Advances in Applied Mathematics 应用数学进展, 2024, 13(11), 5089-5099
Published Online November 2024 in Hans. https://www.hanspub.org/journal/aam
https://doi.org/10.12677/aam.2024.1311491

Hans汉斯

单边多线性极大算子在Campanato 空间上的有界性

李 婷

西北师范大学数学与统计学院，甘肃 兰州

收稿日期：2024年10月27日；录用日期：2024年11月21日；发布日期：2024年11月28日

摘 要

本文研究 Campanato 空间上单边多线性极大算子的映射性质。利用 Lebesgue 空间上的有界性，证明了单边多线性 Hardy-Littlewood 极大算子和分数次极大算子是从单边 Morrey 空间到单边 Campanato 空间上有界的。

关键词

单边 Morrey 空间，单边 Campanato 空间，单边多线性 Hardy-Littlewood 极大算子，单边多线性分数次极大算子

Boundedness of One-Sided Multilinear Maximal Operators on Campanato Spaces

Ting Li

College of Mathematics and Statistics, Northwest Normal University, Lanzhou Gansu

Received: Oct. 27th, 2024; accepted: Nov. 21st, 2024; published: Nov. 28th, 2024

文章引用: 李婷. 单边多线性极大算子在Campanato 空间上的有界性[J]. 应用数学进展, 2024, 13(11): 5089-5099.
DOI: 10.12677/aam.2024.1311491

Abstract

This paper studies the mapping properties of one-sided multilinear maximal operators on Campanato spaces. Using the boundedness on Lebesgue spaces, we prove that the one-sided multilinear Hardy-Littlewood maximal operators and fractional maximal operators are bounded from one-sided Morrey spaces to one-sided Campanato spaces.

Keywords

One-Sided Morrey Space, One-Sided Campanato Spaces, One-Sided Multilinear Hardy-Littlewood Maximal Operators, One-Sided Multilinear Fractional Maximal Operators

1. 引言及主要结果

Sawyer [1]首次引入了单边分数次算子, 并研究了单边 Hardy-Littlewood 极大算子在 Lebesgue 上的有界性. 随后, 单边算子理论受到人们的广泛研究 [2–5].

1927 年, Hardy 和 Littlewood [6]证明了分数次积分在 Lebesgue 空间的有界性.

定理 A [6]　设 $0 < s < n, 1 < p < q < \infty$ 且满足 $\frac{1}{p} - \frac{1}{q} = \frac{s}{n}$, 则存在常数 $C(n, s, p)$ 使得

$$\|I_s(f)\|_{L^q(\mathbb{R}^n)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)}.$$

其中

$$I_s(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-s}} dy.$$

设 $f_i : \mathbb{R} \longrightarrow \mathbb{R}$ 是可测函数, $i = 1, 2, \cdots, m$, $\overrightarrow{f} := (f_1, f_2, \cdots, f_m)$, p 是满足条件 $\frac{1}{p} = \sum_{i=1}^{m} \frac{1}{p_i}$ 的常数, 其中 $1 < p_i < \infty$, $i = 1, 2, \cdots, m, (m \in \mathbb{N})$. 1999年, Kenig 和 Stein [7]引入了多线性分数次

积分算子

$$\mathcal{I}_\gamma(\overrightarrow{f})(x)=\int_{(\mathbb{R}^n)^m}\frac{f_1(y_1)\cdots f_m(y_m)}{(|x-y_1|+\cdots+|x-y_m|)^{mn-\gamma}}d\overrightarrow{y}.$$

Moen [8]中证明了 $\mathcal{I}_\gamma$ 的 L^p-有界性.

定理 B [8] 设 $0<\gamma<mn, 1<p_i<\frac{mn}{\gamma}, f_i\in L^{p_i}(\mathbb{R}^n), \frac{1}{q}=\sum_{i=1}^m\frac{1}{q_i}$, 则存在与 f_i 无关的常数 $C>0$ 使得

$$\|\mathcal{I}_\gamma(\overrightarrow{f})\|_{L^q(\mathbb{R}^n)}\leqslant C\prod_{i=1}^m\|f_i\|_{L^{p_i}(\mathbb{R}^n)}.$$

单边分数次积分定义为 [1,9]

$$I_\alpha^+f(x)=\int_x^\infty\frac{f(y)}{(y-x)^{1-\alpha}}dy.$$

2023 年, Giorgi 和 Alexander [10]证明了单边多线性分数次极大算子在 L^p 上的有界性.

定理 C [10] 设 $0<\alpha<m, 1<p_i<\frac{m}{\alpha}, \frac{1}{q}=\sum_{i=1}^m\frac{1}{q_i}, \frac{1}{q_i}=\frac{1}{p_i}-\frac{\alpha}{m}, f_i\in L^{p_i}(\mathbb{R})$, 则存在常数 $C>0$ 使得

$$\|\mathcal{M}_\alpha^+(\overrightarrow{f})\|_{L^q(\mathbb{R})}\leqslant C\prod_{i=1}^m\|f_i\|_{L^{p_i}(\mathbb{R})},$$

$$\|\mathcal{M}_\alpha^-(\overrightarrow{f})\|_{L^q(\mathbb{R})}\leqslant C\prod_{i=1}^m\|f_i\|_{L^{p_i}(\mathbb{R})}.$$

其中

$$\mathcal{M}_\alpha^+(\overrightarrow{f})(x)=\sup_{h>0}\frac{1}{h^{1-\frac{\alpha}{m}}}\prod_{i=1}^m\int_x^{x+h}|f_i(y_i)|dy_i, \tag{1}$$

$$\mathcal{M}_\alpha^-(\overrightarrow{f})(x)=\sup_{h>0}\frac{1}{h^{1-\frac{\alpha}{m}}}\prod_{i=1}^m\int_{x-h}^{x}|f_i(y_i)|dy_i. \tag{2}$$

当 $m=1$ 时, 单边分数次极大算子为

$$M_\alpha^+(f)=\sup_{h>0}\frac{1}{h^{1-\alpha}}\int_x^{x+h}|f(y)|dy, \tag{3}$$

$$M_\alpha^-(f)=\sup_{h>0}\frac{1}{h^{1-\alpha}}\int_{x-h}^{x}|f(y)|dy. \tag{4}$$

单边算子的研究会伴着单边空间的出现, 更多的结果可参见 [11–18].

定义 1 [19] 设 $-\frac{1}{p}\leq\beta<0, 1\leq p<\infty$, 单边 Morrey 空间定义为

$$L_{p,\beta}^+(\mathbb{R})=\{f:\|f\|_{L_{p,\beta}^+(\mathbb{R})}<\infty\}$$

其中

$$\|f\|_{L^{+}_{p,\beta}(\mathbb{R})} = \sup_{x_0\in\mathbb{R}} \sup_{h>0} \frac{1}{h^{\beta}} \Big(\frac{1}{|(x_0-h,x_0)|} \int_{x_0}^{x_0+h} |f(x)|^p dx \Big)^{\frac{1}{p}}.$$

定义 2 [19] 设 $-\frac{1}{p} \leq \beta < 1, 1 \leq p < \infty$, 单边 Campanato 空间定义为

$$C^{+}_{p,\beta}(\mathbb{R}) = \{f : \|f\|_{C^{+}_{p,\beta}(\mathbb{R})} < \infty\},$$

其中

$$\|f\|_{C^{+}_{p,\beta}(\mathbb{R})} = \sup_{x_0\in\mathbb{R}} \sup_{h>0} \frac{1}{h^{\beta}} \Big(\frac{1}{|(x_0-h,x_0)|} \int_{x_0}^{x_0+h} |f(x) - f_{(x_0,x_0+h)}|^p dx \Big)^{\frac{1}{p}},$$

$$f_{(x_0,x_0+h)} = \frac{1}{|(x_0,x_0+h)|} \int_{x_0}^{x_0+h} f(x)dx.$$

石少广和傅尊伟 [19]证明了单边 Hardy-Littlewood 极大算子

$$M^{+}(f)(x) = \sup_{h>0} \frac{1}{h} \int_{x}^{x+h} |f(y)|dy, \;\; M^{-}(f)(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f(y)|dy$$

是从单边 Morrey 空间 $L^{+}_{p,\beta}(\mathbb{R})$ 到单边 Campanato 空间 $C^{+}_{p,\beta}(\mathbb{R})$ 上有界.

单边多线性 Hardy-Littlewood 极大算子定义为

$$\mathcal{M}^{+}(\overrightarrow{f})(x) = \sup_{h>0} \frac{1}{h^m} \prod_{i=1}^{m} \int_{x}^{x+h} |f_i(y_i)|dy_i, \tag{5}$$

$$\mathcal{M}^{-}(\overrightarrow{f})(x) = \sup_{h>0} \frac{1}{h^m} \prod_{i=1}^{m} \int_{x-h}^{x} |f_i(y_i)|dy_i. \tag{6}$$

受文 [10, 19]的启发, 本文主要目的是研究单边多线性 Hardy-Littlewood 极大算子和分数次极大算子是从单边 Morrey 空间到单边 Campanato 空间上有界的. 主要结果如下.

定理 1 设 $1 < p_i < \infty, -\frac{1}{p_i} \leqslant \beta_i < 0, \beta = \sum_{i=1}^{m} \beta_i, \frac{1}{p} = \sum_{i=1}^{m} \frac{1}{p_i}$, 单边多线性 Hardy-Littlewood 极大算子 $\mathcal{M}^{+}$ 由(5)式定义, 那么当 $1 < p_i < \frac{1}{1+\beta_i} (i = 1, 2, \cdots, m)$ 时, $\mathcal{M}^{+}$ 是从 $L^{+}_{p_1,\beta_1}(\mathbb{R}) \times \cdots L^{+}_{p_m,\beta_m}(\mathbb{R})$ 到单边 Campanato 空间 $C^{+}_{p,\beta}(\mathbb{R})$ 上有界的.

定理 2 设 $0 < \alpha < m, -\frac{1}{p_i} \leqslant \beta_i < 0, \frac{1}{q} = \sum_{i=1}^{m} \frac{1}{q_i}, \frac{1}{q_i} = \frac{1}{p_i} - \frac{\alpha}{m}, \beta_1 = \sum_{i=1}^{m} \beta_{1i}, \beta_2 = \sum_{1}^{m} \beta_{2i}, \beta_{2i} = \beta_{1i} - \frac{\alpha}{m}, i = 1, 2, \cdots, m$, 单边多线性分数次极大算子 $\mathcal{M}^{+}_{\alpha}$ 由(1)式定义, 那么当 $1 < p_i < q_i < \frac{1}{1+\beta_i} (i = 1, 2, \cdots, m)$ 时, $\mathcal{M}^{+}_{\alpha}$ 是从 $L^{+}_{p_1,\beta_{21}}(\mathbb{R}) \times \cdots L^{+}_{p_m,\beta_{2m}}(\mathbb{R})$ 到单边 Campanato 空间 $C^{+}_{q,\beta_1}(\mathbb{R})$ 上有界的.

2. 定理的证明

为了证明定理, 我们需要以下引理.

引理 1 [10] 设 $0<\alpha<1, 1<p<\frac{1}{\alpha}, \frac{1}{q}=\frac{1}{p}-\alpha$, 那么存在与 f 无关的常数 $C>0$ 使得

$$\Big(\int_{\mathbb{R}}|M_\alpha^+(f)|^q dx\Big)^{\frac{1}{q}} \leqslant C\Big(\int_{\mathbb{R}}|f(x)|^p dx\Big)^{\frac{1}{p}}.$$

引理 2 [10] 设 $0<\alpha<1, 1<p<\frac{1}{\alpha}, \frac{1}{q}=\frac{1}{p}-\alpha$, 那么存在与 f 无关的常数 $C>0$ 使得

$$\Big(\int_{\mathbb{R}}|M_\alpha^-(f)|^q dx\Big)^{\frac{1}{q}} \leqslant C\Big(\int_{\mathbb{R}}|f(x)|^p dx\Big)^{\frac{1}{p}}.$$

引理 3 [16] 设 $0<p<\infty$, 那么存在常数 $C>0$ 使得

$$\|M^+f\|_{L^p(\mathbb{R})} \leqslant C\|f\|_{L^p(\mathbb{R})}.$$

设 $x_0\in\mathbb{R}$，$h,\lambda>0$, 以下证明中, 记 $I=(x_0,x_0+h), I^+=(x_0+h,x_0+2h), I^-=(x_0-h,x_0)$, $\lambda I=(x_0,x_0+\lambda h)$.

定理 1的证明 只证 $m=2$, $m>2$ 的情形类似.

因为当 $\beta<0$ 时, $L^+_{p,\beta}(\mathbb{R})\subseteq C^+_{p,\beta}(\mathbb{R})$, 则只需证明存在 $C>0$, 使得

$$\frac{1}{h^\beta}\Big(\frac{1}{|(x_0-h,x_0)|}\int_{x_0}^{x_0+h}|\mathcal{M}^+(\overrightarrow{f})(x)|^p dx\Big)^{\frac{1}{p}} \leqslant C\prod_{i=1}^{2}\|f_i\|_{L^+_{p_i,\beta_i}(\mathbb{R})}.$$

记 $f_i=f_i^0+f_i^\infty$, 其中 $f_i^0=f\chi_{2I}, i=1,2$. 则

$$\begin{aligned}
&\frac{1}{h^\beta}\Big(\frac{1}{|(x_0-h,x_0)|}\int_{x_0}^{x_0+h}|\mathcal{M}^+(\overrightarrow{f})(x)|^p dx\Big)^{\frac{1}{p}}\\
\leqslant\ & h^{-\beta}h^{-\frac{1}{p}}\Big(\int_{x_0}^{x_0+h}|\mathcal{M}^+(f_1^0,f_2^0)(x)|^p dx\Big)^{\frac{1}{p}}+h^{-\beta}h^{-\frac{1}{p}}\Big(\int_{x_0}^{x_0+h}|\mathcal{M}^+(f_1^0,f_2^\infty)(x)|^p dx\Big)^{\frac{1}{p}}\\
+\ & h^{-\beta}h^{-\frac{1}{p}}\Big(\int_{x_0}^{x_0+h}|\mathcal{M}^+(f_1^\infty,f_2^0)(x)|^p dx\Big)^{\frac{1}{p}}+h^{-\beta}h^{-\frac{1}{p}}\Big(\int_{x_0}^{x_0+h}|\mathcal{M}^+(f_1^\infty,f_2^\infty)(x)|^p dx\Big)^{\frac{1}{p}}\\
=\ & I_1+I_2+I_3+I_4.
\end{aligned}$$

I_2 和 I_3 的估计方法类似, 只需估计 I_1,I_2,I_4. 由引理 5 和 $H\ddot{o}lder$ 不等式, 有

$$\begin{aligned}
I_1 \leqslant\ & h^{-\beta}h^{-\frac{1}{p}}\Big(\int_{\mathbb{R}}|\mathcal{M}^+(f_1^0,f_2^0)(x)|^p dx\Big)^{\frac{1}{p}}\\
\leqslant\ & Ch^{-\beta}h^{-\frac{1}{p}}\Big(\int_{x_0}^{x_0+2h}|f_1(x_1)|^{p_1}dx_1\Big)^{\frac{1}{p_1}}\Big(\int_{x_0}^{x_0+2h}|f_2(x_2)|^{p_2}dx_2\Big)^{\frac{1}{p_2}}\\
\leqslant\ & C\frac{1}{h^{\beta_1+\beta_2}}\Big(\frac{1}{|(x_0-2h,x_0)|}\int_{x_0}^{x_0+2h}|f_1(x_1)|^{p_1}dx_1\Big)^{\frac{1}{p_1}}\Big(\frac{1}{|(x_0-2h,x_0)|}\int_{x_0}^{x_0+2h}|f_2(x_2)|^{p_2}dx_2\Big)^{\frac{1}{p_2}}\\
\leqslant\ & C\prod_{i=1}^{2}\|f_i\|_{L^+_{p_i,\beta_i}(\mathbb{R})}.
\end{aligned}$$

对于 I_2, 根据 $\mathcal{M}^+(\overrightarrow{f})(x)$ 的定义可定义

$$\mathcal{M}^+(f_1^0, f_2^\infty)(x) = \sup_{h>0} \frac{1}{h}\int_{x_1}^{x_1+h} |f_1^0(x_1)|dx_1 \frac{1}{h}\int_{x_2}^{x_2+h} |f_2^\infty(x_2)|dx_2$$

则由定义及 $Hölder$ 不等式, 有

$$\begin{aligned}
&\sup_{h>0} \frac{1}{2h}\int_{x_0}^{x_0+2h} |f_1(x_1)|dx_1 \sum_{j=1}^{\infty} \frac{1}{2^j h}\int_{x_0+2h}^{x_0+2^{j+1}h} |f_2(x_2)|dx_2 \\
\leqslant\ & \sup_{h>0}(2h)^{1-\frac{1}{p_1}}\frac{1}{2h}\Big(\int_{x_0}^{x_0+2h} |f_1(x_1)|^{p_1}dx_1\Big)^{\frac{1}{p_1}} \sum_{j=1}^{\infty}(2^j h)^{1-\frac{1}{p_2}}\frac{1}{2^j h}\Big(\int_{x_0+2h}^{x_0+2^{j+1}h} |f_2(x_2)|^{p_2}dx_2\Big)^{\frac{1}{p_2}} \\
=\ & \sup_{h>0}(2h)^{-\frac{1}{p_1}}\Big(\int_{x_0}^{x_0+2h} |f_1(x_1)|^{p_1}dx_1\Big)^{\frac{1}{p_1}} \sum_{j=1}^{\infty} h^{-\frac{1}{p_2}}\Big(\frac{1}{2^j}\int_{x_0+2h}^{x_0+2^{j+1}h} |f_2(x_2)|^{p_2}dx_2\Big)^{\frac{1}{p_2}}
\end{aligned}$$

将上式代入 I_2 中, 有

$$\begin{aligned}
I_2 \quad \leqslant\ & \sup_{h>0} h^{-\beta}h^{-\frac{1}{p}}(2h)^{-\frac{1}{p_1}}\Big(\int_{x_0}^{x_0+2h} |f_1(x_1)|^{p_1}dx_1\Big)^{\frac{1}{p_1}} \sum_{j=1}^{\infty} h^{-\frac{1}{p_2}}\Big(\frac{1}{2^j}\int_{x_0+2h}^{x_0+2^{j+1}h} |f_2(x_2)|^{p_2}dx_2\Big)^{\frac{1}{p_2}} \\
=\ & \sup_{h>0} h^{-\beta}(2h)^{-\frac{1}{p_1}}(2h)^{\beta_1}\frac{1}{(2h)^{\beta_1}}\Big(\frac{|(x_0-2h,x_0)|}{|(x_0-2h,x_0)|}\int_{x_0}^{x_0+2h} |f_1(x)|^{p_1}dx_1\Big)^{\frac{1}{p_1}} \\
\times\ & \sum_{j=1}^{\infty} h^{-\frac{1}{p_2}}(2^j)^{-\frac{1}{p_2}}(2^j h)^{\beta_2}\frac{1}{(2^j h)^{\beta_2}}\Big(\frac{|(x_0-2^j h,x_0)|}{|(x_0-2^j h,x_0)|}\int_{x_0+2h}^{x_0+2^{j+1}h} |f_2(x_2)|^{p_2}dx_2\Big)^{\frac{1}{p_2}} \\
\leqslant\ & C\sup_{h>0}\sum_{j=1}^{\infty}(2^j)^{\beta_2}h^{\beta_1+\beta_2-\beta}\prod_{i=1}^{2}\|f_i\|_{L^+_{p_i,\beta_i}(\mathbb{R})} \\
\leqslant\ & C\prod_{i=1}^{2}\|f_i\|_{L^+_{p_i,\beta_i}(\mathbb{R})}.
\end{aligned}$$

同样, 有

$$\begin{aligned}
\mathcal{M}^+(f_1^\infty, f_2^\infty)(x) \leqslant\ & \sup_{h>0}\sum_{j=1}^{\infty}\frac{1}{2^j h}\int_{x_0+2h}^{x_0+2^{j+1}h} |f_1(x_1)|dx_1 \sum_{j=1}^{\infty}\frac{1}{2^j h}\int_{x_0+2h}^{x_0+2^{j+1}h} |f_2(x_2)|dx_2 \\
\leqslant\ & \sup_{h>0}\sum_{j=1}^{\infty}(2^j h)^{1-\frac{1}{p_1}}\frac{1}{2^j h}\Big(\int_{x_0+2h}^{x_0+2^{j+1}h} |f_1(x_1)|^{p_1}dx_1\Big)^{\frac{1}{p_1}} \\
\times\ & \sum_{j=1}^{\infty}(2^j h)^{1-\frac{1}{p_2}}\frac{1}{2^j h}\Big(\int_{x_0+2h}^{x_0+2^{j+1}h} |f_2(x_2)|^{p_2}dx_2\Big)^{\frac{1}{p_2}} \\
=\ & \sup_{h>0}\sum_{j=1}^{\infty}(h)^{-\frac{1}{p_1}}\Big(\frac{1}{2^j}\int_{x_0+2h}^{x_0+2^{j+1}h} |f_1(x_1)|^{p_1}dx_1\Big)^{\frac{1}{p_1}} \\
\times\ & \sum_{j=1}^{\infty} h^{-\frac{1}{p_2}}\Big(\frac{1}{2^j}\int_{x_0+2h}^{x_0+2^{j+1}h} |f_2(x_2)|^{p_2}dx_2\Big)^{\frac{1}{p_2}}
\end{aligned}$$

代入 I_4 中, 有

$$\begin{aligned}
I_4 &\leqslant \sup_{h>0} h^{-\beta} h^{-\frac{1}{p}} h^{-\frac{1}{p_1}} \sum_{j=1}^{\infty} \Big(\frac{1}{2^j} \int_{x_0+2h}^{x_0+2^{j+1}h} |f_1(x_1)|^{p_1} dx_1\Big)^{\frac{1}{p_1}} \\
&\times \sum_{j=1}^{\infty} h^{-\frac{1}{p_2}} h^{\frac{1}{p}} h^{-\frac{1}{p_1}} \Big(\frac{1}{2^j} \int_{x_0+2h}^{x_0+2^{j+1}h} |f_2(x_2)|^{p_2} dx_2\Big)^{\frac{1}{p_2}} \\
&\leqslant \sup_{h>0} \sum_{j=1}^{\infty} h^{-\beta} (2h)^{-\frac{1}{p_1}} (2^j h)^{\beta_1} \frac{1}{(2^j h)^{\beta_1}} \Big(\frac{|(x_0-2^j h, x_0)|}{|(x_0-2^j h, x_0)|} \int_{x_0+2h}^{x_0+2^{j+1}h} |f_1(x_1)|^{p_1} dx_1\Big)^{\frac{1}{p_1}} \\
&\times \sum_{j=1}^{\infty} h^{-\frac{1}{p_2}} (2^j)^{-\frac{1}{p_2}} (2^j h)^{\beta_2} \frac{1}{(2^j h)^{\beta_2}} \Big(\frac{|(x_0-2^j h, x_0)|}{|(x_0-2^j h, x_0)|} \int_{x_0+2h}^{x_0+2^{j+1}h} |f_2(x_2)|^{p_2} dx_2\Big)^{\frac{1}{p_2}} \\
&\leqslant C \sup_{h>0} \sum_{j=1}^{\infty} (2^j)^{\beta} h^{\beta_1+\beta_2-\beta} \prod_{i=1}^{2} \|f_i\|_{L^+_{p_i,\beta_i}(\mathbb{R})} \leqslant C \prod_{i=1}^{2} \|f_i\|_{L^+_{p_i,\beta_i}(\mathbb{R})}.
\end{aligned}$$

因此定理 1 证毕.

定理 2的证明 同定理1 的证明, 只证 $m=2$ 的情形. 由于当 $\beta<0$ 时, $L^+_{p,\beta}(\mathbb{R}) \subseteq C^+_{p,\beta}(\mathbb{R})$, 则只需证明存在 $C>0$, 使得

$$\frac{1}{h^{\beta_1}} \Big(\frac{1}{|(x_0-h, x_0)|} \int_{x_0}^{x_0+h} |\mathcal{M}^+_{\alpha}(\overrightarrow{f})(x)|^q dx\Big)^{\frac{1}{q}} \leqslant C \prod_{i=1}^{2} \|f_i\|_{L^+_{p_i,\beta_{2i}}(\mathbb{R})}.$$

现将 f_i 分解为 $f_i = f_i^0 + f_i^{\infty}$, 其中 $f_i^0 = f_{\chi_{2I}}, i=1,2$. 则

$$\begin{aligned}
&\frac{1}{h^{\beta_1}} \Big(\frac{1}{|(x_0-h, x_0)|} \int_{x_0}^{x_0+h} |\mathcal{M}^+_{\alpha}(\overrightarrow{f})(x)|^q dx\Big)^{\frac{1}{q}} \\
\leqslant\ & h^{-\beta_1} h^{-\frac{1}{q}} \Big(\int_{x_0}^{x_0+h} |\mathcal{M}^+_{\alpha}(f_1^0, f_2^0)(x)|^q dx\Big)^{\frac{1}{q}} + h^{-\beta_1} h^{-\frac{1}{q}} \Big(\int_{x_0}^{x_0+h} |\mathcal{M}^+_{\alpha}(f_1^0, f_2^{\infty})(x)|^q dx\Big)^{\frac{1}{q}} \\
+\ & h^{-\beta_1} h^{-\frac{1}{q}} \Big(\int_{x_0}^{x_0+h} |\mathcal{M}^+_{\alpha}(f_1^{\infty}, f_2^0)(x)|^q dx\Big)^{\frac{1}{q}} + h^{-\beta_1} h^{-\frac{1}{q}} \Big(\int_{x_0}^{x_0+h} |\mathcal{M}^+_{\alpha}(f_1^{\infty}, f_2^{\infty})(x)|^q dx\Big)^{\frac{1}{q}} \\
=\ & \widetilde{I}_1 + \widetilde{I}_2 + \widetilde{I}_3 + \widetilde{I}_4.
\end{aligned}$$

$\widetilde{I}_2$ 和 $\widetilde{I}_3$ 的估计方法类似, 只需估计 $\widetilde{I}_1, \widetilde{I}_2, \widetilde{I}_4$. 由引理 3 , 有

$$\begin{aligned}
\widetilde{I}_1 &\leqslant h^{-\beta_1} h^{-\frac{1}{q}} \Big(\int_{\mathbb{R}} |\mathcal{M}^+_{\alpha}(f_1^0, f_2^0)(x)|^q dx\Big)^{\frac{1}{q}} \\
&\leqslant C h^{-\beta_1} h^{-\frac{1}{q}} \Big(\int_{x_0}^{x_0+2h} |f_1(x_1)|^{p_1} dx_1\Big)^{\frac{1}{p_1}} \Big(\int_{x_0}^{x_0+2h} |f_2(x_2)|^{p_2} dx_2\Big)^{\frac{1}{p_2}} \\
&\leqslant C h^{\beta_2-\beta_1+\alpha} \prod_{i=1}^{2} \|f_i\|_{L^+_{p_i,\beta_{2i}}(\mathbb{R})} \\
&\leqslant C \prod_{i=1}^{2} \|f_i\|_{L^+_{p_i,\beta_{2i}}(\mathbb{R})}.
\end{aligned}$$

$$
\begin{aligned}
\mathcal{M}_\alpha^+(f_1^0,f_2^\infty)(x) &\leqslant \sup_{h>0}\frac{1}{(2h)^{1-\alpha}}\int_{x_0}^{x_0+2h}|f_1(x_1)|dx_1\sum_{j=1}^\infty\frac{1}{(2^jh)^{1-\frac{\alpha}{2}}}\int_{x_0+2h}^{x_0+2^{j+1}h}|f_2(x_2)|dx_2\\
&\leqslant \sup_{h>0}(2h)^{1-\frac{1}{p_1}}\frac{1}{(2h)^{1-\alpha}}\Big(\int_{x_0}^{x_0+2h}|f_1(x_1)|^{p_1}dx_1\Big)^{\frac{1}{p_1}}\\
&\times\sum_{j=1}^\infty(2^jh)^{1-\frac{1}{p_2}}\frac{1}{(2^jh)^{1-\frac{\alpha}{2}}}\Big(\int_{x_0+2h}^{x_0+2^{j+1}h}|f_2(x_2)|^{p_2}dx_2\Big)^{\frac{1}{p_2}}\\
&= \sup_{h>0}(2h)^{\alpha-\frac{1}{p_1}}\Big(\int_{x_0}^{x_0+2h}|f_1(x_1)|^{p_1}dx_1\Big)^{\frac{1}{p_1}}\\
&\times\sum_{j=1}^\infty h^{\frac{\alpha}{2}-\frac{1}{p_2}}(2^j)^{\frac{\alpha}{2}}\Big(\frac{1}{2^j}\int_{x_0+2h}^{x_0+2^{j+1}h}|f_2(x_2)|^{p_2}dx_2\Big)^{\frac{1}{p_2}}
\end{aligned}
$$

代入 $\widetilde{I}_2$中, 有

$$
\begin{aligned}
\widetilde{I}_2 &\leqslant \sup_{h>0}h^{-\beta_1}h^{-\frac{1}{q}}(2h)^{\alpha-\frac{1}{p_1}}\Big(\int_{x_0}^{x_0+2h}|f_1(x_1)|^{p_1}dx_1\Big)^{\frac{1}{p_1}}\\
&\times\sum_{j=1}^\infty h^{\frac{\alpha}{2}-\frac{1}{p_2}}(2^j)^{\frac{\alpha}{2}}h^{\frac{1}{q}}\Big(\frac{1}{2^j}\int_{x_0+2h}^{x_0+2^{j+1}h}|f_2(x_2)|^{p_2}dx_2\Big)^{\frac{1}{p_2}}\\
&\leqslant C\sup_{h>0}h^{-\beta_1+\alpha-\frac{1}{p_1}}((2h)^{\beta_{21}}\frac{1}{(2h)^{\beta_{21}}}\Big(\frac{|(x_0-2h,x_0)|}{|(x_0-2h,x_0)|}\int_{x_0}^{x_0+2h}|f_1(x)|^{p_1}dx_1\Big)^{\frac{1}{p_1}}\\
&\times\sum_{j=1}^\infty h^{\frac{\alpha}{2}-\frac{1}{p_2}}(2^j)^{\frac{\alpha}{2}-\frac{1}{p_2}}(2^jh)^{\beta_{22}}\frac{1}{(2^jh)^{\beta_{22}}}\Big(\frac{|(x_0-2^jh,x_0)|}{|(x_0-2^jh,x_0)|}\int_{x_0+2h}^{x_0+2^{j+1}h}|f_2(x_2)|^{p_2}dx_2\Big)^{\frac{1}{p_2}}\\
&\leqslant C\sum_{j=1}^\infty(2^j)^{\beta_{22}+\frac{\alpha}{2}}\prod_{i=1}^2\|f_i\|_{L^+_{p_i,\beta_{2i}}(\mathbb{R})}\leqslant C\sum_{j=1}^\infty(2^j)^{\beta_{12}}\prod_{i=1}^2\|f_i\|_{L^+_{p_i,\beta_{2i}}(\mathbb{R})}\\
&\leqslant C\prod_{i=1}^2\|f_i\|_{L^+_{p_i,\beta_{2i}}(\mathbb{R})}.
\end{aligned}
$$

同样, 有

$$
\begin{aligned}
\mathcal{M}_\alpha^+(f_1^\infty,f_2^\infty)(x) &\leqslant \sup_{h>0}\sum_{j=1}^\infty\frac{1}{(2^jh)^{1-\alpha}}\int_{x_0+2h}^{x_0+2^{j+1}h}|f_1(x_1)|dx_1\sum_{j=1}^\infty\frac{1}{(2^jh)^{1-\frac{\alpha}{2}}}\int_{x_0+2h}^{x_0+2^{j+1}h}|f_2(x_2)|dx_2\\
&\leqslant \sup_{h>0}\sum_{j=1}^\infty(2^jh)^{1-\frac{1}{p_1}}\frac{1}{(2^jh)^{1-\alpha}}\Big(\int_{x_0+2h}^{x_0+2^{j+1}h}|f_1(x_1)|^{p_1}dx_1\Big)^{\frac{1}{p_1}}\\
&\times\sum_{j=1}^\infty(2^jh)^{1-\frac{1}{p_2}}\frac{1}{(2^jh)^{1-\frac{\alpha}{2}}}\Big(\int_{x_0+2h}^{x_0+2^{j+1}h}|f_2(x_2)|^{p_2}dx_2\Big)^{\frac{1}{p_2}}\\
&= \sup_{h>0}\sum_{j=1}^\infty(h)^{\alpha-\frac{1}{p_1}}(2^j)^{\alpha}\Big(\frac{1}{2^j}\int_{x_0+2h}^{x_0+2^{j+1}h}|f_1(x_1)|^{p_1}dx_1\Big)^{\frac{1}{p_1}}\\
&\times\sum_{j=1}^\infty h^{\frac{\alpha}{2}-\frac{1}{p_2}}(2^j)^{\frac{\alpha}{2}}\Big(\frac{1}{2^j}\int_{x_0+2h}^{x_0+2^{j+1}h}|f_2(x_2)|^{p_2}dx_2\Big)^{\frac{1}{p_2}}
\end{aligned}
$$

代入 $\widetilde{I}_4$ 中, 有

$$
\begin{aligned}
\widetilde{I}_4 \quad &\leqslant \quad \sup_{h>0} h^{-\beta_1} h^{-\frac{1}{q}} \sum_{j=1}^{\infty} (h)^{\alpha-\frac{1}{p_1}} (2^j)^{\alpha} \left(\frac{1}{2^j} \int_{x_0+2h}^{x_0+2^{j+1}h} |f_1(x_1)|^{p_1} dx_1 \right)^{\frac{1}{p_1}} \\
&\quad \times \quad \sum_{j=1}^{\infty} h^{\frac{\alpha}{2}-\frac{1}{p_2}} (2^j)^{\frac{\alpha}{2}} \left(\frac{1}{2^j} \int_{x_0+2h}^{x_0+2^{j+1}h} |f_2(x_2)|^{p_2} dx_2 \right)^{\frac{1}{p_2}} \\
&\leqslant \quad C \sup_{h>0} \sum_{j=1}^{\infty} h^{-\beta_1} (2^j h)^{\alpha-\frac{1}{p_1}} (2^j h)^{\beta_{21}} \frac{1}{(2^j h)^{\beta_{21}}} \left(\frac{|(x_0-2^j h, x_0)|}{|(x_0-2^j h, x_0)|} \int_{x_0+2h}^{x_0+2^{j+1}h} |f_1(x_1)|^{p_1} dx_1 \right)^{\frac{1}{p_1}} \\
&\quad \times \quad \sum_{j=1}^{\infty} (2^j h)^{\frac{\alpha}{2}-\frac{1}{p_2}} (2^j h)^{\beta_{22}} \frac{1}{(2^j h)^{\beta_{22}}} \left(\frac{|(x_0-2^j h, x_0)|}{|(x_0-2^j h, x_0)|} \int_{x_0+2h}^{x_0+2^{j+1}h} |f_2(x_2)|^{p_2} dx_2 \right)^{\frac{1}{p_2}} \\
&\leqslant \quad C \sum_{j=1}^{\infty} (2^j)^{\beta_{21}+\alpha+\beta_{22}+\frac{\alpha}{2}} \prod_{i=1}^{2} \|f_i\|_{L^+_{p_i,\beta_{2i}}(\mathbb{R})} \\
&\leqslant \quad C \sum_{j=1}^{\infty} (2^j)^{\beta_1} \prod_{i=1}^{2} \|f_i\|_{L^+_{p_i,\beta_{2i}}(\mathbb{R})} \\
&\leqslant \quad C \prod_{i=1}^{2} \|f_i\|_{L^+_{p_i,\beta_{2i}}(\mathbb{R})}.
\end{aligned}
$$

定理 2 证毕.

本文利用 Lebesgue 空间中的有界性证明了单边多线性 Hardy-Littlewood 极大算子和分数次极大算子是从单边 Morrey 空间到单边 Campanato 空间上有界的. 值得一提地是, 本文利用极大算子的有界性证明了它在 (x_0, x_0+2h) 外的估计, 但是本文并没有直接从单边 Campanato 空间去证明, 而是利用空间的包含性去证明了. 在后续的研究中, 可以研究加权的情形.

基金项目

国家自然科学基金项目 (No:12361018).

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