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高阶分数阶微分方程边值问题解的唯一性

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摘　要

分数阶微积分是整数阶微积分的推广，主要研究任意阶微分和积分的理论和应用问题，由于其更加适合描述具有遗传和记忆特质的材料与过程，因此广泛用于解决混沌与湍流、随机游走、统计与随机过程、粘弹性力学、电化学等诸多领域所面临的问题。目前，高阶分数阶微分方程边值问题解的存在唯一性是研究的重点课题之一。本文将借助泛函分析等相关工具，对非线性项含未知函数分数阶导数的边值问题进行深入探讨。首先通过降阶将原边值问题转变成非线性项不含导数的等价边值问题，接着分析Green函数的性质，然后利用混合单调算子不动点定理得到边值问题解的唯一性结果，最后举例说明结果的正确性。

关键词

高阶分数阶微分方程，边值问题，不动点定理，唯一性

Uniqueness of Solutions to Boundary Value Problems of Higher Order Fractional Differential Equations

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Abstract

Fractional calculus is a generalization of integer calculus, mainly studies the theory and application of differential and integral of arbitrary order, because it is more suitable to describe materials and processes with genetic and memory characteristics, so it is widely used to solve problems in many fields such as chaos and turbulence, random walk, statistics and random process, viscoe-

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lastic mechanics, electrochemistry and so on. At present, the existence and uniqueness of solutions to boundary value problem of higher order fractional differential equation is one of the key research topics. In this paper, with the help of functional analysis and other relevant tools, the boundary value problem of nonlinear terms including fractional derivatives of unknown functions is deeply discussed. First, the original boundary value problem is transformed into an equivalent boundary value problem without derivative by reducing the order, then the properties of Green function are analyzed, and then the uniqueness of the solution of the boundary value problem is obtained by using the fixed point theorem of mixed monotone operator. Finally, an example is given to illustrate the correctness of the results.

Keywords

Higher Order Fractional Differential Equation, Boundary Value Problem, Fixed Point Theorem, Uniqueness

1. 引言

分数阶微积分的研究历史几乎与整数阶相同，在发展初期阶段，因为缺少实际应用背景的支撑以及发展前景不明朗等多方面的原因，分数阶微积分长期停留在理论和猜想阶段，因而发展的比较缓慢。近几十年来，有可能包含分形现象在内的物理、工程等诸多应用学科领域应用的拓展，激发了科研人员对分数阶微积分的巨大热情[1]-[3]。在求解实际问题的过程中，从准确的建立问题模型到求解方程都是分数阶微分方程理论所面临的主要问题，一般来说，求解方法可以分为两类，一是使用逐次逼近法用近似解不断的逼近精确解，二是将解的存在性问题转化为某一变换的不动点的存在性问题。目前，分数阶微分方程边值问题解的存在唯一性已有大量的研究成果[4]-[14]。譬如，Lakoud 等人[12]于 2017 年运用 Krasnoselskii 不动点定理研究了下述边值问题

$$\begin{cases} {}^{C}D_{1-}^{\alpha}D_{0+}^{\beta}u(t)+f\left(t,u(t)\right)=0,\ 0<t<1, \\ u(0)=u'(0)=u(1)=0 \end{cases} \tag{1}$$

解的存在性，其中，${}^{C}D_{1-}^{\alpha}$ 表示右 Caputo 导数，D_{0+}^{β} 表示左 Riemann-Liouville 导数，$0<\alpha<1$，$1<\beta\le 2$，$f:[0,1]\times\mathbb{R}\to\mathbb{R}$。

2021 年，Liu 等人在文献[13]中利用 $\varphi-(h,e)$ -凹算子以及 $t-\sigma(t)$ 型混合单调算子的不动点定理分别得到了分数阶微分方程边值问题

$$\begin{cases} {}^{C}D_{1-}^{\alpha}D_{0+}^{\beta}x(t)+f\left(t,x(t)\right)=b,\ 0<t<1, \\ x(0)=0,\ x'(0)=D_{0+}^{\beta}x(1)=0 \end{cases} \tag{2}$$

解的存在唯一性，其中，$0<\alpha\le 1$，$1<\beta\le 2$，$\alpha+\beta>2$，$f:[0,1]\times\mathbb{R}\to\mathbb{R}$ 连续，$b>0$ 是一个常数。

2019 年，Zhang 等[14]考虑了非线性项含有未知函数分数阶导数的边值问题

$$\begin{cases} D_{0+}^{\alpha}u(t)+f\left(t,u(t),D_{0+}^{\beta_1}u(t),D_{0+}^{\beta_2}u(t),\cdots,D_{0+}^{\beta_{n-1}}u(t)\right)=0,\ 0<t<1, \\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=D_{0+}^{\beta}u(1)=0, \end{cases} \tag{3}$$

其中，$n-1<\alpha\le n$，$i-1<\beta_i\le i$ $(i=1,2,\cdots,n-1)$，$\alpha-\beta_{n-1}>\alpha-\beta>1$，$f:[0,1]\times\mathbb{R}^n\to\mathbb{R}$ 连续，利用拓扑度理论获得了边值问题(3)非平凡解的存在性结果。

受上述文献的启发，运用 $t-\sigma(t)$ 型混合单调算子的不动点定理[6]研究下述非线性项含有未知函数分数阶导数的高阶分数阶微分方程边值问题

$$\begin{cases} D_{0+}^{\alpha}u(t)+f\left(t,u(t),D_{0+}^{\beta_1}u(t),D_{0+}^{\beta_2}u(t),\cdots,D_{0+}^{\beta_{n-1}}u(t)\right)=b,\ 0<t<1, \\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=D_{0+}^{\beta}u(1)=0 \end{cases} \tag{4}$$

解的唯一性。总是假定 $n\in\mathrm{N}$，$n\ge 2$，$n-1<\alpha\le n$，$i-1<\beta_i\le i$ $(i=1,2,\cdots,n-1)$，$\alpha-\beta_{n-1}>\alpha-\beta>1$，$b>0$ 是一个常数。

2. 预备知识

首先给出与分数阶微分方程相关的一些基本定义和引理。为此，假定 $\gamma>0$ 是一个常数，$[\gamma]$ 表示 γ 的整数部分，而 N 是大于等于 γ 的最小整数。

2.1. 定义 1 [1]

$[0,1]$ 上的 γ 阶 Riemann-Liouville 分数阶积分 $I_{0+}^{\gamma}u$ 和 $I_{1-}^{\gamma}u$ 分别定义为

$$\left(I_{0+}^{\gamma}u\right)(t):=\frac{1}{\Gamma(\gamma)}\int_0^t\frac{u(s)\mathrm{d}s}{(t-s)^{1-\gamma}}$$

和

$$\left(I_{1-}^{\gamma}u\right)(t):=\frac{1}{\Gamma(\gamma)}\int_0^t\frac{u(s)\mathrm{d}s}{(s-t)^{1-\gamma}},$$

其中，

$$\Gamma(\gamma)=\int_0^{+\infty}s^{\gamma-1}\mathrm{e}^{-s}\mathrm{d}s.$$

2.2. 定义 2 [1]

$[0,1]$ 上的 γ 阶 Riemann-Liouville 分数阶导数 $D_{0+}^{\gamma}u$ 和 $D_{1-}^{\gamma}u$ 分别定义为

$$\begin{aligned}\left(D_{0+}^{\gamma}u\right)(t)&:=\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m\left(I_{0+}^{m-\gamma}u\right)(t)\\ &=\frac{1}{\Gamma(m-\gamma)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m\int_0^t\frac{u(s)\mathrm{d}s}{(t-s)^{\gamma-m+1}}\end{aligned}$$

和

$$\begin{aligned}\left(D_{1-}^{\gamma}u\right)(t)&:=\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m\left(I_{1-}^{m-\gamma}u\right)(t)\\ &=\frac{1}{\Gamma(m-\gamma)}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^m\int_t^1\frac{u(s)\mathrm{d}s}{(s-t)^{\gamma-m+1}},\end{aligned}$$

其中，$m=[\gamma]+1$。

2.3. 引理 1 [7]

设 $u\in C(0,1)\cap L(0,1)$，则分数阶微分方程

$$D_{0+}^{\gamma}u(t)=0$$

有唯一解

$$u(t)=C_1t^{\gamma-1}+C_2t^{\gamma-2}+\cdots+C_Nt^{\gamma-N},$$

其中，$C_i\in\mathbb{R}$，$i=1,2,\cdots,N$。

2.4. 引理 2 [7]

设 $u\in C(0,1)\cap L(0,1)$ 具有 γ 阶导数且 $D_{0+}^{\gamma}u\in C(0,1)\cap L(0,1)$，则

$$I_{0+}^{\gamma}D_{0+}^{\gamma}u(t)=u(t)+C_1t^{\gamma-1}+C_2t^{\gamma-2}+\cdots+C_Nt^{\gamma-N},$$

其中，$C_i\in\mathbb{R}$，$i=1,2,\cdots,N$。

2.5. 引理 3

设 u 是边值问题(4)的解，则 $v=D_{0+}^{\beta_{n-1}}u$ 是边值问题

$$\begin{cases}D_{0+}^{\alpha-\beta_{n-1}}v(t)+f\left(t,I_{0+}^{\beta_{n-1}}v(t),I_{0+}^{\beta_{n-1}-\beta_1}v(t),\cdots,I_{0+}^{\beta_{n-1}-\beta_{n-2}}v(t),v(t)\right)=b,\ 0<t<1,\\ I_{0+}^{\beta_{n-1}-n+2}v(0)=D_{0+}^{\beta-\beta_{n-1}}v(1)=0\end{cases}\tag{5}$$

的解。反之，设 v 是边值问题(5)的解，则 $u=I_{0+}^{\beta_{n-1}}v$ 是边值问题(4)的解。

证明 类似于文献[15]中的引理 2.9，此处省略。

鉴于引理 3，本文的剩余部分致力于讨论边值问题(5)的解。

2.6. 引理 4 [14]

设 $y\in C[0,1]$ 是给定的函数，则边值问题

$$\begin{cases}D_{0+}^{\alpha-\beta_{n-1}}v(t)+y(t)=0,\ 0<t<1,\\ I_{0+}^{\beta_{n-1}-n+2}v(0)=D_{0+}^{\beta-\beta_{n-1}}v(1)=0\end{cases}$$

有唯一解

$$v(t)=\int_0^1G(t,s)y(s)\mathrm{d}s,$$

其中

$$G(t,s)=\frac{1}{\Gamma(\alpha-\beta_{n-1})}\begin{cases}t^{\alpha-\beta_{n-1}-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-\beta_{n-1}-1}, & 0\le s\le t\le 1,\\ t^{\alpha-\beta_{n-1}-1}(1-s)^{\alpha-\beta-1}, & 0\le s\le t\le 1.\end{cases}$$

2.7. 引理 5

$G(t,s)$ 满足如下性质：

1) $G(t,s)$ 在 $[0,1]\times[0,1]$ 上连续；

2) $\dfrac{t^{\alpha-\beta_{n-1}-1}(1-s)^{\alpha-\beta-1}\left[1-(1-s)^{\beta-\beta_{n-1}}\right]}{\Gamma(\alpha-\beta_{n-1})}\le G(t,s)\le\dfrac{t^{\alpha-\beta_{n-1}-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta_{n-1})}$，$(t,s)\in[0,1]\times[0,1]$。

证明 1) 显然成立；

既然右端的不等式是显然的，故只需要证明左端的不等式成立即可。当 $0\le s\le t\le 1$ 时，

$$G(t,s)=\frac{t^{\alpha-\beta_{n-1}-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-\beta_{n-1}-1}}{\Gamma(\alpha-\beta_{n-1})}$$
$$\geq\frac{t^{\alpha-\beta_{n-1}-1}(1-s)^{\alpha-\beta-1}-(t-ts)^{\alpha-\beta_{n-1}-1}}{\Gamma(\alpha-\beta_{n-1})}$$
$$=\frac{t^{\alpha-\beta_{n-1}-1}(1-s)^{\alpha-\beta-1}\left[1-(1-s)^{\beta-\beta_{n-1}}\right]}{\Gamma(\alpha-\beta_{n-1})};$$

而当$0\leq t\leq s\leq 1$时，

$$G(t,s)=\frac{t^{\alpha-\beta_{n-1}-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta_{n-1})}$$
$$\geq\frac{t^{\alpha-\beta_{n-1}-1}(1-s)^{\alpha-\beta-1}-(t-ts)^{\alpha-\beta_{n-1}-1}}{\Gamma(\alpha-\beta_{n-1})}$$
$$=\frac{t^{\alpha-\beta_{n-1}-1}(1-s)^{\alpha-\beta-1}\left[1-(1-s)^{\beta-\beta_{n-1}}\right]}{\Gamma(\alpha-\beta_{n-1})}.$$

本节的剩余部分介绍t-$\sigma(t)$型混合单调算子的定义以及关于$t-\sigma(t)$型混合单调算子的不动点定理。

设$(E,\|\cdot\|)$是一个实Banach空间，P是E中的锥，“$\precsim$”是由P诱导的半序，即$x\precsim y$当且仅当$y-x\in P$，为方便起见，$x\precsim y$有时也写成$y\succsim x$。如果$x\precsim y$且$x\neq y$，则记$x<y$或$y>x$。θ表示E中的零元。如果存在常数$L>0$使得当$\theta\precsim x\precsim y$时必有$\|x\|\leq L\|y\|$，则称$P$为正规锥。给定$h>\theta$，定义集合$P_h=\{x\in E\,|$ 存在$\lambda>0$，$\mu>0$使得$\lambda h\precsim x\precsim \mu h\}$。

2.8. 定义 3 [5]

设D是实Banach空间E的一个子集且$T:D\times D\to E$。如果$T(u,v)$关于u是增的且关于v是减的，即对于任意的$v\in D$，如果$u_1\leq u_2\ (u_1,u_2\in D)$蕴含着$T(u_1,v)\leq T(u_2,v)$；对于任意的$u\in D$，如果$v_1\leq v_2\ (v_1,v_2\in D)$蕴含着$T(u,v_1)\geq T(u,v_2)$，那么称$T$为混合单调算子。如果$T(u^*,u^*)=u^*$，则称$u^*\in D$为$T$的一个不动点。

2.9. 定义 4 [6]

设$T:P\times P\to E$是一个混合单调算子，若对于任意的$0<t<1$，存在$0<\sigma=\sigma(t)<1$使得

$$T\left(tu,\frac{1}{t}v\right)\succsim t^{\sigma(t)}T(u,v)$$

对于任意的$u,v\in P$都成立，则称T为$t-\sigma(t)$型混合单调算子。

2.10. 引理 6 [6]

设P是实Banach空间E中的正规锥，$T:P_h\times P_h\to P_h$是一个$t-\sigma(t)$型混合单调算子，则T有一个唯一的不动点$v^*\in P_h$。更进一步，任取$u_0,v_0\in P_h$，构造序列$u_n=T(u_{n-1},v_{n-1})$，$v_n=T(v_{n-1},u_{n-1})$，$n=1,2,\cdots$，必有$\lim\limits_{n\to\infty}\|u_n-v^*\|=0$且$\lim\limits_{n\to\infty}\|v_n-v^*\|=0$。

3. 主要结果

在后续部分，记$E=C[0,1]$，令$\|u\|=\max\limits_{t\in[0,1]}|u(t)|$，$u\in E$，$P=\{u\in E\,|\,u(t)\geq 0,t\in[0,1]\}$，则$E$是Banach

空间，P 是 E 中的正规锥。显然，$u \precsim v$ 当且仅当 $u(t) \le v(t), t \in [0,1]$。

令 $H = \dfrac{b}{(\alpha-\beta)\Gamma(\alpha-\beta_{n-1})}$，$h(t) = Ht^{\alpha-\beta_{n-1}-1}$，$t \in [0,1]$，则有 $h > \theta$。

为方便起见，记

$$\omega = \min\left\{1, \frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha)}, \frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta_1)}, \frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta_2)}, \cdots, \frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta_{n-2})}\right\}.$$

定理 1

假设下述条件成立：

(H1) 存在两个连续函数 $\phi, \psi : [0,1] \times (0,+\infty)^n \to [0,+\infty)$ 满足对于所有的 $0 < v_i \le u_i (i = 1,2,\cdots,n)$ 都有 $\phi(t, v_1, v_2, \cdots, v_n) \le \phi(t, u_1, u_2, \cdots, u_n)$，$\psi(t, v_1, v_2, \cdots, v_n) \ge \psi(t, u_1, u_2, \cdots, u_n)$ 使得

$$f(t, v_1, v_2, \cdots, v_n) - b = \phi(t, v_1, v_2, \cdots, v_n) + \psi(t, v_1, v_2, \cdots, v_n) \ge 0,\ t \in [0,1];$$

(H2) 对于任意的 $t \in [0,1]$，存在 $\sigma \in (0,1)$ 使得对于任意的 $v_i > 0 (i = 1,2,\cdots,n)$ 以及 $\eta \in (0,1)$ 都有 $\phi(t, \eta v_1, \eta v_2, \cdots, \eta v_n) \ge \eta^\sigma \phi(t, v_1, v_2, \cdots, v_n)$，$\psi(t, \eta^{-1} v_1, \eta^{-1} v_2, \cdots, \eta^{-1} v_n) \ge \eta^\sigma \psi(t, v_1, v_2, \cdots, v_n)$；

(H3) $\omega^{-\sigma} \int_0^1 t^{-\sigma(\alpha-1)} \psi(t, H, H, \cdots, H) \mathrm{d}t < +\infty$，则边值问题(5)有唯一解 $v^* \in P_h$。

证明 定义算子 $T : P_h \times P_h \to E$ 如下：

$$\begin{aligned} T(u,v)(t) &= \int_0^1 G(t,s)\left[f\left(s, I_{0+}^{\beta_{n-1}} u(s), I_{0+}^{\beta_{n-1}-\beta_1} u(s), \cdots, I_{0+}^{\beta_{n-1}-\beta_{n-2}} u(s), u(s)\right) - b\right] \mathrm{d}s \\ &= \int_0^1 G(t,s)\left[\phi\left(s, I_{0+}^{\beta_{n-1}} u(s), I_{0+}^{\beta_{n-1}-\beta_1} u(s), \cdots, I_{0+}^{\beta_{n-1}-\beta_{n-2}} u(s), u(s)\right)\right. \\ &\quad \left. + \psi\left(s, I_{0+}^{\beta_{n-1}} v(s), I_{0+}^{\beta_{n-1}-\beta_1} v(s), \cdots, I_{0+}^{\beta_{n-1}-\beta_{n-2}} v(s), v(s)\right)\right] \mathrm{d}s,\ t \in [0,1], \end{aligned}$$

因此 v 是边值问题(5)的解当且仅当 v 满足 $v = T(v,v)$。

为方便起见，在本节剩余部分，令 $\beta_0 = 0$。

由 P 的定义知 P 是一个正规锥。首先证明 $T : P_h \times P_h \to P_h$。

事实上，若 $u, v \in P_h$，则存在 $M_1, M_2 > 0$ 使得

$$\frac{1}{M_1} h(t) \le u(t) \le M_1 h(t),\ \frac{1}{M_2} h(t) \le v(t) \le M_2 h(t),\ t \in [0,1]. \tag{6}$$

设 $M = \max\{M_1, M_2\}$，则有 $M > 1$ 且

$$\frac{1}{M} h(t) \le u(t) \le M h(t),\ \frac{1}{M} h(t) \le v(t) \le M h(t),\ t \in [0,1].$$

根据(H2)，通过简单计算可知，对于任意的 $\eta > 1$，$v_i > 0 (i = 1,2,\cdots,n)$，

$$\phi(t, \eta v_1, \eta v_2, \cdots, \eta v_n) \le \eta^\sigma \phi(t, v_1, v_2, \cdots, v_n),\ t \in [0,1], \tag{7}$$

$$\psi(t, \eta^{-1} v_1, \eta^{-1} v_2, \cdots, \eta^{-1} v_n) \le \eta^\sigma \psi(t, v_1, v_2, \cdots, v_n),\ t \in [0,1]. \tag{8}$$

既然

$$0 \le I_{0+}^{\beta_{n-1}-\beta_i} h(t) = H \frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta_i)} t^{\alpha-\beta_i-1} \le H,\ t \in [0,1],\ i = 0,1,2,\cdots,n-1,$$

由(6)、(7)以及(8)，鉴于(H1)可得

$$\begin{aligned}&\phi\left(t,I_{0+}^{\beta_{n-1}}u(t),I_{0+}^{\beta_{n-1}-\beta_1}u(t),\cdots,I_{0+}^{\beta_{n-1}-\beta_{n-2}}u(t),u(t)\right)\\&\le\phi\left(t,MI_{0+}^{\beta_{n-1}}h(t),MI_{0+}^{\beta_{n-1}-\beta_1}h(t),\cdots,MI_{0+}^{\beta_{n-1}-\beta_{n-2}}h(t),Mh(t)\right)\\&\le\phi(t,MH,MH,\cdots,MH,MH)\\&\le M^{\sigma}\phi(t,H,H,\cdots,H,H)\end{aligned}\tag{9}$$

以及

$$\begin{aligned}&\psi\left(t,I_{0+}^{\beta_{n-1}}v(t),I_{0+}^{\beta_{n-1}-\beta_1}v(t),\cdots,I_{0+}^{\beta_{n-1}-\beta_{n-2}}v(t),v(t)\right)\\&\ge\psi\left(t,MI_{0+}^{\beta_{n-1}}h(t),MI_{0+}^{\beta_{n-1}-\beta_1}h(t),\cdots,MI_{0+}^{\beta_{n-1}-\beta_{n-2}}h(t),Mh(t)\right)\\&\ge\psi(t,MH,MH,\cdots,MH,MH)\\&\ge M^{-\sigma}\psi(t,H,H,\cdots,H,H).\end{aligned}\tag{10}$$

既然

$$I_{0+}^{\beta_{n-1}-\beta_i}h(t)=H\frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta_i)}t^{\alpha-\beta_i-1}\ge H\omega t^{\alpha-1},\ t\in[0,1],\ i=0,1,2,\cdots,n-1,$$

则由(9)、(10)，鉴于(H2)可得

$$\begin{aligned}&\phi\left(t,I_{0+}^{\beta_{n-1}}u(t),I_{0+}^{\beta_{n-1}-\beta_1}u(t),\cdots,I_{0+}^{\beta_{n-1}-\beta_{n-2}}u(t),u(t)\right)\\&\ge\phi\left(t,M^{-1}I_{0+}^{\beta_{n-1}}h(t),M^{-1}I_{0+}^{\beta_{n-1}-\beta_1}h(t),\cdots,M^{-1}I_{0+}^{\beta_{n-1}-\beta_{n-2}}h(t),M^{-1}h(t)\right)\\&\ge\phi\left(t,M^{-1}H\omega t^{\alpha-1},M^{-1}H\omega t^{\alpha-1},\cdots,M^{-1}H\omega t^{\alpha-1},M^{-1}H\omega t^{\alpha-1}\right)\\&\ge M^{-\sigma}\omega^{\sigma}t^{\sigma(\alpha-1)}\phi(t,H,H,\cdots,H,H)\end{aligned}\tag{11}$$

以及

$$\begin{aligned}&\psi\left(t,I_{0+}^{\beta_{n-1}}v(t),I_{0+}^{\beta_{n-1}-\beta_1}v(t),\cdots,I_{0+}^{\beta_{n-1}-\beta_{n-2}}v(t),v(t)\right)\\&\le\psi\left(t,M^{-1}I_{0+}^{\beta_{n-1}}h(t),M^{-1}I_{0+}^{\beta_{n-1}-\beta_1}h(t),\cdots,M^{-1}I_{0+}^{\beta_{n-1}-\beta_{n-2}}h(t),M^{-1}h(t)\right)\\&\le\psi\left(t,M^{-1}H\omega t^{\alpha-1},M^{-1}H\omega t^{\alpha-1},\cdots,M^{-1}H\omega t^{\alpha-1},M^{-1}H\omega t^{\alpha-1}\right)\\&\le M^{\sigma}\omega^{-\sigma}t^{-\sigma(\alpha-1)}\psi(t,H,H,\cdots,H,H).\end{aligned}\tag{12}$$

令

$$\frac{1}{m_1}=\frac{1}{H\Gamma(\alpha-\beta_{n-1})}\int_0^1(1-s)^{\alpha-\beta-1}\left(1-(1-s)^{\beta-\beta_{n-1}}\right)\left[\omega^{\sigma}s^{\sigma(\alpha-1)}\phi(s,H,H,\cdots,H,H)+\psi(s,H,H,\cdots,H,H)\right]\mathrm{d}s,$$

$$m_2=\frac{1}{H\Gamma(\alpha-\beta_{n-1})}\int_0^1(1-s)^{\alpha-\beta-1}\left[\phi(s,H,H,\cdots,H,H)+\omega^{-\sigma}s^{-\sigma(\alpha-1)}\psi(s,H,H,\cdots,H,H)\right]\mathrm{d}s.$$

由(11)、(12)，鉴于(H1)以及引理 5 可得

$$\begin{aligned}T(u,v)(t)&=\int_0^1G(t,s)\Big[\phi\left(s,I_{0+}^{\beta_{n-1}}u(s),I_{0+}^{\beta_{n-1}-\beta_1}u(s),\cdots,I_{0+}^{\beta_{n-1}-\beta_{n-2}}u(s),u(s)\right)\\&\quad+\psi\left(s,I_{0+}^{\beta_{n-1}}v(s),I_{0+}^{\beta_{n-1}-\beta_1}v(s),\cdots,I_{0+}^{\beta_{n-1}-\beta_{n-2}}v(s),v(s)\right)\Big]\mathrm{d}s\\&\ge\int_0^1\frac{t^{\alpha-\beta_{n-1}-1}(1-s)^{\alpha-\beta-1}\left[1-(1-s)^{\beta-\beta_{n-1}}\right]}{\Gamma(\alpha-\beta_{n-1})}\Big[M^{-\sigma}\omega^{\sigma}s^{\sigma(\alpha-1)}\phi(s,H,H,\cdots,H,H)\\&\quad+M^{-\sigma}\psi(s,H,H,\cdots,H,H)\Big]\mathrm{d}s\\&=\frac{1}{m_1M^{\sigma}}h(t),\ t\in[0,1]\end{aligned}$$

以及

$$
\begin{aligned}
T(u,v)(t) &= \int_0^1 G(t,s)\Big[\phi\Big(s, I_{0+}^{\beta_{n-1}}u(s), I_{0+}^{\beta_{n-1}-\beta_1}u(s), \cdots, I_{0+}^{\beta_{n-1}-\beta_{n-2}}u(s), u(s)\Big) \\
&\quad + \psi\Big(s, I_{0+}^{\beta_{n-1}}v(s), I_{0+}^{\beta_{n-1}-\beta_1}v(s), \cdots, I_{0+}^{\beta_{n-1}-\beta_{n-2}}v(s), v(s)\Big)\Big]\mathrm{d}s \\
&\le \int_0^1 \frac{t^{\alpha-\beta_{n-1}-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta_{n-1})}\Big[M^{\sigma}\phi(s,H,H,\cdots,H,H) \\
&\quad + M^{\sigma}\omega^{-\sigma}s^{-\sigma(\alpha-1)}\psi(s,H,H,\cdots,H,H)\Big]\mathrm{d}s \\
&= m_2 M^{\sigma} h(t),\ t\in[0,1].
\end{aligned}
$$

结合(H3)，可得

$$
0 < \frac{1}{m_1 M^{\sigma}} \le T(u,v)(t) \le m_2 M^{\sigma} h(t) < +\infty,
$$

从而，$T: P_h \times P_h \to P_h$。

接着证明 T 是一个混合单调算子。

根据(H1)，对于任意的 $u_1 \lesssim u_2\ (u_1,u_2\in P_h)$， $v\in P_h$，

$$
\begin{aligned}
T(u_1,v)(t) &= \int_0^1 G(t,s)\Big[\phi\Big(s, I_{0+}^{\beta_{n-1}}u_1(s), I_{0+}^{\beta_{n-1}-\beta_1}u_1(s), \cdots, I_{0+}^{\beta_{n-1}-\beta_{n-2}}u_1(s), u_1(s)\Big) \\
&\quad + \psi\Big(s, I_{0+}^{\beta_{n-1}}v(s), I_{0+}^{\beta_{n-1}-\beta_1}v(s), \cdots, I_{0+}^{\beta_{n-1}-\beta_{n-2}}v(s), v(s)\Big)\Big]\mathrm{d}s \\
&\le \int_0^1 G(t,s)\Big[\phi\Big(s, I_{0+}^{\beta_{n-1}}u_2(s), I_{0+}^{\beta_{n-1}-\beta_1}u_2(s), \cdots, I_{0+}^{\beta_{n-1}-\beta_{n-2}}u_2(s), u_2(s)\Big) \\
&\quad + \psi\Big(s, I_{0+}^{\beta_{n-1}}v(s), I_{0+}^{\beta_{n-1}-\beta_1}v(s), \cdots, I_{0+}^{\beta_{n-1}-\beta_{n-2}}v(s), v(s)\Big)\Big]\mathrm{d}s \\
&= T(u_2,v)(t),\ t\in[0,1],
\end{aligned}
$$

对于任意的 $v_1 \lesssim v_2\ (v_1,v_2\in P_h)$， $u\in P_h$，

$$
\begin{aligned}
T(u_1,v)(t) &= \int_0^1 G(t,s)\Big[\phi\Big(s, I_{0+}^{\beta_{n-1}}u_1(s), I_{0+}^{\beta_{n-1}-\beta_1}u_1(s), \cdots, I_{0+}^{\beta_{n-1}-\beta_{n-2}}u_1(s), u_1(s)\Big) \\
&\quad + \psi\Big(s, I_{0+}^{\beta_{n-1}}v(s), I_{0+}^{\beta_{n-1}-\beta_1}v(s), \cdots, I_{0+}^{\beta_{n-1}-\beta_{n-2}}v(s), v(s)\Big)\Big]\mathrm{d}s \\
&\le \int_0^1 G(t,s)\Big[\phi\Big(s, I_{0+}^{\beta_{n-1}}u_2(s), I_{0+}^{\beta_{n-1}-\beta_1}u_2(s), \cdots, I_{0+}^{\beta_{n-1}-\beta_{n-2}}u_2(s), u_2(s)\Big) \\
&\quad + \psi\Big(s, I_{0+}^{\beta_{n-1}}v(s), I_{0+}^{\beta_{n-1}-\beta_1}v(s), \cdots, I_{0+}^{\beta_{n-1}-\beta_{n-2}}v(s), v(s)\Big)\Big]\mathrm{d}s \\
&= T(u_2,v)(t),\ t\in[0,1].
\end{aligned}
$$

因此，由定义 3 可知，T 是一个混合单调算子。

最后证明 T 是一个 $t-\sigma(t)$ 型混合单调算子。

对任意的 $t\in[0,1]$，存在 $\sigma\in(0,1)$ 使得对所有的 $u,v>0$ 及 $\eta\in(0,1)$， 由(H2)有

$$
\begin{aligned}
&\phi\Big(t, I_{0+}^{\beta_{n-1}}\eta u(t), I_{0+}^{\beta_{n-1}-\beta_1}\eta u(t), \cdots, I_{0+}^{\beta_{n-1}-\beta_{n-2}}\eta u(t), \eta u(t)\Big) \\
&\ge \eta^{\sigma}\phi\Big(t, I_{0+}^{\beta_{n-1}}u(t), I_{0+}^{\beta_{n-1}-\beta_1}u(t), \cdots, I_{0+}^{\beta_{n-1}-\beta_{n-2}}u(t), u(t)\Big)
\end{aligned}
\tag{13}
$$

以及

$$
\begin{aligned}
&\psi\Big(t, I_{0+}^{\beta_{n-1}}\eta^{-1}v(t), I_{0+}^{\beta_{n-1}-\beta_1}\eta^{-1}v(t), \cdots, I_{0+}^{\beta_{n-1}-\beta_{n-2}}\eta^{-1}v(t), \eta^{-1}v(t)\Big) \\
&\ge \eta^{\sigma}\psi\Big(t, I_{0+}^{\beta_{n-1}}v(t), I_{0+}^{\beta_{n-1}-\beta_1}v(t), \cdots, I_{0+}^{\beta_{n-1}-\beta_{n-2}}v(t), v(t)\Big)
\end{aligned}
\tag{14}
$$

由(13)、(14)可知

$$\begin{aligned}T\left(\eta u,\eta^{-1}v\right)(t)&=\int_0^1 G(t,s)\Big[\phi\left(s,I_{0+}^{\beta_{n-1}}\eta u(s),I_{0+}^{\beta_{n-1}-\beta_1}\eta u(s),\cdots,I_{0+}^{\beta_{n-1}-\beta_{n-2}}\eta u(s),\eta u(s)\right)\\&\quad+\psi\left(s,I_{0+}^{\beta_{n-1}}\eta^{-1}v(s),I_{0+}^{\beta_{n-1}-\beta_1}\eta^{-1}v(s),\cdots,I_{0+}^{\beta_{n-1}-\beta_{n-2}}\eta^{-1}v(s),\eta^{-1}v(s)\right)\Big]\mathrm{d}s\\&\geq\eta^{\sigma}\int_0^1 G(t,s)\Big[\phi\left(s,I_{0+}^{\beta_{n-1}}u(s),I_{0+}^{\beta_{n-1}-\beta_1}u(s),\cdots,I_{0+}^{\beta_{n-1}-\beta_{n-2}}u(s),u(s)\right)\\&\quad+\psi\left(s,I_{0+}^{\beta_{n-1}}v(s),I_{0+}^{\beta_{n-1}-\beta_1}v(s),\cdots,I_{0+}^{\beta_{n-1}-\beta_{n-2}}v(s),v(s)\right)\Big]\mathrm{d}s\\&=\eta^{\sigma}T(u,v)(t).\end{aligned}$$

由定义 4，T 是一个 $t-\sigma(t)$ 型混合单调算子。

综上所述，引理 6 的所有条件均满足，故由引理 6 可知，算子 T 在 P_h 中存在唯一的不动点 v^*，即边值问题(5)存在唯一解 $v^*\in P_h$。更进一步，任取 $u_0,v_0\in P_h$，构造序列

$$u_n=T\left(u_{n-1},v_{n-1}\right),\ v_n=T\left(v_{n-1},u_{n-1}\right),\ n=1,2,\cdots,$$

必有 $\lim\limits_{n\to\infty}\left\|u_n-v^*\right\|=0$， $\lim\limits_{n\to\infty}\left\|v_n-v^*\right\|=0$。

4. 实例分析

对于边值问题(5)，令 $n=2$， $\alpha=\frac{3}{2}$， $\beta_1=\frac{1}{8}$， $\beta=\frac{1}{4}$， $b=1$，考虑下述边值问题

$$\begin{cases}D_{0+}^{\frac{3}{2}}u(t)+\left[t^{\frac{1}{2}}\left(u(t)^{\frac{1}{2}}+\left(1/u(t)\right)^{\frac{1}{2}}\right)+\frac{1}{4}\right]+\left[t^{\frac{1}{3}}\left(u(t)^{\frac{1}{3}}+\left(1/u(t)\right)^{\frac{1}{3}}\right)+\frac{1}{4}\right]\\\quad+\left[t^{\frac{1}{2}}\left(D_{0+}^{\frac{1}{8}}u(t)^{\frac{1}{2}}+\left(1\Big/D_{0+}^{\frac{1}{8}}u(t)\right)^{\frac{1}{2}}\right)+\frac{1}{4}\right]+\left[t^{\frac{1}{3}}\left(D_{0+}^{\frac{1}{8}}u(t)^{\frac{1}{3}}+\left(1\Big/D_{0+}^{\frac{1}{8}}u(t)\right)^{\frac{1}{3}}\right)+\frac{1}{4}\right]=1,\ 0<t<1,\\u(0)=D_{0+}^{\frac{1}{4}}u(1)=0.\end{cases}\tag{15}$$

设 $u(t)=I_{0+}^{\frac{1}{8}}v(t)$， $t\in[0,1]$，则边值问题(15)可以转化为：

$$\begin{cases}D_{0+}^{\frac{11}{8}}u(t)+\left[t^{\frac{1}{2}}\left(I_{0+}^{\frac{1}{8}}v(t)^{\frac{1}{2}}+\left(1\Big/I_{0+}^{\frac{1}{8}}v(t)\right)^{\frac{1}{2}}\right)+\frac{1}{4}\right]+\left[t^{\frac{1}{3}}\left(I_{0+}^{\frac{1}{8}}v(t)^{\frac{1}{3}}+\left(1\Big/I_{0+}^{\frac{1}{8}}v(t)\right)^{\frac{1}{3}}\right)+\frac{1}{4}\right]\\\quad+\left[t^{\frac{1}{2}}\left(v(t)^{\frac{1}{2}}+\left(1/v(t)\right)^{\frac{1}{2}}\right)+\frac{1}{4}\right]+\left[t^{\frac{1}{3}}\left(v(t)^{\frac{1}{3}}+\left(1/v(t)\right)^{\frac{1}{2}}\right)+\frac{1}{4}\right]=1,\ 0<t<1,\\I_{0+}^{\frac{1}{8}}v(0)=D_{0+}^{\frac{1}{8}}u(1)=0.\end{cases}\tag{16}$$

对于任意的 $x_1,x_2>0$，令

$$f(t,x_1,x_2)-1=\sum_{i=1}^{2}\left[t^{\frac{1}{2}}\left[x_i^{\frac{1}{2}}+\left(1/x_i\right)^{\frac{1}{2}}\right]+t^{\frac{1}{3}}\left[x_i^{\frac{1}{3}}+\left(1/x_i\right)^{\frac{1}{3}}\right]\right],\ t\in[0,1],$$

$$\phi(t,x_1,x_2)=t^{\frac{1}{2}}\left(x_1^{\frac{1}{2}}+x_2^{\frac{1}{2}}\right)+t^{\frac{1}{3}}\left(x_1^{\frac{1}{3}}+x_2^{\frac{1}{3}}\right)\geq 0,\ t\in[0,1],$$

$$\psi\left(t,x_1,x_2\right)=t^{\frac{1}{2}}\left(\left(1/x_1\right)^{\frac{1}{2}}+\left(1/x_2\right)^{\frac{1}{2}}\right)+t^{\frac{1}{3}}\left(\left(1/x_1\right)^{\frac{1}{3}}+\left(1/x_2\right)^{\frac{1}{3}}\right)\geq 0,\ t\in[0,1].$$

容易验证下述条件成立：

1) 显然 $\phi,\psi:[0,1]\times(0,+\infty)^2\to[0,+\infty)$ 连续且任给 $0<x_i\leq y_i\ (i=1,2)$ 都有 $\phi(t,x_1,x_2)\leq\phi(t,y_1,y_2)$，$\psi(t,x_1,x_2)\geq\psi(t,y_1,y_2)$ 使得

$$f\left(t,x_1,x_2\right)-1=\phi\left(t,x_1,x_2\right)+\psi\left(t,x_1,x_2\right)\geq 0,\ t\in[0,1].$$

2) 任给 $\eta\in(0,1)$，取 $\sigma=\frac{1}{2}\in(0,1)$ 使得对于任意的 $x_i>0\ (i=1,2,\cdots,n)$，$t\in[0,1]$ 有

$$\begin{aligned}\phi\left(t,\eta x_1,\eta x_2\right)&=t^{\frac{1}{2}}\eta^{\frac{1}{2}}\left(x_1^{\frac{1}{2}}+x_2^{\frac{1}{2}}\right)+\eta^{\frac{1}{3}}t^{\frac{1}{3}}\left(x_1^{\frac{1}{3}}+x_2^{\frac{1}{3}}\right)\\&\geq\eta^{\frac{1}{2}}\left[t^{\frac{1}{2}}\left(x_1^{\frac{1}{2}}+x_2^{\frac{1}{2}}\right)+t^{\frac{1}{3}}\left(x_1^{\frac{1}{3}}+x_2^{\frac{1}{3}}\right)\right]\\&=\eta^{\frac{1}{2}}\phi\left(t,x_1,x_2\right),\end{aligned}$$

$$\begin{aligned}\psi\left(t,\eta^{-1}x_1,\eta^{-1}x_2\right)&=t^{\frac{1}{2}}\eta^{\frac{1}{2}}\left(\left(1/x_1\right)^{\frac{1}{2}}+\left(1/x_2\right)^{\frac{1}{2}}\right)+t^{\frac{1}{3}}\eta^{\frac{1}{3}}\left(\left(1/x_1\right)^{\frac{1}{3}}+\left(1/x_2\right)^{\frac{1}{3}}\right)\\&\geq\eta^{\frac{1}{2}}\left[t^{\frac{1}{2}}\left(\left(1/x_1\right)^{\frac{1}{2}}+\left(1/x_2\right)^{\frac{1}{2}}\right)+t^{\frac{1}{3}}\left(\left(1/x_1\right)^{\frac{1}{3}}+\left(1/x_2\right)^{\frac{1}{3}}\right)\right]\\&=\eta^{\frac{1}{2}}\psi\left(t,x_1,x_2\right).\end{aligned}$$

3) 通过简单计算可知

$$H=0.9,\ \omega=1,\ \psi\left(t,H,H\right)=2\times\left[\left(t/0.9\right)^{\frac{1}{2}}+\left(t/0.9\right)^{\frac{1}{3}}\right],$$

因此，

$$\omega^{-\sigma}\int_0^1 s^{-\sigma(\alpha-1)}\psi\left(t,H,H\right)\mathrm{d}s=3.5987<+\infty.$$

定理 1 的条件全部满足，边值问题(15)有唯一解。

5. 结论与展望

本文主要研究的是一类高阶分数阶微分方程边值问题解的唯一性。

通过文献资料的学习并受到文献[13]和[14]的启发，进而确定了本文的问题模型。首先，将原边值问题通过降阶的方法转变成一个非线性项不含导数的等价边值问题，这样就可以继续在原有的空间 $C[0,1]$ 中求解问题。其次，分析 Green 函数的性质并给出相应等价问题具有唯一解的充分条件。最后利用集合 P_h 中的 $t-\sigma(t)$ 型混合单调算子不动点定理证明边值问题存在唯一解并构造出逼近此解的迭代序列。

同参考文献[14]相比，本文在方程部分加了一个大于 0 的常数 b，这就需要引入集合 P_h 中的混合单调算子不动点定理去研究解的性质，得到了边值问题解的唯一性。分数阶微分方程的重要程度不言而喻，随着其不断发展应用到自然科学和生产生活的各个领域，对边值问题解或正解的研究有待进一步探索。接下来可以将原边界条件进一步扩展为更具一般性的 Riemann-Stieltjes 积分边界条件并对其展开研究。

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