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关于复合微分算子的W-留数

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摘 要

本文首先综述近年来关于微分算子W-留数的一系列研究进展。之后研究一类复合微分算子 $D+c(X)$ 的基本结构，并在法坐标系下导出了微分算子的主符号表示，最终结合Lichnerowicz公式给出了5维带边流形上复合微分算子的W-留数表示。

关键词

复合微分算子，Lichnerowicz公式，非交换留数

The W-Residues of Complex Differential Operators

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Abstract

In this paper, a series of research advances on W-residue of differential operators in recent years are reviewed. Then the basic structure of a class of complex differential operators $D+c(X)$ is studied, and the principal symbolic representation of differential operators is derived in normal coordinate system. Finally, the W-residue representation of complex differential operators on 5-dimensional manifolds with edges is given by combining Lichnerowicz formula.

Keywords

Complex Differential Operator, Lichnerowicz Formula, Non-Commutative Residue

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1. 引言

近年来，非交换留数研究有了很多重要的结果，微分算子的 Lichnerowicz 公式及其带边流形上的留数研究获得了细致而深入的理解。本文在以往工作的基础上，探讨一类复合微分算子 $D+c(X)$ 的基本结构，对该微分算子的 Lichnerowicz 公式和 W-留数进行刻画。

20 世纪 90 年代，Wodzicki 对椭圆拟微分算子的 zeta 函数理论作了深入的研究，在非交换几何框架下给出了以他的名字命名的 Wodzicki 定理(参见文献[1])。Connes [2]应用非交换留数导出了四维情形下的微分算子的 Polyakov 作用。进一步，Connes [3]猜想 Dirac 算子逆的平方的非交换留数与 Einstein-Hilbert 作用成正比。Kastler [4]，Kalau 和 Walze [5]从不同的角度证明了这一结论，称之为 Kastler-Kalau-Walze 定理。Fedosov 等人[6]结合紧致流形 M 上 Wodzicki 的相关理论将这种留数推广到 Boutet 代数上，并给出了带边流形 W-留数的结构。结合 FGLS 定理[6]，带边流形上的 W-留数 $\widetilde{Wres}$ [7]，可以分为内部项和边界项两部分来刻画。Sitarz 和 Zajac [8]探讨了扰动 Dirac 算子的谱作用，Iochum 和 Levy [9]计算了具有单形式扰动的 Dirac 算子谱作用渐进展开中的热核系数。结合 Dirac 算子、Signature 算子，Wang [10]证明了低维带边流形的 Kastler-Kalau-Walze 型定理。Wang [11]定义了具有边界的自旋流形的低维体积并计算了具有边界的 5 维和 6 维自旋流形的低维体积。进一步，相关文献[12]-[14]探讨了扭化 Dirac 算子的 Lichnerowicz 型公式，建立了高维带边流形上微分算子的 Kastler-Kalau-Walze 型定理。

本文主要考虑一类复合微分算子 $D+c(X)$，结合 Lichnerowicz 公式计算出 5 维带边流形与复合微分 Dirac 算子相关的 W-留数 $\widetilde{Wres}\left[\pi^+\left(D+c(X)\right)^{-1}\circ\pi^+\left(D+c(X)\right)^{-1}\right]$。

2. 复合微分算子的 Lichnerowicz 公式

设 $\left(M,g^M\right)$ 为 n 维带边流形，边界为 ∂M。记 ∇^L 为 Levi-Civita 联络，在局部坐标 $\{x_i;1\le i\le n\}$ 和标准正交坐标系 $\{\tilde{E}_1,\cdots,\tilde{E}_n\}$ 下，联络矩阵 $(\omega_{s,t})$ 定义为

$$\nabla^L\left(\tilde{E}_1,\cdots,\tilde{E}_n\right)^t=\left(w_{s,t}\right)\left(\tilde{E}_1,\cdots,\tilde{E}_n\right)^t \tag{2.1}$$

定义 2.1：记 M 为具有黎曼度量 g 的 n 维定向自旋流形，Dirac 算子在切丛 TM 的正交框架 e_i，$1\le i\le n$ 和自然框架 ∂_i 下，

$$D=\sum_{i,j}g^{i,j}c(\partial_i)\nabla^S_{\partial_i}=\sum_i c(e_i)\nabla^S_{e_i}, \tag{2.2}$$

其中 $c(e_i)$ 表示 *Clifford* 作用，满足关系式 $c(e_i)c(e_j)+c(e_j)c(e_i)=-2\delta_i^j$。

定义 2.2：记复合微分算子 $\tilde{D}$ 为 Dirac 算子与微分形式的组合，

$$\tilde{D}=D+c(X)=\sum_i c(e_i)\nabla^S_{e_i}+c(X). \tag{2.3}$$

其中 $c(X)$ 为 M 微分一形式，X 为向量场。

结合复合微分算子 $\tilde{D}$ 与 $\tilde{D}^2$ 的运算

$$\tilde{D}^2=\left(D+c(X)\right)^2=D^2+Dc(X)+c(X)D+c^2(X), \tag{2.4}$$

得到复合微分算子 $\tilde{D}$ 的 Lichnerowicz 公式。

命题 2.3：复合微分算子 $\tilde{D}^2$ 可表示为

$$\begin{aligned}\tilde{D}^2 &= -g^{ij}\partial_i\partial_j + \left[-2\sigma^j + \Gamma^j + c\left(\partial^j\right)c(X) + c(X)c\left(\partial^j\right)\right]\partial_j \\ &+ \sum_{i,j} g^{ij}\left[-\left(\partial_i\sigma_j\right) - \sigma_i\sigma_j + \Gamma_{ij}^k\sigma_k + c\left(\partial_i\right)\partial_j\left(c(X)\right) + c\left(\partial_i\right)\sigma_j c(X) + c(X)c\left(\partial_i\right)\sigma_j\right] \\ &+ \frac{1}{4}s + c^2(X).\end{aligned} \tag{2.5}$$

结合命题 2.3，得到如下结论。

定理 2.4：对于偶数 n 维的紧致无边流形，复合微分算子 $\tilde{D}$ 的非交换留数可表示为

$$Wres\left(\tilde{D}^{-n+2}\right) = \frac{(n-2)\pi^{\frac{n}{2}}\dim\left(S(TM)\right)}{\left(\frac{n}{2}-1\right)!}\int_M -\frac{5s}{12}\mathrm{d}vol_M. \tag{2.6}$$

3. 带边流形上复合微分算子 $\tilde{D}$ 的 W-留数

定理 3.1 [6]：对于带边流形 x，边界为 ∂x， $\dim V \geq 3$， $A = \begin{pmatrix} \pi^+P+G & K \\ T & S \end{pmatrix} \in B$，记 p,b 和 s 分别表示 P,B 和 S 的局部表示，微分算子的 W-留数定义为

$$\begin{aligned}\widetilde{Wres}(A) &= \int_X\int_S tr_E\left[p_{-n}(x,\xi)\right]\sigma(\xi)\mathrm{d}x \\ &+ 2\pi\int_X\int_{S'}\left\{tr_E\left[\left(trb_{-n}\right)(x',\xi')\right] + tr_F\left[s_{1-n}(x',\xi')\right]\right\}\sigma(\xi')\mathrm{d}x'.\end{aligned} \tag{3.1}$$

a) $\widetilde{Wres}([A,B]) = 0$，对于任意的 $A,B \in \mathrm{B}$；b) 是 $\mathrm{B}/\mathrm{B}^{-\infty}$ 上唯一的连续迹。

记 $\sigma_l(A)$ 表示微分算子 A 的 l 阶符号，则带边流形微分算子 A 的 W-留数为：

$$\widetilde{Wres}\left[\pi^+\tilde{D}^{-p_1}\circ\pi^+\tilde{D}^{-p_2}\right] = \int_X\int_{|\xi|=1} tr_{S(TM)}\left[\sigma_{-n}\left(\tilde{D}^{-p_1-p_2}\right)\right]\sigma(\xi)\mathrm{d}x + \int_{\partial M}\Phi, \tag{3.2}$$

其中

$$\begin{aligned}\Phi &= \int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{j,k=0}^{\infty}\sum\frac{(-i)^{|a|+j+k+1}}{a!(j+k+1)}trace_{S(TM)}\left[\partial_{x_n}^j\partial_{\xi'}^a\partial_{\xi_n}^k\sigma_r^+\left(\tilde{D}^{-p_1}\right)(x',0,\xi',\xi_n)\right. \\ &\left.\times\partial_{x'}^a\partial_{\xi_n}^{j+1}\partial_{x_n}^k\sigma_l\left(\tilde{D}^{-p_2}\right)(x',0,\xi',\xi_n)\right]\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x',\end{aligned} \tag{3.3}$$

$$r-k+|a|+l-j-1 = -n, r \leq -p_1, l \leq -p_2.$$

依据命题 3.5 [11]可知， $p_1+p_2 \equiv n \bmod 1$， $Vol_n^{(p_1,p_2)}M = \int_{\partial M}\Phi$，则 5 维带边流形情形下复合微分算子 $\tilde{D}$ 的 W-留数为 $\widetilde{Wres}\left[\pi^+\tilde{D}^{-1}\circ\pi^+\tilde{D}^{-1}\right] = \int_{\partial M}\Phi$ 。

引理 3.2：复合微分算子 $\tilde{D}$ 的主符号

$$\sigma_{-1}\left(\tilde{D}^{-1}\right) = q_{-1} = \frac{\sqrt{-1}c(\xi)}{|\xi|^2};$$

$$\begin{aligned}\sigma_{-2}\left(\tilde{D}^{-1}\right) = q_{-2} &= \frac{c(\xi)p_0c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6}\sum_j c\left(d_{x_j}\right)\left[\partial_{x_j}\left(c(\xi)\right)|\xi|^2 - c(\xi)\partial x_j\left(|\xi|^2\right)\right] \\ &= \sigma_{-2}\left(D^{-1}\right) + \frac{c(X)}{|\xi|^2} - \frac{2g(X,\xi)c(\xi)}{|\xi|^4};\end{aligned}$$

$$\sigma_{-3}\left(\tilde{D}^{-1}\right)=q_{-3}=-\frac{1}{p_1}\left[p_0q_{-2}+\sum_{j=1}^{n-1}c\left(dx_j\right)\partial_{x_j}q_{-2}+c\left(dx_n\right)\partial_{x_n}q_{-2}\right];$$

其中， $p_0\left(x\right)=-h'\left(0\right)c\left(dx_n\right)+c\left(X\right)$ 。

引理 3.3：复合微分算子 $\tilde{D}$ 的-3 阶基本形式为

$$\begin{aligned}\sigma_{-3}\left(\tilde{D}^{-1}\right)\left(x_0\right)\Big|_{|\xi'|=1}&=\sigma_{-3}\left(D^{-1}\right)\left(x_0\right)\Big|_{|\xi'|=1}+\Bigg[-q_1c\left(X\right)\sigma_{-2}\left(D^{-1}\right)-q_1\frac{c\left(X\right)}{|\xi|^2}+q_1\frac{2g\left(X,\xi\right)c\left(\xi\right)}{|\xi|^4}\\&\quad-q_1\sum_{j=1}^{n-1}c\left(dx_j\right)\partial_{x_j}\left(\frac{c\left(X\right)}{|\xi|^2}\right)+q_1\sum_{j=1}^{n-1}c\left(dx_j\right)\partial_{x_j}\left(\frac{2g\left(X,\xi\right)c\left(\xi\right)}{|\xi|^4}\right)\\&\quad-q_1c\left(dx_n\right)\partial_{x_n}\left(\frac{c\left(X\right)}{|\xi|^2}\right)+q_1\partial_{x_n}\left(\frac{2g\left(X,\xi\right)}{|\xi|^4}\right)\Bigg]\left(x_0\right)\Big|_{|\xi'|=1}\\&=\sigma_{-3}\left(D^{-1}\right)\left(x_0\right)\Big|_{|\xi'|=1}+R_{-3}\left(x_0\right)\Big|_{|\xi'|=1}.\end{aligned}\tag{3.4}$$

对于 5 维带边流形，根据 $-r-l+k+j-|a|-1=-5$， $r,l\le-1$，可得 $\Phi=\sum_{i=1}^{15}\Phi_i$ 。

(1) $r=-1$， $l=-1$， $k=0$， $j=1$， $|a|=1$，

$$\Phi_1=\frac{i}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}trace\left[\partial_{x_n}\partial_{\xi_i}\pi_{\xi_n}^{+}q_{-1}\times\partial_{x_i}\partial_{\xi_n}^{2}q_{-1}\right]\left(x_0\right)\mathrm{d}\xi_n\sigma\left(\xi'\right)\mathrm{d}x'.$$

对于 $j<n$，有

$$\partial_{x_i}q_{-1}\left(x_0\right)=\partial_{x_i}\left(\frac{\sqrt{-1}c\left(\xi\right)}{|\xi|^2}\right)\left(x_0\right)=\frac{\sqrt{-1}\partial_{x_i}\left|c\left(\xi\right)\right|\left(x_0\right)}{|\xi|^2}-\frac{\sqrt{-1}c\left(\xi\right)\partial_{x_i}\left(|\xi|^2\right)\left(x_0\right)}{|\xi|^4}=0,\quad \Phi_1=0.$$

(2) $r=-1$， $l=-1$， $k=0$， $j=2$， $|a|=0$

$$\Phi_2=\frac{i}{6}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{j=2}trace\left[\partial_{x_n}^{2}\pi_{\xi_n}^{+}q_{-1}\times\partial_{\xi_n}^{3}q_{-1}\right]\left(x_0\right)\mathrm{d}\xi_n\sigma\left(\xi'\right)\mathrm{d}x'.$$

直接计算得到

$$\partial_{\xi_n}^{3}q_{-1}\left(x_0\right)\Big|_{|\xi'|=1}=\frac{24\xi_n-24\xi_n^3}{\left(1+\xi_n^2\right)^4}\sqrt{-1}c\left(\xi'\right)+\frac{-6\xi_n^4+36\xi_n^2-6}{\left(1+\xi_n^2\right)^4}\sqrt{-1}c\left(dx_n\right),$$

$$\begin{aligned}\partial x_n^2\pi_{\xi_n}^{+}q_{-1}\left(x_0\right)\Big|_{|\xi'|=1}&=\pi_{\xi_n}^{+}\partial x_n^2\frac{c\left(\xi\right)}{|\xi|^2}\Bigg|_{|\xi'|=1}=\pi_{\xi_n}^{+}\partial x_n\left[\frac{\partial x_n\left(c\left(\xi\right)\right)}{|\xi|^2}-\frac{c\left(\xi\right)\partial x_n\left(|\xi|^2\right)}{|\xi|^4}\right]\\&=\pi_{\xi_n}^{+}\left(\frac{\partial x_n^2\left(c\left(\xi'\right)\right)}{|\xi|^2}-2\frac{h'\left(0\right)\partial x_n\left(c\left(\xi'\right)\right)}{|\xi|^4}-\frac{h''\left(0\right)c\left(\xi\right)}{|\xi|^4}+\frac{c\left(\xi\right)2\left(h'\left(0\right)\right)^2}{|\xi|^6}\right)\\&=\pi_{\xi_n}^{+}\left(\frac{\partial x_n^2\left(c\left(\xi'\right)\right)}{|\xi|^2}\right)-2h'\left(0\right)\partial x_n\left(c\left(\xi'\right)\right)\pi_{\xi_n}^{+}\frac{1}{|\xi|^4}-h''\left(0\right)\pi_{\xi_n}^{+}\frac{c\left(\xi\right)}{|\xi|^4}+2\left(h'\left(0\right)\right)^2\pi_{\xi_n}^{+}\frac{c\left(\xi\right)}{|\xi|^6}\\&=\left(\frac{3}{4}\left(h'\left(0\right)\right)^2-\frac{1}{2}h''\left(0\right)\right)\frac{c\left(\xi'\right)}{2\left(\xi_n-i\right)}-h'\left(0\right)\frac{\xi_n-2i}{4\left(\xi_n-i\right)^2}\partial x_n\left(c\left(\xi'\right)\right)\\&\quad-h''\left(0\right)\frac{\left(\xi_n-2i\right)c\left(\xi'\right)+c\left(dx_n\right)}{4\left(\xi_n-i\right)^2}+2i\left(h'\left(0\right)\right)^2\frac{\left(-3i\xi_n^2-9\xi_n+8i\right)c\left(\xi'\right)-\left(i\xi_n+3\right)c\left(dx_n\right)}{16\left(\xi_n-i\right)^3}.\end{aligned}$$

结合 Clifford 运算及迹性质 $trAB = trBA$，有

$$tr\left[c(\xi')c(dx_n)\right]=0;\, tr\left[c(dx_n)^2\right]=-4;\, tr\left[c(\xi')^2(x_0)\right](x_0)\Big|_{|\xi'|=1}=-4;$$

$$tr\left[\partial_{x_n}c(\xi')c(dx_n)\right]=0;\, tr\left[\partial_{x_n}c(\xi')\times c(\xi')\right](x_0)\Big|_{|\xi'|=1}=-2h'(0).$$

则

$$\begin{aligned}&trace\left[\partial_{x_n}^2\pi_{\xi_n}^2 q_{-1}\times\partial_{\xi_n}^3 q_{-1}\right](x_0)\Big|_{|\xi'|=1}\\&=i\left(h'(0)\right)^2\frac{3\left(33\xi_n^5-75i\xi_n^4-94\xi_n^3+90i\xi_n^2+57\xi_n-3i\right)}{2(\xi_n-i)^3\left(1+\xi_n^2\right)^4}\\&\quad+i\left(h''(0)\right)\frac{6\left(-9\xi_n^4+12i\xi_n^3+14\xi_n^2-12i\xi_n-1\right)}{2(\xi_n-i)^2\left(1+\xi_n^2\right)^4}.\end{aligned}$$

$$\begin{aligned}\Phi_2&=-\frac{1}{6}\left(h'(0)\right)^2\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{3\left(33\xi_n^5-75i\xi_n^4-94\xi_n^3+90i\xi_n^2+57\xi_n-3i\right)}{2(\xi_n-i)^3\left(1+\xi_n^2\right)^4}\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\\&\quad-\frac{1}{6}h''(0)\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{6\left(-9\xi_n^4+12i\xi_n^3+14\xi_n^2-12i\xi_n-1\right)}{2(\xi_n-i)^3\left(1+\xi_n^2\right)^4}\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\\&=-\frac{1}{6}\left(h'(0)\right)^2\Omega_3\int_{\Gamma^+}\frac{3(33\xi_n^5-75i\xi_n^4-94\xi_n^3+90i\xi_n^2+57\xi_n-3i)}{2(\xi_n-i)^3\left(1+\xi_n^2\right)^4}\mathrm{d}\xi_n\mathrm{d}x'\\&\quad-\frac{1}{6}h''(0)\Omega_3\int_{\Gamma^+}\frac{6\left(-9\xi_n^4+12i\xi_n^3+14\xi_n^2-12i\xi_n-1\right)}{2(\xi_n-i)^3\left(1+\xi_n^2\right)^4}\mathrm{d}\xi_n\mathrm{d}x'\\&=\left(\frac{29}{64}\left(h'(0)\right)^2-\frac{3}{8}h''(0)\right)\pi\Omega_3\mathrm{d}x',\end{aligned}$$

其中 Ω_3 是 S^3 的标准体积。

(3) $r=-1$，$l=-2$，$k=0$，$j=0$，$|a|=1$

由(3.3)得：

$$\begin{aligned}\Phi_3&=-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|a|=1}trace\left[\partial_{\xi_i}\pi_{\xi_n}^+q_{-1}\times\partial_{x_i}\partial_{\xi_n}q_{-2}\right](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\\&=\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|a|=1}trace\left[\partial_{\xi_n}\partial_{\xi_i}\pi_{\xi_n}^+q_{-1}\times\partial_{x_i}q_{-2}\right](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'\\&=\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|a|=1}trace\left[\partial_{\xi_n}\partial_{\xi_i}\pi_{\xi_n}^+q_{-1}\times\partial_{x_i}\left(\sigma_{-2}\left(D^{-1}\right)+\frac{c(X)}{|\xi|^2}-\frac{2g(X,\xi)c(\xi)}{|\xi|^4}\right)\right](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'.\end{aligned}$$

在 x_0 点结合 $\pi_{\xi_n}^+$ 运算公式得到

$$\partial_{\xi_i}\pi_{\xi_n}^+q_{-1}(x_0)\Big|_{|\xi'|=1}=\frac{-1}{2(\xi_n-i)^2}c(dx_i)-\xi_i\frac{3i-\xi_n}{2(\xi_n-i)^3}c(\xi')+\xi_i\frac{1}{(\xi_n-i)^3}c(dx_n),$$

$$\partial_{x_i}\left(\frac{c(X)}{|\xi|^2}-\frac{2g(X,\xi)c(\xi)}{|\xi|^4}\right)\Bigg|_{|\xi'|=1}=\frac{1}{|\xi|^2}\partial_{x_i}\left[c(X)\right]-\frac{2c(\xi)}{|\xi|^4}\partial_{x_i}\left[g(X,\xi)\right].$$

计算得

$$\sum_{|a|=1} trace\left[\partial_{\xi_n}\partial_{\xi_i}\pi_{\xi_n}^{+}q_{-1}\times\partial x_i\left(\frac{c(X)}{|\xi|^2}-\frac{2g(X,\xi)c(\xi)}{|\xi|^4}\right)\right](x_0)$$
$$=\frac{2}{(\xi_n-i)^3(\xi_n+i)}\sum_{j-1}^{n-1}\partial_{x_j}\left[g\left(X,dx_j\right)\right]+\frac{-2\xi_n^3+6i\xi_n^2-2\xi_n-2i}{(\xi_n-i)^5(\xi_n+i)^2}\sum_{i,j-1}^{n-1}\xi_i\xi_j\partial_{x_j}\left[g\left(X,dx_j\right)\right].$$

组合以上算式得

$$\Phi_3=\left(\frac{3}{16}-\frac{5}{32}i\right)s_{\partial M}\pi\Omega_3\mathrm{d}x'$$
$$+\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|a|=1} trace\left[\partial_{\xi_n}\partial_{\xi'}^{a}\pi_{\xi_n}^{+}q_{-1}\times\partial_{x'}^{a}\left(\frac{c(X)}{|\xi|^2}-\frac{2g(X,\xi)c(\xi)}{|\xi|^4}\right)\right](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'$$
$$=\left(\frac{3}{16}-\frac{5}{32}i\right)s_{\partial M}\pi\Omega_3\mathrm{d}x'-\frac{9}{16}\pi\Omega_3\sum_{j=1}^{n-1}\partial x_j\left[g\left(X,\mathrm{d}x_j\right)\right]\mathrm{d}x'$$
$$=\left(\frac{3}{16}-\frac{5}{32}i\right)s_{\partial M}\pi\Omega_3\mathrm{d}x'-\frac{9}{16}\pi\Omega_3C_1^1\left(\nabla^{\partial M}\left(X'|_{\partial M}\right)^{*}\right)\mathrm{d}x'.$$

其中向量场 $\left(X'|_{\partial M}\right)^{*}$ 指代 $X'|_{\partial M}$ 的度量对偶，$\nabla^{\partial M}$ 是 ∂M 的 Levi-Civita 联络，C_1^1 是(1,1)张量的缩并。

类似以上方法，计算得到其他情形。

$$\Phi_4=-\frac{5}{16}\left(h'(0)\right)^2\pi\Omega_3\mathrm{d}x',\quad \Phi_5=0,\quad \Phi_6=\left(\frac{29}{64}\left(h'(0)\right)^2-\frac{3}{8}h''(0)\right)\pi\Omega_3\mathrm{d}x',$$

$$\Phi_7=\frac{39}{32}\left(h'(0)\right)^2\pi\Omega_3\mathrm{d}x'-\frac{5}{8}a_nh'(0)\Omega_3\mathrm{d}x',\quad \Phi_8=-\frac{1}{4}s_{\partial M}\pi^3\mathrm{d}x',$$

$$\Phi_9=\left(-\frac{367}{128}h'(0)^2+\frac{103}{64}h''(0)\right)\pi\Omega_3\mathrm{d}x'-\frac{3}{8}\pi\Omega_3\partial_{x_n}(a_n)\mathrm{d}x'+\frac{15}{64}\pi a_nh'(0)\Omega_3\mathrm{d}x',$$

$$\Phi_{10}=\left(-\frac{367}{128}h'(0)^2+\frac{103}{64}h''(0)\right)\pi\Omega_3\mathrm{d}x'-\frac{3}{8}\pi\Omega_3\partial_{x_n}(a_n)\mathrm{d}x'+\frac{15}{64}\pi a_nh'(0)\Omega_3\mathrm{d}x',$$

$$\Phi_{11}=0,\quad \Phi_{12}=\frac{39}{32}\left(h'(0)\right)^2\pi\Omega_3\mathrm{d}x'-\frac{5}{8}a_nh'(0)\pi\Omega_3\mathrm{d}x',$$

$$\Phi_{13}=-\frac{821}{256}\left(h'(0)\right)^2\pi\Omega_3\mathrm{d}x'+\frac{15}{16}\pi a_nh'(0)\Omega_3\mathrm{d}x'+\frac{\pi}{2}|X|_{gTM}^2\Omega_3\mathrm{d}x'$$
$$+\frac{35}{64}\pi|X|_{g\partial M}^2\Omega_3\mathrm{d}x'-\pi a_n^2\Omega_3\mathrm{d}x',$$

$$\Phi_{14}=\left(\frac{239}{64}\left(h'(0)\right)^2-\frac{27}{16}h''(0)-\frac{11}{192}s_{\partial M}\right)\pi\Omega_3\mathrm{d}x'-\frac{5}{8}\pi a_nh'(0)\Omega_3\mathrm{d}x'$$
$$+\frac{3}{4}\pi\partial_{x_n}(a_n)\Omega_3\mathrm{d}x'+\frac{3}{4}\pi C_1^1\left(DX^{*}\right)\mathrm{d}x',$$

$$\Phi_{15}=\left(\frac{239}{64}\left(h'(0)\right)^2-\frac{27}{16}h''(0)-\frac{11}{192}s_{\partial M}\right)\pi\Omega_3\mathrm{d}x'$$
$$-\frac{5}{8}\pi a_nh'(0)\Omega_3\mathrm{d}x'+\frac{3}{4}\pi\partial_{x_n}(a_n)\Omega_3\mathrm{d}x'+\frac{3}{4}\pi\Omega_3C_1^1\left(\nabla^{\partial M}\left(X'|_{\partial M}\right)^{*}\right)\mathrm{d}x',$$

组合上述结果得到

$$\Phi=\left(\frac{399}{256}\left(h'(0)\right)^2-\frac{29}{32}h''(0)-\left(\frac{17}{96}+\frac{5}{32}i\right)s_{\partial M}\right)\pi\Omega_3\mathrm{d}x'+\frac{25}{32}\pi a_n h'(0)\Omega_3\mathrm{d}x' -\frac{\pi}{2}|X|^2_{gTM}\,\Omega_3\mathrm{d}x'+\frac{35\pi}{65}|X|^2_{gTM}\,h'(0)\Omega_3\mathrm{d}x'-\pi a_n^2\Omega_3\mathrm{d}x' -\frac{3}{4}\pi\partial_{x_n}(a_n)\Omega_3\mathrm{d}x'+\frac{15}{16}\pi C_1^1\left(\nabla^{\partial M}\left(X'|_{\partial M}\right)^*\right)\Omega_3\mathrm{d}x' \tag{3.5}$$

4. 结论

依据带边流形的 Einstein-Hilbert 作用[10] [11]：

$$I_{G_r}=\frac{1}{16\pi}\int_M s\mathrm{d}vol_M+2\int_{\partial M}K\mathrm{d}vol_{\partial M}:=I_{G_{r,i}}+I_{G_{r,b}},$$

$$K=\sum_{1\le i,j\le n-1}K_{i,j}g^{i,j}_{\partial M},\quad K_{i,j}(x_0)=-\Gamma^n_{i,j}(x_0).$$

对于 5 维带边流形，$K(x_0)=\sum_{i,j}K_{i,j}(x_0)g^{i,j}_{\partial M}(x_0)=\sum_{i,j}^{4}K_{i,j}(x_0)=-2h'(0)$，有

$$s_M=3\left(h'(0)\right)^2-4h''(0)+s_{\partial M}(x_0). \tag{4.1}$$

定理 4.1：设 M 为 5 维带边流形，边界为 ∂M 。复合微分算子 $\tilde{D}$ 的 W-留数为

$$\widetilde{Wres}\left[\pi^+\left(D+c(X)\right)^{-1}\circ\pi^+\left(D+c(X)\right)^{-1}\right] =\int_{\partial M}\left[\frac{1}{16}\left(\frac{225}{32}K^2+\frac{29}{4}s_M|_{\partial M}-\left(\frac{155}{12}+5i\right)s_{\partial M}\right)-\frac{25}{32}a_nK-|X'|^2_{g\partial M} -\frac{35}{64}|X'|^2_{g\partial M}K-3a_n^2|_{\partial M}-\frac{3}{2}\partial_{x_n}(a_n)|_{\partial M}+\frac{15}{8}C_1^1\left(\nabla^{\partial M}\left(X'|_{\partial M}\right)^*\right)\right]\pi^3\mathrm{d}vol_{\partial M} \tag{4.2}$$

其中 $s_M, s_{\partial M}$ 分别指代 M 和 ∂M 上的数量曲率，向量场 $\left(X'|_{\partial M}\right)^*$ 是 $X'|_{\partial M}$ 的度量对偶，$\nabla^{\partial M}$ 是 ∂M 的 Levi-Civita 联络，C_1^1 是(1,1)张量的缩并。

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