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Hans汉斯

分数布朗运动驱动的SDE梯形数值格式的收敛性分析

王聪淇

辽宁师范大学数学学院，辽宁 大连

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摘　要

分数布朗运动驱动的SDE在金融、物理和工程等领域有广泛应用，但其精确解通常难以获得，因此数值方法的研究至关重要。梯形格式作为一种经典的数值方法，其在分数布朗运动驱动SDE中的应用及收敛性分析具有一定的理论意义和实际价值。本文研究加性分数布朗运动驱动的SDE的梯形数值格式的收敛性分析，其中分数布朗运动的Hurst参数 $H \in \left(\frac{1}{2},1\right)$。利用Malliavin导数和Skorohod积分等工具，得到了该数值格式的强收敛阶为1阶。

关键词

随机微分方程，分数布朗运动，梯形数值格式

Convergence Analysis of Trapezoidal Numerical Scheme for SDE Driven by Fractional Brownian Motion

Congqi Wang

School of Mathematics, Liaoning Normal University, Dalian Liaoning

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Abstract

The SDE driven by Fractional Brownian Motion (FBM) serves as a fundamental stochastic model in fields such as financial mathematics, physics, and engineering. Due to the difficulty in obtaining

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exact solutions for such equations, numerical methods are indispensable tools for studying them. This paper focuses on the convergence analysis of the trapezoidal numerical scheme for SDEs driven by additive Fractional Brownian Motion, where the Hurst parameter H of the Fractional Brownian Motion belongs to $\left(\frac{1}{2},1\right)$. Using tools such as Malliavin derivatives and Skorohod integrals, the strong convergence order of this numerical scheme is obtained to be of first order.

Keywords

Stochastic Differential Equation, Fractional Brownian Motion, Trapezoidal Numerical Scheme

1. 引言

由于带有 Hurst 参数 $H \in (0,1)$ 的分数布朗运动驱动的 SDE 是表征随机现象的基本模型，在多孔介质[1]、金融[2]等领域具有广泛应用。所以，近年来分数布朗运动驱动的 SDE 研究受到了学术界越来越多的关注，特别是在 SDE 的数值方法研究方面取得了重要成果。例如，Huang 和 Wang [3]研究了具有 Hurst 参数 $H \in \left(\frac{1}{3},\frac{1}{2}\right)$ 的分数布朗运动驱动的 SDE 在漂移系数满足一阶、二阶及三阶导数均有界的情况下，Euler 方法具有强阶 2H。Hu 等人[4]得到了由 Hurst 参数 $H > \frac{1}{2}$ 的多维分数布朗运动驱动的 SDE 的 Crank-Nicolson 格式(又称梯形数值格式)的收敛阶。Zhang 和 Yuan [5]利用向后 Euler 方法求解一类具有 Hurst 参数 $H > \frac{1}{2}$ 的分数布朗运动驱动的一维 SDE，得到其收敛阶为 H 阶。受到上述文献的启发，本文在文献[5]的基础上对分数布朗运动驱动的 SDE 构造梯形数值格式，证明该数值方法的强阶增加为 1 阶。

2. 分数布朗运动驱动的 SDE

考虑以下加性分数布朗运动驱动的 SDE

$$\mathrm{d}X_t = a\left(X_t\right)\mathrm{d}t + \sigma \mathrm{d}B_t^H, \qquad t \in [0,T], \tag{1.1}$$

这里初始值 $X_0 > 0$，漂移系数 $a:\mathbb{R} \to \mathbb{R}$ 是无界的，$B^H = \left\{B_t^H\right\}_{t\in[0,T]}$ 是具有 Hurst 参数 $H \in \left(\frac{1}{2},1\right)$ 的分数布朗运动。由文献[6]可知，该方程存在唯一的逐路径解

$$X_t = X_0 + \int_0^t a\left(X_t\right)\mathrm{d}t + \sigma\int_0^t \mathrm{d}B_t^H, \qquad t \in [0,T].$$

下面给出本文需要的定义及引理。文中的 M 为常数，不同行取值不同。

定义 1.1 [5]　带 Hurst 参数 $H \in (0,1)$ 的分数布朗运动 $B^H = \left\{B_t^H, t \in [0,T]\right\}$ 是具有连续路径的中心高斯过程，且协方差函数为

$$\mathbb{E}\left(B_t^H B_s^H\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad s,t \in [0,T]. \tag{1.2}$$

在下文中，总是假设 Hurst 参数 $H \in \left(\frac{1}{2},1\right)$，则分数布朗运动的协方差函数可表示为

$$\mathbb{E}\left(B_t^H B_s^H\right)=\alpha_H\int_0^t\int_0^s|v-u|^{2H-2}\,\mathrm{d}u\mathrm{d}v,\quad s,t\in[0,T].$$

基于上式，定义内积

$$\left\langle I_{[0,t]},I_{[0,s]}\right\rangle_U:=\alpha_H\int_0^T\int_0^T I_{[0,t]}(u)I_{[0,s]}(v)|v-u|^{2H-2}\,\mathrm{d}u\mathrm{d}v=\alpha_H\int_0^t\int_0^s|v-u|^{2H-2}\,\mathrm{d}u\mathrm{d}v,\tag{1.3}$$

其中，$I_{[0,t]}(\cdot)$是示性函数，$\alpha_H=H(2H-1)$。记 Hilbert 空间$\left(U,\langle\cdot,\cdot\rangle_U\right)$是定义在$[0,T]$上的所有阶梯函数关于内积$\langle\cdot,\cdot\rangle_U$生成的闭包，则

$$\langle\varphi,\psi\rangle_U=\alpha_H\int_0^T\int_0^T\varphi_u\psi_v|v-u|^{2H-2}\,\mathrm{d}u\mathrm{d}v,\quad \varphi,\psi\in U.$$

定义 1.2 [7]　对 $f\in C\left([0,T];\mathbb{R}^d\right)$，$\beta\in(0,1]$。定义函数$f$的$\beta$-Hölder 半范数

$$\|f\|_{s,t,\beta}:=\sup_{s\le u<v\le t}\frac{\|f_v-f_u\|}{|v-u|^{\beta}},\qquad 0\le s<t\le T.$$

特别地，简记$\|f\|_\beta:=\|f\|_{0,T,\beta}$。由定义 1.1 可知

$$\sup_{0\le s<t\le T}\frac{\left\|B_t^H-B_s^H\right\|_{L^p(\Omega)}}{|t-s|^H}<\infty,\qquad p\ge1,$$

故B_t^H的轨道 Hölder 连续性指标$\beta<H$。

定义 1.3 [3]　给定一个随机变量

$$G=g\left(B_{t_1},\cdots,B_{t_N}\right),\tag{1.4}$$

其中，$t_1,\cdots,t_N\in[0,T]$，$g:\mathbb{R}^N\to\mathbb{R}$是一个满足任意阶导数有界的有界光滑函数。定义$G$的 Malliavin 导数为$U$-值随机变量$DG$，且

$$D_tG:=\sum_{i=1}^N\frac{\partial g}{\partial x_i}\left(B_{t_1},\cdots,B_{t_N}\right)I_{[0,t_i]}(t),\quad t\in[0,T].$$

此外，对于$p\ge1$，

$$\|G\|_{\mathbb{W}^{1,p}}:=\left(\mathbb{E}\left[|G|^p\right]+\mathbb{E}\left[\|DG\|_U^p\right]\right)^{\frac{1}{p}}.\tag{1.5}$$

其中，Sobolev 空间$\mathbb{W}^{1,p}$是具有(1.4)形式的全体随机变量对该范数生成的闭包。

下面介绍算子D的对偶算子δ。

定义 1.4 [3]　若U-值随机变量$\varphi\in L^2(\Omega;U)$满足

$$\left|\mathbb{E}\left[\langle\varphi,DG\rangle_U\right]\right|\le Q(\varphi)\|G\|_{L^2(\Omega)},\quad G\in\mathbb{W}^{1,2},$$

其中，随机变量$Q:\Omega\to(0,\infty)$，则称φ属于算子δ的定义域，记为$\varphi\in Dom(\delta)$。定义$\delta(\varphi)\in L^2(\Omega)$是满足

$$\mathbb{E}\left[\langle\varphi,DG\rangle_U\right]=\mathbb{E}\left[G\delta(\varphi)\right],\quad G\in\mathbb{W}^{1,2}$$

的随机变量。同时，称$\delta(\varphi)$为φ关于分数布朗运动$\left\{B_t^H\right\}_{t\in[0,T]}$的 Skorohod 积分，记为

$$\delta(\varphi)=\int_0^T\varphi_t\delta B_t,$$

利用示性函数可定义 $\int_0^t \varphi_u \delta B_u := \delta\left(\varphi I_{[0,t]}\right), t\in[0,T]$。根据文献[8]中命题 5.2.3，分数布朗运动的随机积分与 Skorohod 积分有以下关系

$$\int_0^t \varphi_u \mathrm{d}B_u^H = \int_0^t \varphi_u \delta B_u^H + \alpha_H \int_0^t \int_0^T D_v \varphi_u |v-u|^{2H-2} \mathrm{d}v\mathrm{d}u.$$

引理 1.1 [7]　设 $p>1$，记

$$\mathbb{W}^{1,p}\left(|U|\right) := \left\{\varphi \in \mathbb{W}^{1,p}(U): \|\mathbb{E}[\varphi]\|_{|U|}^p + \mathbb{E}\left[\|D\varphi\|_{|U|\otimes|U|}^p\right] < \infty\right\}, \tag{1.6}$$

其中

$$\|\varphi\|_{|U|}^2 := \alpha_H \int_{[0,T]^2} |\varphi_v||\varphi_u||v-u|^{2H-2} \mathrm{d}v\mathrm{d}u,$$

$$\|D\varphi\|_{|U|\otimes|U|}^2 := \alpha_H^2 \int_{[0,T]^4} \left|D_{v_1}\varphi_{u_1}\right|\left|D_{v_2}\varphi_{u_2}\right|\left|v_1-v_2\right|^{2H-2}\left|u_1-u_2\right|^{2H-2} \mathrm{d}v_1\mathrm{d}u_1\mathrm{d}v_2\mathrm{d}u_2.$$

若 $\varphi \in \mathbb{W}^{1,p}\left(|U|\right)$，则存在常数 $M = M(H,p)$，使得

$$\mathbb{E}\left[\left|\int_0^T \varphi_t \delta B_t\right|^p\right] \le M\left(\|\mathbb{E}[\varphi]\|_{|U|}^p + \mathbb{E}\left[\|D\varphi\|_{|U|\otimes|U|}^p\right]\right). \tag{1.7}$$

3. 梯形数值格式

在本节中，我们将利用梯形数值格式来求解方程(1.1)，并分析所得数值格式的强收敛阶。

设 $h=\dfrac{T}{N}$，$t_k = kh$，$k=0,\cdots,N$，对于 $t\in(t_k,t_{k+1}]$，记 $\lfloor t \rfloor := t_k$，$\lceil t \rceil := t_{k+1}$ 和 $\Delta B_{k+1}^H = B_{t_{k+1}}^H - B_{t_k}^H$。将梯形数值格式应用于(1.1)，得到

$$X_{k+1} = X_k + \frac{h}{2}\left[a\left(X_{k+1}\right) + a\left(X_k\right)\right] + \sigma \Delta B_{k+1}^H,\ \ k\in\{0,\cdots,N\}, \tag{2.1}$$

其中，$X_0 = X_{t_0} > 0$。

设存在常数 $K>0$，使得 $a(x)$ 满足

$$a'(x)h \le hK < 2,\quad a''(x) \le K,\quad x\in(0,+\infty). \tag{2.2}$$

类似于文献[5]的证明，方程(2.1)是可解的。

定理 2.1　设 $H\in\left(\dfrac{1}{2},1\right)$，$a:\mathbb{R}\to\mathbb{R}$ 满足(2.2)，则

$$\left(\mathbb{E}\sup_{0\le k\le N-1}\left|X_{t_{k+1}} - X_{k+1}\right|^p\right)^{\frac{1}{p}} \le Mh^H, \tag{2.3}$$

其中，$X_{t_{k+1}}$ 是方程(1.1)的精确解，X_{k+1} 为(2.1)给出的数值解。

证　由(2.1)和积分中值定理

$$\begin{aligned}X_{t_{k+1}} - X_{k+1} &= X_{t_k} - X_k + \int_{\lfloor t \rfloor}^{\lceil t \rceil} a\left(X_s\right)\mathrm{d}s - \frac{h}{2}\left[a\left(X_{k+1}\right) + a\left(X_k\right)\right] \\ &= X_{t_k} - X_k + \frac{h}{2}\left(a\left(X_{t_{k+1}}\right) - a\left(X_{k+1}\right)\right) + \frac{h}{2}\left(a\left(X_{t_k}\right) - a\left(X_k\right)\right) \\ &\quad - \frac{1}{2}\int_{\lfloor t \rfloor}^{\lceil t \rceil}\left(a\left(X_{t_{k+1}}\right) - a\left(X_s\right)\right)\mathrm{d}s - \frac{1}{2}\int_{\lfloor t \rfloor}^{\lceil t \rceil}\left(a\left(X_{t_k}\right) - a\left(X_s\right)\right)\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&= X_{t_k} - X_k + \frac{h}{2}a'\left(X_{k+1} + \zeta_{k+1}\left(X_{t_{k+1}} - X_{k+1}\right)\right)\left(X_{t_{k+1}} - X_{k+1}\right) \\
&\quad + \frac{h}{2}a'\left(X_k + \zeta_k\left(X_{t_k} - X_k\right)\right)\left(X_{t_k} - X_k\right) \\
&\quad - \frac{1}{2}\int_{\lfloor t\rfloor}^{\lceil t\rceil}\left(\int_s^{\lceil t\rceil} a'(X_r)a(X_r)\mathrm{d}r + \sigma\int_s^{\lceil t\rceil} a'(X_r)\mathrm{d}B_r^H\right)\mathrm{d}s \\
&\quad + \frac{1}{2}\int_{\lfloor t\rfloor}^{\lceil t\rceil}\left(\int_{\lfloor t\rfloor}^{s} a'(X_r)a(X_r)\mathrm{d}r + \sigma\int_{\lfloor t\rfloor}^{s} a'(X_r)\mathrm{d}B_r^H\right)\mathrm{d}s \\
&= X_{t_k} - X_k + \frac{h}{2}a'\left(X_{k+1} + \zeta_{k+1}\left(X_{t_{k+1}} - X_{k+1}\right)\right)\left(X_{t_{k+1}} - X_{k+1}\right) \\
&\quad + \frac{h}{2}a'\left(X_k + \zeta_k\left(X_{t_k} - X_k\right)\right)\left(X_{t_k} - X_k\right) \\
&\quad - \frac{1}{2}\left(\int_{\lfloor t\rfloor}^{\lceil t\rceil}(r - t_k)a'(X_r)a(X_r)\mathrm{d}r + \sigma\int_{\lfloor t\rfloor}^{\lceil t\rceil}(r - t_k)a'(X_r)\mathrm{d}B_r^H\right) \\
&\quad + \frac{1}{2}\left(\int_{\lfloor t\rfloor}^{\lceil t\rceil}(t_{k+1} - r)a'(X_r)a(X_r)\mathrm{d}r + \sigma\int_{\lfloor t\rfloor}^{\lceil t\rceil}(t_{k+1} - r)a'(X_r)\mathrm{d}B_r^H\right),
\end{aligned}
\tag{2.4}
$$

其中，$\zeta_k, \zeta_{k+1} \in (0,1)$，最后一步等式由 Fubini 定理得到。令

$$
\begin{aligned}
&\Delta_{k+1} = X_{t_{k+1}} - X_{k+1}, \\
&\Psi_{k+1} = a'\left(X_{k+1} + \zeta_{k+1}\left(X_{t_{k+1}} - X_{k+1}\right)\right), \\
&\Gamma_{k+1} = \frac{1}{2}\left[\int_{\lfloor t\rfloor}^{\lceil t\rceil}(t_{k+1} + t_k - 2r)a'(X_r)a(X_r)\mathrm{d}r + \sigma\int_{\lfloor t\rfloor}^{\lceil t\rceil}(t_{k+1} + t_k - 2r)a'(X_r)\mathrm{d}B_r^H\right],
\end{aligned}
\tag{2.5}
$$

则有

$$
\left(1 - \frac{h}{2}\Psi_{k+1}\right)\Delta_{k+1} = \left(1 + \frac{h}{2}\Psi_k\right)\Delta_k + \Gamma_{k+1}. \tag{2.6}
$$

令 $\tilde{\Delta}_{k+1} = \Theta_{k+1}\Delta_{k+1}$，其中 $\Theta_{k+1} = \prod_{i=0}^{k}\left(1 - \frac{h}{2}\Psi_{i+1}\right)$。因此

$$
\tilde{\Delta}_{k+1} = \left(1 + \frac{h}{2}\Psi_k\right)\tilde{\Delta}_k + \Theta_k\Gamma_{k+1}.
$$

可得 $\left(1 + \frac{h}{2}\Psi_k\right)^{-1}\tilde{\Delta}_{k+1} = \tilde{\Delta}_k + \Theta_k\left(1 + \frac{h}{2}\Psi_k\right)^{-1}\Gamma_{k+1}$。

设 $\hat{\Delta}_{k+1} = \Lambda_k\tilde{\Delta}_{k+1}$，其中 $\Lambda_k = \prod_{i=0}^{k}\left(1 + \frac{h}{2}\Psi_i\right)^{-1}$。所以

$$
\left(1 + \frac{h}{2}\Psi_k\right)^{-1}\Lambda_k^{-1}\hat{\Delta}_{k+1} = \Lambda_k^{-1}\Delta_k + \Theta_k\left(1 + \frac{h}{2}\Psi_k\right)^{-1}\Gamma_{k+1}.
$$

则 $\hat{\Delta}_{k+1} = \hat{\Delta}_k + \Lambda_k\Theta_k\Gamma_{k+1}$。通过迭代可以得到

$$
\hat{\Delta}_{k+1} = \frac{1}{2}\left[\int_0^{\lceil t\rceil}(t_{k+1} + t_k - 2r)\Lambda_{t_k}\Theta_{t_k}a'(X_r)a(X_r)\mathrm{d}r + \sigma\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}}(t_{i+1} + t_i - 2r)\Lambda_i\Theta_i a'(X_r)\mathrm{d}B_r^H\right].
$$

由 $\tilde{\Delta}_k$ 和 $\hat{\Delta}_k$ 的定义有 $\Delta_{k+1} = \frac{\tilde{\Delta}_{k+1}}{\Theta_{k+1}} = \frac{\hat{\Delta}_{k+1}}{\Lambda_k\Theta_{k+1}}$。因此，

$$\Delta_{k+1}=\frac{1}{2}\int_0^{\lceil t\rceil}\left(t_{k+1}+t_k-2r\right)\frac{\Lambda_{t_k}\Theta_{t_k}}{\Lambda_k\Theta_{k+1}}a'\left(X_r\right)a\left(X_r\right)\mathrm{d}r \\ +\frac{\sigma}{2}\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}}\left(t_{i+1}+t_i-2r\right)\frac{\Lambda_i\Theta_i}{\Lambda_k\Theta_{k+1}}a'\left(X_r\right)\mathrm{d}B_r^H. \tag{2.7}$$

通过 Taylor 公式和 $\left(1-\frac{h}{2}\Psi_{i+1}\right)^{-1}\le\frac{2}{2-hK}$ 有

$$\frac{\Lambda_i\Theta_i}{\Lambda_k\Theta_{k+1}}=\frac{\prod_{j=i+1}^{k}\left(1+\frac{h}{2}\Psi_j\right)}{\prod_{j=i}^{k}\left(1-\frac{h}{2}\Psi_{j+1}\right)}\le\frac{\left(1+\frac{h}{2}K\right)^{k-i}}{\left(1-\frac{h}{2}K\right)^{k-i+1}}\le\frac{2}{2-hK}\mathrm{e}^{k\log\frac{2+hK}{2-hK}}\le M\mathrm{e}^{\frac{2KT}{2-hK}}.$$

记 $J(t)=\sigma\int_0^t(1-hK)^k\left(t_{k+1}+t_k-2r\right)a'\left(X_r\right)\mathrm{d}B_r^H$ ，定义 $\tilde{\Lambda}_i\tilde{\Theta}_i=\frac{\Lambda_i\Theta_i}{(1-hK)^i}$ ，则有

$$\begin{aligned}
&\sigma\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}}\left(t_{i+1}+t_i-2r\right)\Lambda_i\Theta_i a'\left(X_r\right)\mathrm{d}B_r^H\\
&=\sigma\sum_{i=0}^{k}\frac{\Lambda_i\Theta_i}{(1-hK)^i}\int_{t_i}^{t_{i+1}}(1-hK)^i\left(t_{i+1}+t_i-2r\right)a'\left(X_r\right)\mathrm{d}B_r^H\\
&=\sum_{i=0}^{k}\tilde{\Lambda}_i\tilde{\Theta}_i\left[\sigma\int_0^{t_{i+1}}(1-hK)^{i+1}\left(t_{i+1}+t_i-2r\right)a'\left(X_r\right)\mathrm{d}B_r^H-\sigma\int_0^{t_i}(1-hK)^i\left(t_i+t_{i-1}-2r\right)a'\left(X_r\right)\mathrm{d}B_r^H\right]\\
&=\sum_{i=0}^{k}\tilde{\Lambda}_i\tilde{\Theta}_i\left(J\left(t_{i+1}\right)-J\left(t_i\right)\right)=\tilde{\Lambda}_k\tilde{\Theta}_kJ\left(t_{k+1}\right)-\sum_{i=1}^{k}\left(\tilde{\Lambda}_i\tilde{\Theta}_i-\tilde{\Lambda}_{i-1}\tilde{\Theta}_{i-1}\right)J\left(t_i\right).
\end{aligned}$$

由于 $\tilde{\Lambda}_k\tilde{\Theta}_k\ge\tilde{\Lambda}_{k-1}\tilde{\Theta}_{k-1}$，所以

$$\begin{aligned}
&\left|\tilde{\Lambda}_k\tilde{\Theta}_kJ\left(t_{k+1}\right)-\sum_{i=1}^{k}\left(\tilde{\Lambda}_i\tilde{\Theta}_i-\tilde{\Lambda}_{i-1}\tilde{\Theta}_{i-1}\right)J\left(t_i\right)\right|\\
&\le\left|\tilde{\Lambda}_k\tilde{\Theta}_k\right|\left|J\left(t_{k+1}\right)\right|+\sum_{i=1}^{k}\left|\tilde{\Lambda}_i\tilde{\Theta}_i\right|\cdot\sup_{1\le i\le k}\left|J\left(t_i\right)\right|\\
&\le(k+1)\left|\tilde{\Lambda}_k\tilde{\Theta}_k\right|\cdot\sup_{1\le i\le k+1}\left|J\left(t_i\right)\right|.
\end{aligned}$$

因此(2.7)可得

$$\left|\Delta_{k+1}\right|\le M\int_0^{t_{k+1}}\left|t_{k+1}+t_k-2r\right|\left|\frac{\Lambda_{t_k}\Theta_{t_k}}{\Lambda_k\Theta_{k+1}}\right|\left|a'\left(X_r\right)\right|\left|a\left(X_r\right)\right|\mathrm{d}r+(k+1)\left|\frac{\tilde{\Lambda}_k\tilde{\Theta}_k}{\Lambda_k\Theta_{k+1}}\right|\cdot\sup_{1\le i\le k+1}\left|J\left(t_i\right)\right|.$$

则

$$\begin{aligned}
\mathbb{E}\left|\Delta_{k+1}\right|^p&\le M\mathbb{E}\left(\int_0^{t_{k+1}}\left|t_{k+1}+t_k-2r\right|\left|\frac{\Lambda_{t_k}\Theta_{t_k}}{\Lambda_k\Theta_{k+1}}\right|\left|a'\left(X_r\right)\right|\left|a\left(X_r\right)\right|\mathrm{d}r+(k+1)\left|\frac{\tilde{\Lambda}_k\tilde{\Theta}_k}{\Lambda_k\Theta_{k+1}}\right|\cdot\sup_{1\le i\le k+1}\left|J\left(t_i\right)\right|\right)^p\\
&\le M\left[\mathbb{E}\left(\int_0^{t_{k+1}}\left|t_{k+1}+t_k-2r\right|\left|a'\left(X_r\right)\right|\left|a\left(X_r\right)\right|\mathrm{d}r\right)^p+\mathbb{E}\sup_{1\le i\le k+1}\left|J\left(t_i\right)\right|^p\right]\\
&:=M\left(A_1+A_2\right).
\end{aligned}$$

对于 A_1 项有

$$\mathbb{E}\left(\int_0^{t_{k+1}}\left|t_i+t_{i-1}-2r\right|\left|a'\left(X_r\right)\right|\left|a\left(X_r\right)\right|\mathrm{d}r\right)^p\le Mh^p\mathbb{E}\int_0^{t_{k+1}}\left|a'\left(X_r\right)\right|^p\left|a\left(X_r\right)\right|^p\mathrm{d}r\le Mh^p.$$

因为 $D_v\left[a'\left(X_r\right)\right]=a''\left(X_r\right)D_vX_r=\sigma a''\left(X_r\right)\exp\left\{\int_0^r a'\left(X_r\right)\mathrm{d}t\right\}I_{[0,r]}(v)$，所以

$$\left|D_v\left[a'\left(X_r\right)\right]\right|\le\sigma\mathrm{e}^{TK}\left|a''\left(X_r\right)\right|.$$

由上式，文献[9]的引理 2.1 和(2.2)，A_2 项可得

$$\begin{aligned}A_2&=\mathbb{E}\left(\sup_{1\le i\le k+1}\left|\sigma\int_0^t(1-hK)^k\left(t_{i+1}+t_i-2r\right)a'\left(X_r\right)\mathrm{d}B_r^H\right|^p\right)\\&\le Mh^p\mathbb{E}\int_0^T\left|(1-hK)^i\left(t_{i+1}+t_i-2r\right)a'\left(X_r\right)\right|^p\mathrm{d}r\\&\quad+Mh^p\mathbb{E}\int_0^T\left(\int_0^T\left|(1-hK)^i\left(t_{i+1}+t_i-2r\right)D_va'\left(X_r\right)\right|^{\frac{1}{H}}\mathrm{d}v\right)^{pH}\mathrm{d}r\\&\quad+Mh^p\mathbb{E}\left[\int_0^T\int_0^T\left|(1-hK)^i\left(t_{i+1}+t_i-2r\right)D_va'\left(X_r\right)\right|\phi(v,r)\mathrm{d}v\mathrm{d}r\right]^p\\&\le Mh^p\mathbb{E}\int_0^T\left|a'\left(X_r\right)\right|^p\mathrm{d}r+Mh^p\mathbb{E}\int_0^T\left|a''\left(X_r\right)\right|^p\mathrm{d}r\le Mh^p.\end{aligned}$$

综上所述，

$$\mathbb{E}\sup_{0\le k\le N-1}\left|X_{t_{k+1}}-X_{k+1}\right|^p\le Mh^p.$$

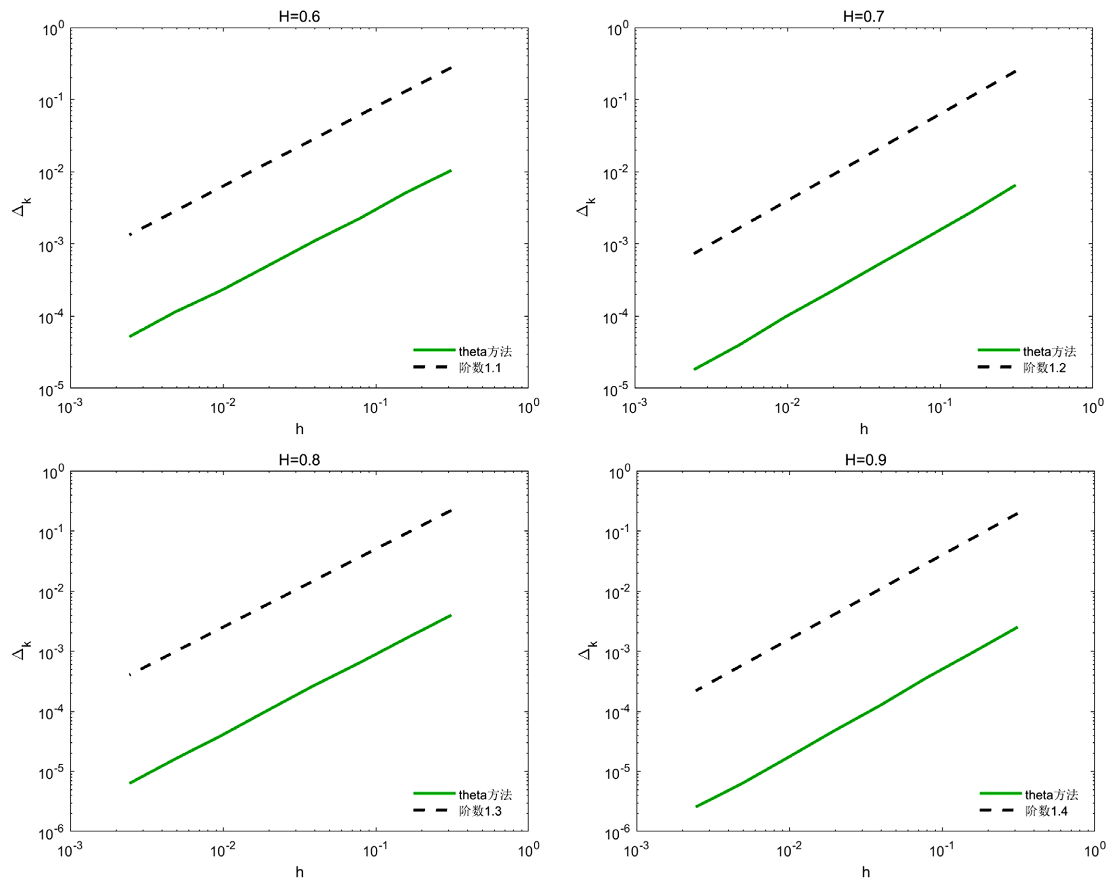

Figure 1. Strong convergence order of trapezoidal numerical scheme
图 1. 梯形数值格式的强收敛阶

4. 数值实验

在本节中，我们展示了 fBMs 驱动的 SDEs 的梯形数值格式的数值模拟。下面，使用漂移项 $a(x)=\frac{1}{2}\kappa\left(\frac{\gamma}{x}-x\right)$ 进行模拟，其中参数 $\kappa=2$，$\gamma=0.5$，这些常数的值不是唯一的，这里的值是为了方便数值实验。

从图 1 中可以看出，当 $H=0.6,0.7,0.8,0.9$ 时，梯形数值格式的强收敛阶数为 $H+\frac{1}{2}$。定理 4.1 中所述的强收敛阶是在数值模拟中观察到的最优强收敛阶的范围内。

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