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耦合KdV方程的相互作用解

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摘 要

随着人们对非线性方程的深入研究，经常通过探求耦合KdV方程的相互作用解来描述水波等自然运动规律。本文提出了一种求耦合KdV方程相互作用解的辅助方程新方法。该新方法可以很容易得到三角函数、指数函数、双曲函数和其他函数的混合函数解。利用该方法，我们成功地得到了耦合KdV方程的相互作用解。这些解在帮助物理学家准确分析相关领域中的特殊现象方面具有十分重要的理论意义和应用价值。

关键词

耦合KdV方程，相互作用解，辅助方程法

Interaction Solutions of the Coupled KdV Equations

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Abstract

In this paper, based on the auxiliary equation method, we obtain new interaction solutions of the coupled KdV equations, these solutions are degenerated to the solitary wave solutions, the triangle function solutions and other function solutions. It is significant to help physicists to analyze special phenomena in their relevant fields accurately.

Keywords

Coupled KdV Equations, Interaction Solution, Auxiliary Equation Method

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1. 引言

非线性偏微分方程的相互作用解在非线性光学、理论物理、等离子体物理、流体动力学、半导体等领域有着重要的应用。非线性偏微分方程的相互作用解包括了三角函数、指数函数、双曲函数和其他函数的混合函数解对于一些物理现象的研究有重要意义，而对这些解析解的研究有助于我们更好地理解复杂的物理现象。近几十年来，人们提出了许多求非线性偏微分方程精确解的方法，如：1) 逆散射理论[1]，Hirotas 双线性方法[2]；相对于逆散射理论方法而言，Hirotas 双线性方法也被称为直接方法，这种方法的优点在于它是一种代数而不是解析的方法，这种方法已经从求 KdV 方程、MKDV 方程等的孤立子解而发展成一种求解一大批非线性偏微分方程孤子解的相当普遍的方法，而这种方法的关键在于寻求相关变量变换。2) 截断 Painlev 展开法[3]、Darboux 变换法[4]；Darboux 变换法相对于截断 Painlev 展开法更加实用，Darboux 变换是求解非线性微分方程的一种非常实用的方法，比多尺度法、双线性变换法等其他方法更实用、更简单。近年来，人们研究了大量求解非线性偏微分方程的有效方法，它们是齐次平衡法[5]、正弦余弦法[6]、sech 函数法[7]、双曲正切函数法[8] [9]、多重 exp 函数法[10] [11]、雅可比椭圆函数展开法[12] [13]。而辅助方程法也是一种重要的方法，它在非线性微分方程的研究中扮演着很重要的角色，它不仅能求解一大批非线性偏微分方程的孤子解，也可以求几种类型的特殊解。它以简洁易懂的特质而备受关注。

众所周知，复杂的自然现象常常用非线性偏微分方程来描述。其中最具代表性的非线性方程是耦合 KdV 方程方程[14] [15]。本文利用一种新的辅助方程方法并结合齐次平衡法研究了耦合 KdV 方程：

$$u_t+\alpha_1 u_{xxx}-\alpha_2 uu_x+\alpha_3 vv_x=0 \tag{1.1}$$

$$v_t+\beta_1 v_{xxx}-\beta_2 uv_x-\beta_3 vu_x=0 \tag{1.2}$$

的精确解。

方程(1.1)和(1.2)描述了具有不同色散关系的两长波的相互作用。其中 $\alpha_1,\alpha_2,\alpha_3$ 和 β_1,β_2,β_3 都是常数。

2. 新辅助方程的新解

假设新的辅助方程为：

$$F''(\xi)=A_1+B_1F(\xi)+C_1F^3(\xi) \tag{2.1}$$

如果 $A_1=B_1=0, C_1\neq 0$，则我们由 maple 软件得到方程(2.1)的新的解为：

$$F_1(\xi)=\sqrt[3]{\frac{2}{C_1}}\left\{\ln\left[\frac{(\tanh(\xi)+\tan(\xi))e^{\xi}}{1+\tan(\xi)\coth(\xi)+\tanh(\xi)+\tan(\xi)}\right]\right\}^{-1} \tag{2.2}$$

$$F_2(\xi)=\sqrt[3]{\frac{2}{C_1}}\left\{\ln\left[\frac{(\tanh(\xi)+\coth(\xi))e^{\xi}}{1+\tanh(\xi)\tan(\xi)+\tan(\xi)+\coth(\xi)}\right]\right\}^{-1} \tag{2.3}$$

$$F_3(\xi)=\sqrt[3]{\frac{2}{C_1}}\left\{\ln\left[\frac{(\tanh(\xi)+\cot(\xi))e^{\xi}}{1+\cot(\xi)\coth(\xi)+\tanh(\xi)+\cot(\xi)}\right]\right\}^{-1} \tag{2.4}$$

$$F_4(\xi)=\sqrt[3]{\frac{2}{C_1}}\left\{\ln\left[\frac{(\cot(\xi)+\coth(\xi))e^{\xi}}{1+\tanh(\xi)\cot(\xi)+\cot(\xi)+\coth(\xi)}\right]\right\} \tag{2.5}$$

$$F_5(\xi)=\sqrt[3]{\frac{2}{C_1}}\left\{\ln\left[\frac{(1+\tan(\xi)\coth(\xi))e^{\xi}}{1+\tanh(\xi)+\tan(\xi)+\tan(\xi)\coth(\xi)}\right]\right\}^{-1} \tag{2.6}$$

$$F_6(\xi)=\sqrt[3]{\frac{2}{C_1}}\left\{\ln\left[\frac{(1+\cot(\xi)\coth(\xi))e^{\xi}}{1+\tanh(\xi)+\cot(\xi)+\cot(\xi)\coth(\xi)}\right]\right\}^{-1} \tag{2.7}$$

$$F_7(\xi)=\sqrt[3]{\frac{2}{C_1}}\left\{\ln\left[\frac{(1+\tanh(\xi)\cot(\xi))e^{\xi}}{1+\tan(\xi)+\coth(\xi)+\tanh(\xi)\tan(\xi)}\right]\right\}^{-1} \tag{2.8}$$

$$F_8(\xi)=\sqrt[3]{\frac{2}{C_1}}\left\{\ln\left[\frac{(1+\tanh(\xi)\cot(\xi))e^{\xi}}{1+\cot(\xi)+\coth(\xi)+\tanh(\xi)\cot(\xi)}\right]\right\}^{-1} \tag{2.9}$$

在求解的过程中我们忽略了一些虚解，因此将该辅助方程应用到非线性偏微分方程中时，我们就仅得到了非线性偏微分方程的实相互作用解。

3. 新的辅助方程法

在上一节中，我们已经求出了辅助方程的新解，为了更好地求解耦合 KdV 方程的相互作用解，本文介绍了一种新的辅助方程法。这种辅助方程法的步骤如下所示：

第一步：对于给定的具有自变量 x,y,t 等的非线性偏微分方程：

$$\begin{cases} P(u_t,u_{xx},u_{yy},v,\cdots)=0, \\ Q(v_t,v_{xx},v_{yy},u,\cdots)=0. \end{cases} \tag{3.1}$$

我们作如下变换：

$$\begin{cases} u=a_0+a_1F^2(\xi)+a_2G^2(\xi)+a_3F(\xi)G(\xi), \\ v=a_{10}+a_{11}F^2(\xi)+a_{12}G^2(\xi)+a_{13}F(\xi)G(\xi). \end{cases},\quad \xi=k_1(x-c_1t),\quad \eta=k_2(x-c_2t) \tag{3.2}$$

其中 a_i 为待定系数， $F(\xi)$ 满足(2.1)， $G(\eta)$ 满足下式。

$$G'(\eta)=A_2+B_2G^2(\eta) \tag{3.3}$$

第二步：将变换(3.2)代入方程(3.1)，再利用(2.1)和(3.1)，则方程左端化为关于 $F^i(\xi)\ G^j(\eta)\ (F'(\xi))^k$ 的多项式，令 $F^i(\xi)\ G^j(\eta)\ (F'(\xi))^k$ 的所有系数都等于零，则得到关于待定系数的代数方程组。因此，a_i 将通过解代数方程组来确定。然后我们将这个方法应用到耦合 KdV 方程中去，就能得到耦合 KdV 方程的相互作用解。

4. 耦合 KdV 方程的新相互作用解

对于求解耦合 KdV 方程的新相互作用解，

我们利用上述方法求解含有待定系数的代数方程组，

得到以下结果：

1)

$$a_0=\frac{1}{\beta_1(\alpha_2-\beta_2)}\left(4\alpha_1B_1k_1^2\beta_1-4B_1\beta_1^2k_1^2-a_{10}\alpha_1\beta_2\sqrt{\frac{\alpha_3\beta_1}{\alpha_1\beta_2-\alpha_1\beta_3+\alpha_2\beta_1}}\right.$$
$$-a_{10}\alpha_1\beta_3\sqrt{\frac{\alpha_3\beta_1}{\alpha_1\beta_2-\alpha_1\beta_3+\alpha_2\beta_1}}+a_{10}\alpha_2\beta_1\sqrt{\frac{\alpha_3\beta_1}{\alpha_1\beta_2-\alpha_1\beta_3+\alpha_2\beta_1}}$$
$$\left.+a_{10}\beta_1\beta_3\sqrt{\frac{\alpha_3\beta_1}{\alpha_1\beta_2-\alpha_1\beta_3+\alpha_2\beta_1}}\right)$$

$$a_1=a_{11}\sqrt{\frac{\alpha_3\beta_1}{\alpha_1\beta_2-\alpha_1\beta_3+\alpha_2\beta_1}},$$

$$C_1=\frac{\alpha_{11}(\beta_2+\beta_3)\sqrt{\frac{\alpha_3\beta_1}{\alpha_1\beta_2-\alpha_1\beta_3+\alpha_2\beta_1}}}{6\beta_1k_1^2},$$

$$A_1=a_{12}=a_{13}=a_2=a_3=0,$$

$$c_1=-\frac{1}{\beta_1(\alpha_2-\beta_2)}\left(4\alpha_1B_1k_1^2\beta_1\beta_2-4\alpha_2B_1\beta_1^2k_1^2-a_{10}\alpha_1\beta_2^2\sqrt{\frac{\alpha_3\beta_1}{\alpha_1\beta_2-\alpha_1\beta_3+\alpha_2\beta_1}}\right.$$
$$-a_{10}\alpha_1\beta_2\beta_3\sqrt{\frac{\alpha_3\beta_1}{\alpha_1\beta_2-\alpha_1\beta_3+\alpha_2\beta_1}}+a_{10}\alpha_2\beta_1\beta_2\sqrt{\frac{\alpha_3\beta_1}{\alpha_1\beta_2-\alpha_1\beta_3+\alpha_2\beta_1}} \tag{4.1}$$
$$\left.+a_{10}\alpha_2\beta_1\beta_3\sqrt{\frac{\alpha_3\beta_1}{\alpha_1\beta_2-\alpha_1\beta_3+\alpha_2\beta_1}}\right)$$

其中，B_1,k_1,a_0,a_{10} 和 a_{11} 都是任意常数。

2)

$$A_2=\frac{N_1}{96k_2\beta_1M},$$

$$B_2=\frac{\sqrt{\frac{a_{12}^2\alpha_3\beta_2^2+2a_{12}^2\alpha_3\beta_2\beta_3+a_{12}^2\alpha_3\beta_3^2}{144\alpha_1\beta_1\beta_2-144\alpha_1\beta_1\beta_3+144\beta_1^2\alpha_2}}}{k_2},$$

$$a_2=\frac{12\beta_1\sqrt{\frac{a_{12}^2\alpha_3\beta_2^2+2a_{12}^2\alpha_3\beta_2\beta_3+a_{12}^2\alpha_3\beta_3^2}{144\alpha_1\beta_1\beta_2-144\alpha_1\beta_1\beta_3+144\beta_1^2\alpha_2}}}{\beta_2+\beta_3},$$

$$c_2=-\frac{N_2}{12M},$$

$$A_1=a_1=a_{11}=a_{13}=a_3=0,$$

其中，

$$M=(\alpha_1\beta_2+\alpha_1\beta_3-\alpha_2\beta_1)\sqrt{\frac{a_{12}^2\alpha_3\beta_2^2+2a_{12}^2\alpha_3\beta_2\beta_3+a_{12}^2\alpha_3\beta_3^2}{144\alpha_1\beta_1\beta_2-144\alpha_1\beta_1\beta_3+144\beta_1^2\alpha_2}}(\alpha_1-\beta_1),$$

$$N_1 = 12a_0\alpha_1\alpha_2\beta_1\beta_2\sqrt{\frac{a_{12}^2\alpha_3\beta_2^2+2a_{12}^2\alpha_3\beta_2\beta_3+a_{12}^2\alpha_3\beta_3^2}{144\alpha_1\beta_1\beta_2-144\alpha_1\beta_1\beta_3+144\beta_1^2\alpha_2}}$$
$$+12a_0\alpha_1\alpha_2\beta_1\beta_3\sqrt{\frac{a_{12}^2\alpha_3\beta_2^2+2a_{12}^2\alpha_3\beta_2\beta_3+a_{12}^2\alpha_3\beta_3^2}{144\alpha_1\beta_1\beta_2-144\alpha_1\beta_1\beta_3+144\beta_1^2\alpha_2}}$$
$$-12a_0\alpha_1\beta_1\beta_2^2\sqrt{\frac{a_{12}^2\alpha_3\beta_2^2+2a_{12}^2\alpha_3\beta_2\beta_3+a_{12}^2\alpha_3\beta_3^2}{144\alpha_1\beta_1\beta_2-144\alpha_1\beta_1\beta_3+144\beta_1^2\alpha_2}}$$
$$-12a_0\alpha_1\beta_1\beta_2\beta_3\sqrt{\frac{a_{12}^2\alpha_3\beta_2^2+2a_{12}^2\alpha_3\beta_2\beta_3+a_{12}^2\alpha_3\beta_3^2}{144\alpha_1\beta_1\beta_2-144\alpha_1\beta_1\beta_3+144\beta_1^2\alpha_2}}$$
$$-12a_0\alpha_2^2\beta_1^2\sqrt{\frac{a_{12}^2\alpha_3\beta_2^2+2a_{12}^2\alpha_3\beta_2\beta_3+a_{12}^2\alpha_3\beta_3^2}{144\alpha_1\beta_1\beta_2-144\alpha_1\beta_1\beta_3+144\beta_1^2\alpha_2}}$$
$$+12a_0\alpha_2\beta_1^2\beta_2\sqrt{\frac{a_{12}^2\alpha_3\beta_2^2+2a_{12}^2\alpha_3\beta_2\beta_3+a_{12}^2\alpha_3\beta_3^2}{144\alpha_1\beta_1\beta_2-144\alpha_1\beta_1\beta_3+144\beta_1^2\alpha_2}}$$
$$-a_{10}a_{12}\alpha_3\alpha_1\beta_2^2-2a_{10}a_{12}\alpha_3\alpha_1\beta_2\beta_3-a_{10}a_{12}\alpha_3\alpha_1\beta_3^2+a_{10}a_{12}\alpha_3\alpha_2\beta_2\beta_1$$
$$+a_{10}a_{12}\alpha_3\alpha_2\beta_1\beta_3+a_{10}a_{12}\alpha_3\beta_2\beta_1\beta_3+a_{10}a_{12}\alpha_3\beta_1\beta_3^2,$$
$$N_2 = 12a_0\alpha_1^2\beta_2^2\sqrt{\frac{a_{12}^2\alpha_3\beta_2^2+2a_{12}^2\alpha_3\beta_2\beta_3+a_{12}^2\alpha_3\beta_3^2}{144\alpha_1\beta_1\beta_2-144\alpha_1\beta_1\beta_3+144\beta_1^2\alpha_2}}$$
$$+12a_0\alpha_1^2\beta_2\beta_3\sqrt{\frac{a_{12}^2\alpha_3\beta_2^2+2a_{12}^2\alpha_3\beta_2\beta_3+a_{12}^2\alpha_3\beta_3^2}{144\alpha_1\beta_1\beta_2-144\alpha_1\beta_1\beta_3+144\beta_1^2\alpha_2}}$$
$$-24a_0\alpha_1\alpha_2\beta_1\beta_2\sqrt{\frac{a_{12}^2\alpha_3\beta_2^2+2a_{12}^2\alpha_3\beta_2\beta_3+a_{12}^2\alpha_3\beta_3^2}{144\alpha_1\beta_1\beta_2-144\alpha_1\beta_1\beta_3+144\beta_1^2\alpha_2}} \tag{4.2}$$
$$-12a_0\alpha_1\beta_1\alpha_2\beta_3\sqrt{\frac{a_{12}^2\alpha_3\beta_2^2+2a_{12}^2\alpha_3\beta_2\beta_3+a_{12}^2\alpha_3\beta_3^2}{144\alpha_1\beta_1\beta_2-144\alpha_1\beta_1\beta_3+144\beta_1^2\alpha_2}}$$
$$+12a_0\alpha_2^2\beta_1^2\sqrt{\frac{a_{12}^2\alpha_3\beta_2^2+2a_{12}^2\alpha_3\beta_2\beta_3+a_{12}^2\alpha_3\beta_3^2}{144\alpha_1\beta_1\beta_2-144\alpha_1\beta_1\beta_3+144\beta_1^2\alpha_2}}$$
$$+a_{10}a_{12}\alpha_3\alpha_1\beta_2^2+a_{10}a_{12}\alpha_3\alpha_1\beta_2\beta_3-a_{10}a_{12}\alpha_3\alpha_2\beta_2\beta_1-a_{10}a_{12}\alpha_3\alpha_2\beta_3\beta_1,$$

其中，$a_0,a_{10},a_{12},k_1,k_2,c_1,B_1,C_1$ 都是任意常数。

利用解 1 和(2.1)的解，我们得到耦合 KdV 方程的相互作用解如下所示：

$$\begin{cases} u_1 = a_0 + a_1\left\{\sqrt[3]{\dfrac{2}{C_1}}\left\{\ln\left[\dfrac{(\tanh(\xi)+\tan(\xi))e^{\xi}}{1+\tan(\xi)\coth(\xi)+\tanh(\xi)+\tan(\xi)}\right]\right\}^{-1}\right\}^2, \\ v_1 = a_{10} + a_{11}\left\{\sqrt[3]{\dfrac{2}{C_1}}\left\{\ln\left[\dfrac{(\tanh(\xi)+\tan(\xi))e^{\xi}}{1+\tan(\xi)\coth(\xi)+\tanh(\xi)+\tan(\xi)}\right]\right\}^{-1}\right\}^2. \end{cases} \tag{4.3}$$

$$\begin{cases} u_2 = a_0 + a_1\left\{\sqrt[3]{\dfrac{2}{C_1}}\left\{\ln\left[\dfrac{(\tanh(\xi)+\coth(\xi))e^{\xi}}{1+\tanh(\xi)\tan(\xi)+\tan(\xi)+\coth(\xi)}\right]\right\}^{-1}\right\}^2, \\ v_2 = a_{10} + a_{11}\left\{\sqrt[3]{\dfrac{2}{C_1}}\left\{\ln\left[\dfrac{(\tanh(\xi)+\coth(\xi))e^{\xi}}{1+\tanh(\xi)\tan(\xi)+\tan(\xi)+\coth(\xi)}\right]\right\}^{-1}\right\}^2. \end{cases} \tag{4.4}$$

$$\begin{cases} u_3 = a_0 + a_1\left\{\sqrt[3]{\dfrac{2}{C_1}}\left\{\ln\left[\dfrac{\left(\tanh(\xi)+\cot(\xi)\right)e^{\xi}}{1+\cot(\xi)\coth(\xi)+\tanh((\xi)+\cot(\xi)}\right]\right\}^{-1}\right\}^2, \\ v_3 = a_{10} + a_{11}\left\{\sqrt[3]{\dfrac{2}{C_1}}\left\{\ln\left[\dfrac{\left(\tanh(\xi)+\cot(\xi)\right)e^{\xi}}{1+\cot(\xi)\coth(\xi)+\tanh((\xi)+\cot(\xi)}\right]\right\}^{-1}\right\}^2. \end{cases} \tag{4.5}$$

$$\begin{cases} u_4 = a_0 + a_1\left\{\sqrt[3]{\dfrac{2}{C_1}}\left\{\ln\left[\dfrac{\left(\cot(\xi)+\coth(\xi)\right)e^{\xi}}{1+\tanh(\xi)\cot(\xi)+\cot(\xi)+\coth(\xi)}\right]\right\}^{-1}\right\}^2, \\ v_4 = a_{10} + a_{11}\left\{\sqrt[3]{\dfrac{2}{C_1}}\left\{\ln\left[\dfrac{\left(\cot(\xi)+\coth(\xi)\right)e^{\xi}}{1+\tanh(\xi)\cot(\xi)+\cot(\xi)+\coth(\xi)}\right]\right\}^{-1}\right\}^2. \end{cases} \tag{4.6}$$

$$\begin{cases} u_5 = a_0 + a_1\left\{\sqrt[3]{\dfrac{2}{C_1}}\left\{\ln\left[\dfrac{\left(1+\tan(\xi)\coth(\xi)\right)e^{\xi}}{1+\tanh(\xi)+\tan(\xi)+\tan(\xi)\coth(\xi)}\right]\right\}^{-1}\right\}^2, \\ v_5 = a_{10} + a_{11}\left\{\sqrt[3]{\dfrac{2}{C_1}}\left\{\ln\left[\dfrac{\left(1+\tan(\xi)\coth(\xi)\right)e^{\xi}}{1+\tanh(\xi)+\tan(\xi)+\tan(\xi)\coth(\xi)}\right]\right\}^{-1}\right\}^2. \end{cases} \tag{4.7}$$

其中，$\xi = k_1(x - c_1 t)$。

至此，我们已经求得了耦合 KdV 方程相互作用解。如(4.3)和(4.7)所示。若用代数方程组的解 2，再利用(3.3)的解，还可得到耦合 KdV 方程的周期解和更多的孤立波解。

5. 总结

本文给出了一种新的辅助方程法，这种辅助方程法在求解耦合 KdV 方程的相互作用解以及其他非线性偏微分方程方面都有重要的应用。

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