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耦合磁场的两相流模型的低马赫数极限

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摘　要

本文研究了耦合磁场的两相流模型的低马赫数极限问题。在适当的初值条件下，证明了当马赫数趋近于零时，可压缩两相流模型的解收敛于不可压缩磁流体方程的强解。

关键词

可压缩两相流模型，不可压缩磁流体方程，低马赫数极限，相对熵

Low Mach Number Limit of a Two-Phase Model with Magnetic Field

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Abstract

In this paper, we study the low Mach number limit of a two-phase model with magnetic field. In the case of well-prepared initial data, we prove that the solution of the compressible two-fluid system converges to the strong solution of the incompressible magnetohydrodynamic equations as the Mach number $\epsilon \to 0$.

Keywords

Compressible Two-Fluid Model, Incompressible MHD Equations, Low Mach Number Limit, Relative Entropy

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1. 引言

研究耦合磁场的两相流模型的低马赫数极限问题具有重要物理意义，该研究不仅丰富了等离子体等应用学科的理论知识，而且有助于从数学角度解释流体相关物理现象，从而对实际应用起到指导作用。例如在一些涉及磁流体和两相流的工程应用场景中，如核电领域中对相关流动现象的研究。

本文研究下述耦合磁场的两相流模型

$$\begin{cases} \partial_t n + \operatorname{div}(n\boldsymbol{u}) = 0, \\ \partial_t \rho + \operatorname{div}(\rho\boldsymbol{u}) = 0, \\ \partial_t \left[(n+\rho)\boldsymbol{u}\right] + \operatorname{div}\left[(\rho+n)\boldsymbol{u}\otimes\boldsymbol{u}\right] + \nabla P(n,\rho) \\ \qquad\qquad = \mu\Delta\boldsymbol{u} + (\mu+\lambda)\nabla\operatorname{div}\boldsymbol{u} + (\nabla\times\boldsymbol{B})\times\boldsymbol{B}, \\ \partial_t \boldsymbol{B} - \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) = -\nabla\times(\nu\nabla\times\boldsymbol{B}), \\ \operatorname{div}\boldsymbol{B} = 0, \end{cases} \tag{1.1}$$

的低马赫数极限。其中，$n(x,t)>0$ 和 $\rho(x,t)>0$ 分别表示两种流体的密度函数，$\boldsymbol{u}(x,t)$ 表示两种流体的速度函数以及 $\boldsymbol{B}(x,t)$ 表示磁场。压力函数 $P(n,\rho)=An^{\gamma}+\rho$，$\gamma>1$，常数 $A>0$，γ 表示绝热指数，粘度系数 μ，λ 满足 $\mu>0$，$\lambda+\frac{2}{3}\mu\geq 0$，$\nu>0$ 表示磁扩散系数。

当 $\rho\equiv 0$ 且不考虑磁场时，模型(1.1)简化为可压缩等熵 Navier-Stokes 方程，Pierre L Lions [1]等证明了当马赫数趋于零时，该方程的整体弱解到不可压缩 Navier-Stokes 方程解的收敛性。

在不考虑磁场影响的情况下，模型(1.1)简化为流体－粒子相互作用模型，王红利[2]等通过相对熵方法和精细的能量估计，证明了在适当初值条件下，当马赫数趋于零时，可压缩流体－粒子相互作用模型的弱解收敛到不可压缩 Navier-Stokes 方程的强解。

当 $\rho\equiv 0$ 时，模型(1.1)简化为可压缩等熵 MHD 方程。黎野平[3]通过相对熵方法和精细的能量估计，证明了当马赫数趋于零时，三维粘性可压缩磁流体方程的光滑解收敛到理想不可压缩磁流体方程的光滑解。胡贤鹏[4]等和江松[5]等通过相对熵方法和精细的能量估计，分别在不同空间中，证明了当马赫数趋于零时，可压缩等熵粘性磁流体方程的弱解收敛到不可压缩粘性磁流体方程的弱解。

最近，温焕尧[6]等通过引入一个新的线性化模型，证明了耦合磁场的粘性可压缩两相流模型的强解的整体存在性和唯一性。本文将基于相对熵方法和精细的能量估计，在适当的初值条件下，证明当马赫数趋于零时，耦合磁场的可压缩两相流模型的解收敛到不可压磁流体方程的强解。

2. 形式推导

为了证明系统(1.1)在三维全空间上的低马赫数极限，进行如下尺度变换

$$n(x,t)=n^{\epsilon}(x,\epsilon t),\ \rho(x,t)=\rho^{\epsilon}(x,\epsilon t),$$
$$\boldsymbol{u}(x,t)=\epsilon\boldsymbol{u}^{\epsilon}(x,\epsilon t),\ \boldsymbol{B}(x,t)=\epsilon\boldsymbol{B}^{\epsilon}(x,\epsilon t)$$

并假设粘度系数 μ，λ 和磁扩散系数 ν 的尺度变换如下

$$\mu\mapsto\epsilon\mu,\ \lambda\mapsto\epsilon\lambda,\ \nu\mapsto\epsilon\nu,$$

其中 $\epsilon \in (0,1)$ 为尺度化后的马赫数，在上述尺度变换下可得

$$
\begin{cases}
\partial_t n^\epsilon + \operatorname{div}\left(n^\epsilon \boldsymbol{u}^\epsilon\right) = 0, \\
\partial_t \rho^\epsilon + \operatorname{div}\left(\rho^\epsilon \boldsymbol{u}^\epsilon\right) = 0, \\
\partial_t \left[\left(n^\epsilon + \rho^\epsilon\right)\boldsymbol{u}^\epsilon\right] + \operatorname{div}\left[\left(n^\epsilon + \rho^\epsilon\right)\boldsymbol{u}^\epsilon \otimes \boldsymbol{u}^\epsilon\right] + \dfrac{1}{\epsilon^2}\nabla P\left(n^\epsilon, \rho^\epsilon\right) \\
\qquad\qquad = \mu \Delta \boldsymbol{u}^\epsilon + \left(\mu + \lambda\right)\nabla \operatorname{div} \boldsymbol{u}^\epsilon + \left(\nabla \times \boldsymbol{B}^\epsilon\right) \times \boldsymbol{B}^\epsilon, \\
\partial_t \boldsymbol{B}^\epsilon - \nabla \times \left(\boldsymbol{u}^\epsilon \times \boldsymbol{B}^\epsilon\right) = -\nabla \times \left(\nu \nabla \times \boldsymbol{B}^\epsilon\right), \\
\operatorname{div} \boldsymbol{B}^\epsilon = 0.
\end{cases}
\tag{2.1}
$$

其中系统(2.1)的初值满足

$$
\left(n^\epsilon, \rho^\epsilon, \boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon\right)(x,0) = \left(n_0^\epsilon, \rho_0^\epsilon, \boldsymbol{u}_0^\epsilon, \boldsymbol{B}_0^\epsilon\right)(x), x \in \mathbb{R}^3. \tag{2.2}
$$

记 $\nabla P\left(n^\epsilon, \rho^\epsilon\right) = \nabla\left[P\left(n^\epsilon, \rho^\epsilon\right) - P\left(n^*, \rho^*\right)\right]$。动量方程表明 n^ϵ，ρ^ϵ 应分别表示为 $n = n^* + O\left(\epsilon^2\right)$，$\rho = \rho^* + O\left(\epsilon^2\right)$，其中，$n^*$ 和 ρ^* 是正的常数。当 $\epsilon \to 0^+$ 时，因为 n^ϵ 和 ρ^ϵ 分别趋向于 n^* 和 ρ^*，期望从(2.1)$_1$ 或(2.1)$_2$ 推导出极限 $\operatorname{div}\boldsymbol{u}^0 = 0$。假设极限 $\boldsymbol{u}^\epsilon \to \boldsymbol{u}^0$，$\boldsymbol{B}^\epsilon \to \boldsymbol{B}^0$ 存在，于是形式上可得下列不可压缩磁流体方程：

$$
\begin{cases}
\operatorname{div} \boldsymbol{u}^0 = 0, \ \operatorname{div} \boldsymbol{B}^0 = 0, \\
\partial_t \boldsymbol{u}^0 + \left(\boldsymbol{u}^0 \cdot \nabla\right)\boldsymbol{u}^0 + \nabla \pi^0 = \dfrac{\mu}{\rho^* + n^*}\Delta \boldsymbol{u}^0 + \dfrac{1}{\rho^* + n^*}\left(\nabla \times \boldsymbol{B}^0\right) \times \boldsymbol{B}^0, \\
\partial_t \boldsymbol{B}^0 - \nabla \times \left(\boldsymbol{u}^0 \times \boldsymbol{B}^0\right) = -\nabla \times \left(\nu \nabla \times \boldsymbol{B}^0\right),
\end{cases}
\tag{2.3}
$$

其中，π^0 是 $\dfrac{P\left(n^\epsilon, \rho^\epsilon\right) - P\left(n^*, \rho^*\right)}{\epsilon^2}$ 的极限。

本文将严格证明上述极限过程，即证明在适当的初值条件下，当 ϵ 趋于 0 时，可压缩两相流模型(2.1)的解收敛于不可压缩磁流体方程(2.3)的解。

在本文中，C 是与 ϵ 无关的正常数，χ 为特征函数。为简单起见，用 $\int f \, dx = \int_{\mathbb{R}^3} f \, dx$ 表示。对于 $1 \le m, k \le \infty$，即：

$$
L^m = L^m\left(\mathbb{R}^3\right), \ W^{k,m} = W^{k,m}\left(\mathbb{R}^3\right), \ H^k = W^{k,2}, \ L^2 = H^0.
$$

3. 主要结果

命题 3.1 [7] [8] 设 $s > \dfrac{3}{2} + 2$。假设不可压 MHD 系统(2.3)的初值 $\left(\boldsymbol{u}^0, \boldsymbol{B}^0\right)\big|_{t=0} = \left(\boldsymbol{u}_0^0, \boldsymbol{B}_0^0\right)$ 满足

$$
\boldsymbol{u}_0^0 \in H^s, \boldsymbol{B}_0^0 \in H^s \text{ 和 } \operatorname{div}\boldsymbol{u}_0^0 = \operatorname{div}\boldsymbol{B}_0^0 = 0.
$$

则存在 $\hat{T}^* \in (0, \infty]$，使得不可压缩磁流体方程(2.3)的唯一解 $\left(\boldsymbol{u}^0, \boldsymbol{B}^0\right)$ 满足 $\left(\boldsymbol{u}^0, \boldsymbol{B}^0\right) \in L^\infty\left(\left[0, \hat{T}^*\right); H^s\right)$，且对于任意 $0 < T < \hat{T}^*$，有

$$
\sup_{0 \le t \le T}\left\{\left\|\left(\boldsymbol{u}^0, \boldsymbol{B}^0\right)(t)\right\|_{H^s} + \left\|\left(\partial_t \boldsymbol{u}^0, \partial_t \boldsymbol{B}^0\right)(t)\right\|_{H^{s-2}} + \left\|\nabla \pi^0(t)\right\|_{H^{s-2}}\right\} \le C. \tag{3.1}
$$

命题 3.2 [6] 假设

$$
\left(n_0^\epsilon - n^*, \left(n_0^\epsilon\right)^\gamma - \left(n^*\right)^\gamma\right) \in H^2\left(\mathbb{R}^3\right), \left(P\left(n_0^\epsilon, \rho_0^\epsilon\right) - P\left(n^*, \rho^*\right), \boldsymbol{u}_0^\epsilon, \boldsymbol{B}_0^\epsilon\right) \in H^2\left(\mathbb{R}^3\right) \cap L^1\left(\mathbb{R}^3\right)
$$

且 $\operatorname{div}\boldsymbol{B}_0^\epsilon = 0$。则存在一个与 ϵ 无关的常数 $M>0$，使得

$$\left\|\left(n_0^\epsilon - n^*, \left(n_0^\epsilon\right)^\gamma - \left(n^*\right)^\gamma, \boldsymbol{u}_0^\epsilon, \boldsymbol{B}_0^\epsilon\right)\right\|_{H^2} \le M,$$

则系统(2.1)~(2.2)存在唯一的整体强解 $\left(n^\epsilon, \rho^\epsilon, \boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon\right)$ 且满足

$$\left(n^\epsilon - n^*, \rho^\epsilon - \rho^*\right) \in C\left(\left[0,\infty\right); H^2\right) \tag{3.2}$$

和

$$\left(\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon\right) \in C\left(\left[0,\infty\right); H^2\right) \cap L^2\left(\left[0,\infty\right); H^3\right). \tag{3.3}$$

为简化描述，用以下等式表示

$$F\left(n^\epsilon\right) = \frac{A}{\gamma-1}\left[\left(n^\epsilon\right)^\gamma - \gamma\left(n^*\right)^{\gamma-1}\left(n^\epsilon - n^*\right) - \left(n^*\right)^\gamma\right],$$

$$G\left(\rho^\epsilon\right) = \rho^\epsilon \ln \rho^\epsilon - \rho^* \ln \rho^* - \left(\ln \rho^* + 1\right)\left(\rho^\epsilon - \rho^*\right).$$

易知，这两个函数都是凸函数，满足以下等式

$$F''\left(n^\epsilon\right) = A\gamma\left(n^\epsilon\right)^{\gamma-2}, G''\left(\rho^\epsilon\right) = \frac{1}{\rho^\epsilon},$$

$$F'\left(n^*\right) = G'\left(\rho^*\right) = 0, F\left(n^*\right) = G\left(\rho^*\right) = 0.$$

本文主要结果如下：

定理 3.1 设 $\left(n^\epsilon, \rho^\epsilon, \boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon\right)$ 是命题 3.2 中系统(2.1)的解。假设初值 $\left(n_0^\epsilon, \rho_0^\epsilon, \boldsymbol{u}_0^\epsilon, \boldsymbol{B}_0^\epsilon\right)$ 满足(3.2)和以下假设成立

$$\frac{1}{\epsilon^2}\int\left[F\left(n_0^\epsilon\right) + G\left(\rho_0^\epsilon\right)\right]\mathrm{d}x \le C\epsilon^{\min\left\{1,\frac{2}{\gamma}\right\}}, \tag{3.4}$$

$$\int\left|\sqrt{n_0^\epsilon + \rho_0^\epsilon}\,\boldsymbol{u}_0^\epsilon - \sqrt{n^* + \rho^*}\,\boldsymbol{u}_0^0\right|^2 \mathrm{d}x \le C\epsilon^{\min\left\{1,\frac{2}{\gamma}\right\}}, \tag{3.5}$$

$$\int\left|\boldsymbol{B}_0^\epsilon - \boldsymbol{B}_0^0\right|^2 \mathrm{d}x \le C\epsilon^{\min\left\{1,\frac{2}{\gamma}\right\}}, \tag{3.6}$$

设 $\left(\boldsymbol{u}^0, \boldsymbol{B}^0\right)$ 是不可压缩磁流体方程(2.3)以 $\left(\boldsymbol{u}_0^0, \boldsymbol{B}_0^0\right)$ 为初值的光滑解且初值满足(3.1)。则对于任意 $t \in \left[0,T\right]$，有

$$\int\left[F\left(n^\epsilon\right) + G\left(\rho^\epsilon\right)\right]\mathrm{d}x \le C\epsilon^{\min\left\{3,2+\frac{2}{\gamma}\right\}}, \tag{3.7}$$

$$\left\|\sqrt{n^\epsilon + \rho^\epsilon}\,\boldsymbol{u}^\epsilon - \sqrt{n^* + \rho^*}\,\boldsymbol{u}^0\right\|_{L^2}^2 \le C\epsilon^{\min\left\{1,\frac{2}{\gamma}\right\}}, \tag{3.8}$$

$$\left\|\boldsymbol{B}^\epsilon - \boldsymbol{B}^0\right\|_{L^2}^2 \le C\epsilon^{\min\left\{1,\frac{2}{\gamma}\right\}}, \tag{3.9}$$

$$\left\|\boldsymbol{u}^\epsilon - \boldsymbol{u}^0\right\|_{L^2}^2 \le C\epsilon^{\min\left\{1,\frac{2}{\gamma}\right\}}, \tag{3.10}$$

$$\left\|\left(n^\epsilon + \rho^\epsilon\right)\left|\boldsymbol{u}^\epsilon\right|^2 - \left(n^* + \rho^*\right)\left|\boldsymbol{u}^0\right|^2\right\|_{L^1} \le C\epsilon^{\min\left\{\frac{1}{2},\frac{1}{\gamma}\right\}}. \tag{3.11}$$

注：当密度的误差小时，它是属于 L^2 范数。当密度的误差较大时，它是属于 L^γ 。导致对初值也有这样的要求。

引理 3.1 设 $\left(n^\epsilon,\rho^\epsilon,\boldsymbol{u}^\epsilon,\boldsymbol{B}^\epsilon\right)$ 是系统(2.1)~(2.2)在 $\left(0,T^*\right)$ 上的光滑解。则有如下的能量恒等式

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\left[\frac{1}{2}\left(n^\epsilon+\rho^\epsilon\right)\left|\boldsymbol{u}^\epsilon\right|^2+\frac{1}{\epsilon^2}F\left(n^\epsilon\right)+\frac{1}{\epsilon^2}G\left(\rho^\epsilon\right)+\frac{1}{2}\left|\boldsymbol{B}^\epsilon\right|^2\right]\mathrm{d}x\\&+\int\left[\mu\left|\nabla\boldsymbol{u}^\epsilon\right|^2+\left(\mu+\lambda\right)\left|\mathrm{div}\boldsymbol{u}^\epsilon\right|^2+\nu\left|\nabla\times\boldsymbol{B}^\epsilon\right|^2\right]\mathrm{d}x=0,t\in\left(0,T^*\right),\end{aligned}\tag{3.12}$$

证明 将方程$(2.1)_1$，$(2.1)_2$，$(2.1)_3$，$(2.1)_4$两边分别与 $\frac{1}{\epsilon^2}F'\left(n^\epsilon\right)-\frac{1}{2}\left|\boldsymbol{u}^\epsilon\right|^2$， $\frac{1}{\epsilon^2}G'\left(\rho^\epsilon\right)-\frac{1}{2}\left|\boldsymbol{u}^\epsilon\right|^2$， $\boldsymbol{u}^\epsilon$， $\boldsymbol{B}^\epsilon$ 在 $\mathbb{R}^3$ 中作 L^2 内积，之后求和即得(3.12)。

对于极限系统，由命题 3.1 易得系统(2.3)的能量和能量耗散有如下的能量恒等式：

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\left[\frac{1}{2}\left(n^*+\rho^*\right)\left|\boldsymbol{u}^0\right|^2+\frac{1}{2}\left|\boldsymbol{B}^0\right|^2\right]\mathrm{d}x+\int\left(\mu\left|\nabla\boldsymbol{u}^0\right|^2+\nu\left|\nabla\times\boldsymbol{B}^0\right|^2\right)\mathrm{d}x=0.\tag{3.13}$$

根据引理 3.1 易知对于几乎所有 $t\in[0,T^*]$，有如下不等式成立

$$\begin{aligned}&\int\left[\frac{1}{2}\left(n^\epsilon+\rho^\epsilon\right)\left|\boldsymbol{u}^\epsilon\right|^2+\frac{1}{\epsilon^2}F\left(n^\epsilon\right)+\frac{1}{\epsilon^2}G\left(\rho^\epsilon\right)+\frac{1}{2}\left|\boldsymbol{B}^\epsilon\right|^2\right]\mathrm{d}x\\&+\int_0^t\int\left[\mu\left|\nabla\boldsymbol{u}^\epsilon\right|^2+\left(\mu+\lambda\right)\left|\mathrm{div}\boldsymbol{u}^\epsilon\right|^2+\nu\left|\nabla\times\boldsymbol{B}^\epsilon\right|^2\right]\mathrm{d}x\mathrm{d}s\\&\le\int\left[\frac{1}{2}\left(n_0^\epsilon+\rho_0^\epsilon\right)\left|\boldsymbol{u}_0^\epsilon\right|^2+\frac{1}{\epsilon^2}F\left(n_0^\epsilon\right)+\frac{1}{\epsilon^2}G\left(\rho_0^\epsilon\right)+\frac{1}{2}\left|\boldsymbol{B}_0^\epsilon\right|^2\right]\mathrm{d}x\le C.\end{aligned}\tag{3.14}$$

那么从(3.14)可得

$$\frac{1}{\epsilon^2}F\left(n^\epsilon\right)\text{在 }L^\infty\left(\left[0,T\right];L^1\right)\text{ 上有界，}\tag{3.15}$$

$$\frac{1}{\epsilon^2}G\left(\rho^\epsilon\right)\text{在 }L^\infty\left(\left[0,T\right];L^1\right)\text{ 上有界。}\tag{3.16}$$

引理 3.2 设 $\left(n^\epsilon,\rho^\epsilon,\boldsymbol{u}^\epsilon,\boldsymbol{B}^\epsilon\right)$ 是系统(2.1)~(2.2)的光滑解，则存在常数 $C>0$，使得

$$\int\left|n^\epsilon-n^*\right|^2\chi_{\left(\left|n^\epsilon-n^*\right|\le\frac{n^*}{2}\right)}\mathrm{d}x+\int\left|n^\epsilon-n^*\right|^\gamma\chi_{\left(\left|n^\epsilon-n^*\right|>\frac{n^*}{2}\right)}\mathrm{d}x\le C\epsilon^2,\tag{3.17}$$

证明：

1) 如果 $1<\gamma<2$，若 $\left|n^\epsilon-n^*\right|\le\frac{n^*}{2}$，易知

$$\frac{A\left(\frac{3}{2}\right)^{\gamma-2}}{2}\left|n^\epsilon-n^*\right|^2\le F\left(n^\epsilon\right),$$

若 $\left|n^\epsilon-n^*\right|>\frac{n^*}{2}$，易知不等式 $c_0\left|n^\epsilon-n^*\right|^\gamma\le F\left(n^\epsilon\right)$ 成立，其中

$$c_0=\frac{A\left(3^\gamma-\left(\gamma+2\right)2^{\gamma-1}\right)}{\gamma-1},$$

即不等式(3.17)成立。

2) 如果 $\gamma=2$， $F\left(n^{\epsilon}\right)=\frac{A}{2}\left(n^{\epsilon}-n^{*}\right)^{2}$，结果(3.17)由(3.15)直接得出。

3) 如果 $\gamma>2$，当 $\left|n^{\epsilon}-n^{*}\right|\le\frac{n^{*}}{2}$ 时，易证

$$\mathcal{F}_{1}\left(n^{\epsilon}\right)=F\left(n^{\epsilon}\right)-\frac{A}{(\gamma-1)2^{\gamma-2}}\left|n^{\epsilon}-n^{*}\right|^{2}$$

在 $\left(\frac{n^{*}}{2},\frac{3n^{*}}{2}\right)$ 上是凸的且满足

$$\mathcal{F}_{1}\left(n^{*}\right)=\mathcal{F}_{1}'\left(n^{*}\right)=0,$$

即 $\mathcal{F}_{1}\left(n^{\epsilon}\right)\ge 0$ 恒成立。当 $\left|n^{\epsilon}-n^{*}\right|>\frac{n^{*}}{2}$ 时，由函数

$$\mathcal{F}_{2}\left(n^{\epsilon}\right)=F\left(n^{\epsilon}\right)-\frac{A}{\gamma-1}\left|n^{\epsilon}-n^{*}\right|^{\gamma}$$

得

$$\mathcal{F}_{2}'\left(n^{\epsilon}\right)=\frac{A\gamma}{\gamma-1}\left[\left(n^{\epsilon}\right)^{\gamma-1}-\left(n^{*}\right)^{\gamma-1}-\left|n^{\epsilon}-n^{*}\right|^{\gamma-1}\cdot\operatorname{sgn}\left(n^{\epsilon}-n^{*}\right)\right]\tag{3.18}$$

和

$$\mathcal{F}_{2}''\left(n^{\epsilon}\right)=A\gamma\left[\left(n^{\epsilon}\right)^{\gamma-2}-\left|n^{\epsilon}-n^{*}\right|^{\gamma-2}\right].\tag{3.19}$$

根据(3.18)~(3.19)，可得 $\mathcal{F}_{2}\left(n^{\epsilon}\right)$ 在 $\left(0,\frac{n^{*}}{2}\right)$ 上是凹的，在 $\left(\frac{3n^{*}}{2},+\infty\right)$ 上是凸的，且满足 $\mathcal{F}_{2}(0)>0$，$\mathcal{F}_{2}\left(\frac{n^{*}}{2}\right)>0$，$\mathcal{F}_{2}\left(\frac{3n^{*}}{2}\right)>0$ 和 $\mathcal{F}_{2}'\left(\frac{3n^{*}}{2}\right)>0$，即 $\mathcal{F}_{2}\left(n^{\epsilon}\right)\ge 0$ 成立。则由(3.15)可知不等式(3.17)在 $\gamma>2$ 的情况下成立。

4. 定理 3.1 的证明

引入相对能量函数：

$$\mathcal{H}^{\epsilon}(t)=\int\left[\frac{1}{2}\left(n^{\epsilon}+\rho^{\epsilon}\right)\left|\boldsymbol{u}^{\epsilon}-\boldsymbol{u}^{0}\right|^{2}+\frac{1}{\epsilon^{2}}F\left(n^{\epsilon}\right)+\frac{1}{\epsilon^{2}}G\left(\rho^{\epsilon}\right)+\frac{1}{2}\left|\boldsymbol{B}^{\epsilon}-\boldsymbol{B}^{0}\right|^{2}\right]\mathrm{d}x,$$

其中 $\left(\boldsymbol{u}^{0},\boldsymbol{B}^{0}\right)$ 是不可压缩 MHD 系统(2.3)的光滑解。

引理 4.1. 设 $\left(n^{\epsilon},\rho^{\epsilon},\boldsymbol{u}^{\epsilon},\boldsymbol{B}^{\epsilon}\right)$ 是系统(2.1)和(2.2)的一个光滑解，其中 $t\in\left(0,T^{*}\right)$。那么存在常数 $C>0$，有如下不等式成立

$$\left\|\boldsymbol{u}^{\epsilon}-\boldsymbol{u}^{0}\right\|_{L^{2}}^{2}\le C\mathcal{H}^{\epsilon}(t)+C\epsilon^{2}.$$

证明 注意到(3.3)中的 $\boldsymbol{u}^{\epsilon}\in C\left([0,\infty);H^{2}\right)\cap L^{2}\left([0,\infty);H^{3}\right)$，由引理 3.2，Cauchy 不等式和 Sobolev 不等式，得

$$\begin{aligned}\left\|\boldsymbol{u}^{\epsilon}-\boldsymbol{u}^{0}\right\|_{L^{2}}^{2}&=\frac{1}{n^{*}+\rho^{*}}\int\left|\sqrt{n^{\epsilon}+\rho^{\epsilon}}\left(\boldsymbol{u}^{\epsilon}-\boldsymbol{u}^{0}\right)-\left(\sqrt{n^{\epsilon}+\rho^{\epsilon}}-\sqrt{n^{*}+\rho^{*}}\right)\left(\boldsymbol{u}^{\epsilon}-\boldsymbol{u}^{0}\right)\right|^{2}\mathrm{d}x\\&\le C\mathcal{H}^{\epsilon}(t)+C\left\|\sqrt{n^{\epsilon}+\rho^{\epsilon}}-\sqrt{n^{*}+\rho^{*}}\right\|_{L^{2}}^{2}\cdot\left(\left\|\boldsymbol{u}^{\epsilon}\right\|_{H^{2}}^{2}+\left\|\boldsymbol{u}^{0}\right\|_{H^{2}}^{2}\right)\le C\mathcal{H}^{\epsilon}(t)+C\epsilon^{2}.\end{aligned}$$

引理 4.2. 设 $\gamma > 1$。则

$$\mathcal{H}^{\epsilon}(t) \le C\epsilon$$

在 $[0,T]$ 上一致成立。

证明 利用系统(2.1)和系统(2.3)的能量和能量耗散的定义，可以将 $\mathcal{H}^{\epsilon}(t)$ 改写为如下形式：

$$\begin{aligned}&\mathcal{H}^{\epsilon}(t)+\int_0^t\int\left[\mu\left|\nabla\boldsymbol{u}^{\epsilon}-\nabla\boldsymbol{u}^0\right|^2+(\mu+\lambda)\left|\mathrm{div}\boldsymbol{u}^{\epsilon}\right|^2+\nu\left|\nabla\times\left(\boldsymbol{B}^{\epsilon}-\boldsymbol{B}^0\right)\right|^2\right]\mathrm{d}x\mathrm{d}s\\&=\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4+\mathcal{I}_5+\mathcal{I}_6,\end{aligned} \tag{4.1}$$

其中

$$\mathcal{I}_1=\left(E^{\epsilon}+E^0\right)+\int_0^t\left(D^{\epsilon}+D^0\right)(s)\mathrm{d}s,$$

$$\mathcal{I}_2=\frac{1}{2}\int\left[\left(n^{\epsilon}+\rho^{\epsilon}\right)-\left(n^*+\rho^*\right)\right]\left|\boldsymbol{u}^0\right|^2\mathrm{d}x,$$

$$\mathcal{I}_3=-\int\left(n^{\epsilon}+\rho^{\epsilon}\right)\boldsymbol{u}^{\epsilon}\cdot\boldsymbol{u}^0\mathrm{d}x,$$

$$\mathcal{I}_4=-\int\boldsymbol{B}^{\epsilon}\cdot\boldsymbol{B}^0\mathrm{d}x,$$

$$\mathcal{I}_5=-2\mu\int_0^t\int\nabla\boldsymbol{u}^{\epsilon}:\nabla\boldsymbol{u}^0\mathrm{d}x\mathrm{d}s,$$

$$\mathcal{I}_6=-2\nu\int_0^t\int\left(\nabla\times\boldsymbol{B}^{\epsilon}\right)\cdot\left(\nabla\times\boldsymbol{B}^0\right)\mathrm{d}x\mathrm{d}s.$$

现在来估计 $\frac{\mathrm{d}\mathcal{H}^{\epsilon}(t)}{\mathrm{d}t}$。为此，需要逐项处理积分 $\mathcal{I}_k\,(k=1,2,3,4,5,6)$ 的导数。

对于 $\frac{\mathrm{d}\mathcal{I}_1}{\mathrm{d}t}$，由(3.12)~(3.13)，可得

$$\frac{\mathrm{d}\mathcal{I}_1}{\mathrm{d}t}=0. \tag{4.2}$$

对于 $\frac{\mathrm{d}\mathcal{I}_3}{\mathrm{d}t}$，利用$(2.1)_1$，$(2.1)_3$，$(2.3)_2$ 并运用分部积分法，得

$$\begin{aligned}\frac{\mathrm{d}\mathcal{I}_3}{\mathrm{d}t}=&\,\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3+\mathcal{J}_4+\mathcal{J}_5+\mathcal{J}_6-\int\boldsymbol{u}^{\epsilon}\cdot\left[\left(\nabla\times\boldsymbol{B}^0\right)\times\boldsymbol{B}^0\right]\mathrm{d}x\\&-\int\boldsymbol{u}^0\cdot\left[\left(\nabla\times\boldsymbol{B}^{\epsilon}\right)\times\boldsymbol{B}^{\epsilon}\right]\mathrm{d}x+2\mu\int\nabla\boldsymbol{u}^{\epsilon}:\nabla\boldsymbol{u}^0\mathrm{d}x.\end{aligned}$$

其中

$$\mathcal{J}_1=-\int\left[\left(n^{\epsilon}+\rho^{\epsilon}\right)\left(\boldsymbol{u}^{\epsilon}-\boldsymbol{u}^0\right)\otimes\left(\boldsymbol{u}^{\epsilon}-\boldsymbol{u}^0\right)\right]:\nabla\boldsymbol{u}^0\mathrm{d}x\le C\mathcal{H}^{\epsilon}(t),$$

$$\mathcal{J}_2=-\frac{\mathrm{d}\mathcal{I}_2}{\mathrm{d}t}+\frac{1}{2}\int\left(n^{\epsilon}-n^*\right)\partial_t\left|\boldsymbol{u}^0\right|^2\mathrm{d}x+\frac{1}{2}\int\left(\rho^{\epsilon}-\rho^*\right)\partial_t\left|\boldsymbol{u}^0\right|^2\mathrm{d}x\le-\frac{\mathrm{d}\mathcal{I}_2}{\mathrm{d}t}+C\epsilon^{\min\left\{1,\frac{2}{\gamma}\right\}},$$

$$\mathcal{J}_3=\int\left[\left(n^{\epsilon}+\rho^{\epsilon}\right)\boldsymbol{u}^0\otimes\boldsymbol{u}^0\right]:\nabla\boldsymbol{u}^0\mathrm{d}x\le C\epsilon^{\min\left\{1,\frac{2}{\gamma}\right\}},$$

$$\mathcal{J}_4=\int\left(n^{\epsilon}+\rho^{\epsilon}\right)\boldsymbol{u}^{\epsilon}\cdot\nabla\pi^0\mathrm{d}x\le\frac{\mathrm{d}}{\mathrm{d}t}\int\left[\left(n^{\epsilon}+\rho^{\epsilon}\right)-\left(n^*+\rho^*\right)\right]\pi^0\mathrm{d}x+C\epsilon^{\min\left\{1,\frac{2}{\gamma}\right\}},$$

$$\mathcal{J}_5=\frac{\mu}{n^*+\rho^*}\int\left[\left(n^{\epsilon}+\rho^{\epsilon}\right)-\left(n^*+\rho^*\right)\right]\boldsymbol{u}^{\epsilon}\cdot\Delta\boldsymbol{u}^0\mathrm{d}x\le C\epsilon^{\min\left\{1,\frac{2}{\gamma}\right\}},$$

$$\mathcal{J}_6=-\frac{1}{n^*+\rho^*}\int\left[\left(n^\epsilon+\rho^\epsilon\right)-\left(n^*+\rho^*\right)\right]\boldsymbol{u}^\epsilon\cdot\left[\left(\nabla\times\boldsymbol{B}^0\right)\times\boldsymbol{B}^0\right]\mathrm{d}x\le C\epsilon^{\min\left\{1,\frac{2}{\gamma}\right\}},$$

故有

$$\begin{aligned}\frac{\mathrm{d}\left(\mathcal{I}_2+\mathcal{I}_3\right)}{\mathrm{d}t}\le&\frac{\mathrm{d}}{\mathrm{d}t}\int\left[\left(n^\epsilon+\rho^\epsilon\right)-\left(n^*+\rho^*\right)\right]\pi^0\mathrm{d}x+C\mathcal{H}^\epsilon(t)+C\epsilon^{\min\left\{1,\frac{2}{\gamma}\right\}}\\&-\int\boldsymbol{u}^\epsilon\cdot\left[\left(\nabla\times\boldsymbol{B}^0\right)\times\boldsymbol{B}^0\right]\mathrm{d}x-\int\boldsymbol{u}^0\cdot\left[\left(\nabla\times\boldsymbol{B}^\epsilon\right)\times\boldsymbol{B}^\epsilon\right]\mathrm{d}x+2\mu\int\nabla\boldsymbol{u}^\epsilon:\nabla\boldsymbol{u}^0\mathrm{d}x.\end{aligned}\tag{4.3}$$

由式子(2.1)$_4$和(2.3)$_3$得

$$\frac{\mathrm{d}\mathcal{I}_4}{\mathrm{d}t}=\int\boldsymbol{u}^\epsilon\cdot\left[\left(\nabla\times\boldsymbol{B}^0\right)\times\boldsymbol{B}^\epsilon\right]\mathrm{d}x+\int\boldsymbol{u}^0\cdot\left[\left(\nabla\times\boldsymbol{B}^\epsilon\right)\times\boldsymbol{B}^0\right]\mathrm{d}x+2\nu\int\left(\nabla\times\boldsymbol{B}^\epsilon\right)\cdot\left(\nabla\times\boldsymbol{B}^0\right)\mathrm{d}x.\tag{4.4}$$

进一步可得

$$\frac{\mathrm{d}\mathcal{I}_5}{\mathrm{d}t}=-2\mu\int\nabla\boldsymbol{u}^\epsilon:\nabla\boldsymbol{u}^0\mathrm{d}x,\tag{4.5}$$

$$\frac{\mathrm{d}\mathcal{I}_6}{\mathrm{d}t}=-2\nu\int\left(\nabla\times\boldsymbol{B}^\epsilon\right)\cdot\left(\nabla\times\boldsymbol{B}^0\right)\mathrm{d}x.\tag{4.6}$$

将(4.2)~(4.6)代入(4.1)得

$$\begin{aligned}&\frac{\mathrm{d}\mathcal{H}(t)}{\mathrm{d}t}+\int\left[\mu\left|\nabla\boldsymbol{u}^\epsilon-\nabla\boldsymbol{u}^0\right|^2+(\mu+\lambda)\left|\mathrm{div}\boldsymbol{u}^\epsilon\right|^2+\nu\left|\nabla\times\boldsymbol{B}^\epsilon-\nabla\times\boldsymbol{B}^0\right|^2\right]\mathrm{d}x\\&\le\frac{\mathrm{d}}{\mathrm{d}t}\int\left[\left(n^\epsilon+\rho^\epsilon\right)-\left(n^*+\rho^*\right)\right]\pi^0\mathrm{d}x+C\mathcal{H}^\epsilon(t)+C\epsilon^{\min\left\{1,\frac{2}{\gamma}\right\}}\\&\quad+\int\left(\boldsymbol{u}^\epsilon-\boldsymbol{u}^0\right)\cdot\left[\left(\nabla\times\boldsymbol{B}^0\right)\times\left(\boldsymbol{B}^\epsilon-\boldsymbol{B}^0\right)\right]\mathrm{d}x\\&\quad-\int\boldsymbol{u}^0\cdot\left[\left(\nabla\times\boldsymbol{B}^\epsilon-\nabla\times\boldsymbol{B}^0\right)\times\left(\boldsymbol{B}^\epsilon-\boldsymbol{B}^0\right)\right]\mathrm{d}x.\end{aligned}$$

由引理 4.1 和 Cauchy 不等式得

$$\int\left(\boldsymbol{u}^\epsilon-\boldsymbol{u}^0\right)\cdot\left[\left(\nabla\times\boldsymbol{B}^0\right)\times\left(\boldsymbol{B}^\epsilon-\boldsymbol{B}^0\right)\right]\mathrm{d}x\le C\mathcal{H}^\epsilon(t)+C\epsilon^2,$$

和

$$-\int\boldsymbol{u}^0\cdot\left[\left(\nabla\times\boldsymbol{B}^\epsilon-\nabla\times\boldsymbol{B}^0\right)\times\left(\boldsymbol{B}^\epsilon-\boldsymbol{B}^0\right)\right]\mathrm{d}x\le\frac{\nu}{2}\int\left|\nabla\times\boldsymbol{B}^\epsilon-\nabla\times\boldsymbol{B}^0\right|^2\mathrm{d}x+C\mathcal{H}^\epsilon(t).$$

为得到最终估计，还需要为调整能量引入修正项：

$$\mathcal{H}_{corr}^\epsilon(t)=-\int\left[\left(n^\epsilon+\rho^\epsilon\right)-\left(n^*+\rho^*\right)\right]\pi^0\mathrm{d}x.\tag{4.7}$$

于是，有

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{H}^\epsilon(t)+\mathcal{H}_{corr}^\epsilon(t)\right)+\int\left[\mu\left|\nabla\boldsymbol{u}^\epsilon-\nabla\boldsymbol{u}^0\right|^2+(\mu+\lambda)\left|\mathrm{div}\boldsymbol{u}^\epsilon\right|^2+\frac{\nu}{2}\left|\nabla\times\left(\boldsymbol{B}^\epsilon-\boldsymbol{B}^0\right)\right|^2\right]\mathrm{d}x\\&\le C\mathcal{H}^\epsilon(t)+C\epsilon^{\min\left\{1,\frac{2}{\gamma}\right\}}.\end{aligned}\tag{4.8}$$

对(4.8)应用 Gronwall 不等式，得

$$\mathcal{H}^\epsilon(t)\le\mathcal{H}^\epsilon(0)-\mathcal{H}_{corr}^\epsilon(t)+\mathcal{H}_{corr}^\epsilon(0)+C\epsilon^{\min\left\{1,\frac{2}{\gamma}\right\}}.$$

利用初始条件(3.4)~(3.5)，运用引理 3.2 中的证明方法，可得 $\mathcal{H}^{\epsilon}(0)\le C\epsilon^{\min\left\{1,\frac{2}{\gamma}\right\}}$。利用 Hölder 不等式，可证修正项

$$\mathcal{H}_{corr}^{\epsilon}(t)\le C\epsilon^{\min\left\{1,\frac{2}{\gamma}\right\}},\ \mathcal{H}_{corr}^{\epsilon}(0)\le C\epsilon^{\min\left\{1,\frac{2}{\gamma}\right\}}.$$

因此对于 $t\in\left[0,T^{*}\right)$，得到

$$\mathcal{H}^{\epsilon}(t)\le C\epsilon^{\min\left\{1,\frac{2}{\gamma}\right\}},$$

即引理 4.2 成立。由引理 3.2，易证式子(3.7)和式子(3.9)成立。由引理 3.2 和引理 4.2 可得

$$\left\|\sqrt{n^{\epsilon}+\rho^{\epsilon}}\boldsymbol{u}^{\epsilon}-\sqrt{n^{*}+\rho^{*}}\boldsymbol{u}^{0}\right\|_{L^2}^{2}\le C\epsilon^{\min\left\{1,\frac{2}{\gamma}\right\}},$$

即(3.8)成立。进一步，由引理 4.1 和 4.2 可知，结论(3.10)成立。

最后，利用引理 3.2 和引理 4.2，Cauchy 不等式，Minkowski 不等式，可得

$$\begin{aligned}
&\left\|\left(n^{\epsilon}+\rho^{\epsilon}\right)\left|\boldsymbol{u}^{\epsilon}\right|^{2}-\left(n^{*}+\rho^{*}\right)\left|\boldsymbol{u}^{0}\right|^{2}\right\|_{L^1}\\
&\le C\left(\left\|\sqrt{n^{\epsilon}+\rho^{\epsilon}}\left(\boldsymbol{u}^{\epsilon}-\boldsymbol{u}^{0}\right)\right\|_{L^2}+\left\|\left(\sqrt{n^{\epsilon}+\rho^{\epsilon}}-\sqrt{n^{*}+\rho^{*}}\right)\boldsymbol{u}^{0}\right\|_{L^2}\right)\\
&\quad\cdot\left\|\boldsymbol{u}^{0}\right\|_{L^\infty}\left(\left\|\sqrt{n^{\epsilon}+\rho^{\epsilon}}-\sqrt{n^{*}+\rho^{*}}\right\|_{L^2}+C\right)\\
&\le C\epsilon^{\min\left\{\frac{1}{2},\frac{1}{\gamma}\right\}},
\end{aligned}$$

这表明(3.11)成立。至此完成了定理 3.1 的所有证明。

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