lambda}}-1\right)}{\ln\left(1+\lambda\ln(1+\ln(1+t))\right)}\left(1+\lambda\ln(1+\ln(1+t))\right)^{\frac{x}{\lambda}}$$
$$=\lambda\sum_{k\ge1}\left(\frac{1}{\lambda}\right)_k\frac{\lambda^k\ln^k(1+\ln(1+t))}{k!}\frac{1}{\ln\left(1+\lambda\ln(1+\ln(1+t))\right)}\left(1+\lambda\ln(1+\ln(1+t))\right)^{\frac{x}{\lambda}}$$
$$=\lambda\sum_{k\ge0}\left(\frac{1}{\lambda}\right)_{k+1}\frac{1}{k+1}\frac{\lambda^k\ln^k(1+\ln(1+t))}{k!}\frac{\lambda\ln(1+\ln(1+t))}{\ln\left(1+\lambda\ln(1+\ln(1+t))\right)}\left(1+\lambda\ln(1+\ln(1+t))\right)^{\frac{x}{\lambda}}$$
$$=\lambda\left(\sum_{k\ge0}\left(\frac{1}{\lambda}\right)_{k+1}\frac{\lambda^k}{k+1}\frac{\ln^k(1+\ln(1+t))}{k!}\right)\left(\sum_{k\ge0}C_k\left(\frac{x}{\lambda}\right)\frac{\lambda^k\ln^k(1+\ln(1+t))}{k!}\right)$$
$$=\sum_{k\ge0}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{1}{\lambda}\right)_{j+1}\frac{\lambda^{k+1}}{j+1}C_{k-j}\left(\frac{x}{\lambda}\right)\frac{\ln^k(1+\ln(1+t))}{k!}$$
$$=\sum_{k\ge0}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{1}{\lambda}\right)_{j+1}\frac{\lambda^{k+1}}{j+1}C_{k-j}\left(\frac{x}{\lambda}\right)\sum\beta(n,k)\frac{t^n}{n!}$$
$$=\sum_{n\ge0}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{1}{\lambda}\right)_{j+1}\frac{\lambda^{k+1}}{j+1}C_{k-j}\left(\frac{x}{\lambda}\right)\beta(n,k)\frac{t^n}{n!}$$
$$=\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{1}{\lambda}\right)_{j+1}\frac{\lambda^{k+1}}{j+1}C_{k-j}\left(\frac{x}{\lambda}\right)\beta(n,k)$$

比较等式两边 $\frac{t^n}{n!}$ 的系数，可完成定理的证明。

由式(17)和定理 13，立即可得如下结论。

定理 14 设 n 为非负整数，则有

$$\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{i=0}^{j}\binom{j}{i}\left(\frac{1}{\lambda}\right)_{i+1}\frac{\lambda^{j+1}}{i+1}C_{j-i}\left(\frac{x}{\lambda}\right)\beta(k,j)S(n,k)=C_{n,\lambda,3}(x).\quad(29)$$

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