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四阶线性方程的隐显式全离散局部间断Galerkin方法

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摘 要

针对一维四阶线性方程，研究了一种隐显式Runge-Kutta全离散局部间断Galerkin方法的稳定性和最优误差估计。空间离散采用局部间断Galerkin方法，时间离散选用强稳定显式Runge-Kutta方法和具有L稳定对角隐式Runge-Kutta方法相结合的三阶隐显式Runge-Kutta方法，数值流通量采用广义交替数值流通量，从而得到全离散LDG格式，分析了该格式的稳定性，同时引入全局Gauss-Radau投影，证明该格式具有 $k+1$ 阶收敛。最后通过数值实验验证理论结果的正确性。

关键词

四阶线性方程，显隐式Runge-Kutta，局部间断Galerkin方法

Implicit-Explicit Fully Discrete Local Discontinuity Galerkin Method for Fourth-Order Linear Equations

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Abstract

The stability and error estimation of an implicit-explicit Runge-Kutta fully discrete local discontinuous Galerkin method for one-dimensional fourth-order linear equations are studied. The local

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discontinuity Galerkin method is used for spatial discretization, and the third-order implicit-explicit Runge-Kutta method combining the strong-stability-preserving explicit Runge-Kutta method and the implicit Runge-Kutta method with L-stable diagonal implicit is used for time marching, and the numerical circulation adopts the generalized alternating numerical flux, so as to obtain the fully discrete LDG scheme, and the stability of the scheme is analyzed, and the generalized Gauss-Radau projection is introduced to prove that the scheme has $k+1$ order convergence. Finally, the theoretical results are verified by numerical experiments.

Keywords

Fourth-Order Linear Equations, Implicit-Explicit Runge-Kutta, Locally Discontinuous Galerkin Method

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1. 引言

自然界中的很多现象都可以被抽象成偏微分方程模型，因此对于偏微分方程的研究一直是学者们重点关注的领域。四阶方程就是其中一类重要方程，具有广泛应用，通常用于描述一些复杂的物理现象，比如弹性波动、热传导等，因此对于它们的求解有很高的理论价值和应用价值。

间断 Galerkin (Discontinuous Galerkin，简称 DG)方法最早出现在 1973 年，是在研究中子运输问题时由 Reed [1]和 Hill 提出来的，此方法易解决复杂初边值问题，稳定性强，而且具有高精度等特点。局部间断 Galerkin (Local Discontinuous Galerkin，简称 LDG)是对 DG 方法的扩展，是受 Bassi [2]和 Rebay 关于可压缩 Navier-Stokes 方程的工作的启发，由 Cockburn [3]和 Shu 针对二维对流扩散问题，首次提出并发展了 LDG 方法，并给出了 LDG 方法的基本框架以及一些理论分析。此方法不仅继承了 DG 方法的优点还可以解决 DG 方法解决不了的高阶偏微分方程。因此，LDG 方法对于求解高阶偏微分方程备受众多学者关注，之后被用于求解 Kdv 方程[4]、Hunter-Saxton 方程[5]、Cahn-Hilliard 方程[6]和波动方程[7] [8]等多种方程。

近年来，越来越多的学者开始研究全离散 LDG 方法，即就是在空间上用 LDG 方法离散与时间离散方法相结合。Wang 等[9]针对一维具有 Dirichlet 边界条件的线性对流扩散方程，采用三阶显式总变差递减 Runge-Kutta 方法和 LDG 方法相结合得到全离散数值格式，利用能量分析得到了空间和时间上的最优误差估计。We 等[10]针对四阶时间分数阶问题，基于时间上的有限差分格式和空间上的局部间断 Galerkin 方法，提出并分析了一种全离散的 LDG 方法。Wang 等[11]将 LDG 方法与三种特定的隐显式 Runge-Kutta 时间离散化方法结合，针对一维瞬态线性四阶方程进行稳定性分析和误差估计。Wang 等[12]分析了求解对流扩散问题的基于广义交替数值流通量隐显式 Runge-Kutta 时间推进的 LDG 方法。Bi 等[13]针对四阶线性调和方程，采用三阶对角隐式 Runge-Kutta 方法，研究了全离散 LDG 方法的稳定性和误差估计。Xiao 等[14]针对变阶四阶方程，将基于广义交替数值流通量的 LDG 方法和时间上采取有限差分法相结合得到全离散 LDG 格式，分析其稳定性和误差估计。用 LDG 方法解决高阶方程时，数值流通量常用纯迎风数值流通量 $\hat{u}_h = u_h^-, \hat{q}_h = q_h^+$，而广义交替数值流通量 $\hat{u}_h = \theta u_h^- + (1-\theta) u_h^+$ 能为数值格式提供更大的灵活度，并会减少在计算过程中产生数值耗散，而且权重 θ 可以偏向单元边界两侧中迎风侧数值解的值使其满足迎风原理，所以深受学者关注。由此，本文针对一类一维线性四阶方程，研究了一种隐显式全离散 LDG 方法的稳定性和误差分析，时间离散选用将强稳定显式 Runge-Kutta 方法和具有 L 稳定对角隐式 Runge-

Kutta 方法相结合的三阶隐显式 Runge-Kutta 方法，空间离散采用局部间断 Galerkin 方法，数值流通量采用广义交替数值流通量，从而得到全离散 LDG 格式，在理论上分析了该格式的稳定性并建立了最优误差估计，最终也通过数值算例验证理论结果。

2. LDG 方法和全离散格式

2.1. 预备知识

首先对于区域 $I=(a,b)$ 进行剖分，$a=x_{1/2}<x_{3/2}<\cdots<x_{N-1/2}<x_{N+1/2}=b$，剖分成 N 个单元，剖分所得集合记为 $\mathcal{I}_h=\left\{I_j=\left(x_{j-1/2},x_{j+1/2}\right)\right\}_{j=1}^{N}$，其中 I_j 为一个单元，记作 $I_j=\left(x_{j-1/2},x_{j+1/2}\right)$，单元的中点记作 $x_j=\left(x_{j+1/2}+x_{j-1/2}\right)/2$，单元的长度记作 $h_j=x_{j+1/2}-x_{j-1/2}$，并且 $h=\max h_j$。然后定义有限元空间

$$V_h\equiv V_h^k=\left\{v_h\in L^2(I):v_h\big|_{I_j}\in P^k\left(I_j\right),\ j=1,2,\cdots,N\right\}$$

其中 $P^k\left(I_j\right)$ 表示区间 I_j 上次数不超过 k 次的多项式集合。如果 $v(x)\in V_h$，定义 $[\![v]\!]=v^+-v^-$ 表示 v 在 x 点的跳跃，$[\![v]\!]^{(\theta)}=\theta v^++(1-\theta)v^-$ 表示 v 在 x 点的加权平均，这里 θ 为常数，用 $x_{j+1/2}^-$ 和 $x_{j+1/2}^+$ 表示在 $x_{j+1/2}$ 的左极限和右极限。

对于正整数 $\ell\geq1$，$H^{\ell}(\mathcal{I})$ 表示函数本身及其 ℓ 阶导数在 $\mathcal{I}$ 上都是平方可积的空间，然后定义分裂的 Sobolev 空间为 $H^{\ell}\left(\mathcal{I}_h\right)=\left\{v\in L^2(I):v\big|_{I_j}\in H^{\ell}\left(I_j\right),\ j=1,2,\cdots,N\right\}$。

接下来介绍在进行误差估计的过程中要用到的全局 Gauss-Radau 投影。

对于 $\forall u\in H^1\left(\mathcal{I}_h\right)$，全局 Gauss-Radau 投影 $P_hu\in V_h$ 满足如下式子：

$$\begin{aligned}&\int_{I_j}\left(u-P_hu\right)v\mathrm{d}x=0,\qquad \forall v\in P^{k-1}\left(I_j\right),\\&\theta\left(u-P_hu\right)_{j+\frac{1}{2}}^{\pm}+(1-\theta)\left(u-P_hu\right)_{j+\frac{1}{2}}^{\mp}=0,\end{aligned}\tag{1}$$

这里 $j=1,2,\cdots,N$。

根据文献[15]，能够得到全局 Gauss-Radau 投影有如下性质：

引理 1.1 对于 $u\in H^{\ell}\left(\mathcal{I}_h\right)$ 且正整数 $\ell\geq1$，全局 Gauss-Radau 投影 P_hu 如上定义，可得

$$\left\|u-P_hu\right\|+h^{1/2}\left\|u-P_hu\right\|_{\Gamma_h}\leq Ch^{\min\{k+1,\ell\}}\|u\|_{H^{\ell}\left(\mathcal{I}_h\right)}$$

对于 $\forall v\in V_h$，$\|v\|_{\Gamma_h}^2=\sum_{j=1}^{N}\|v\|_{\partial I_j}^2$，$\|v\|_{\partial I_j}=\sqrt{\left(v_{j-1/2}^+\right)^2+\left(v_{j+1/2}^-\right)^2}$，$C>0$ 而且取值和 u 无关。

2.2. 四阶线性方程的半离散 LDG 格式

考虑如下带有周期边界条件的一维四阶线性方程：

$$\begin{aligned}&u_t+u_x+u_{xxxx}=u_{xx},\qquad (x,t)\in(a,b)\times(0,T],\\&u(x,0)=u_0(x),\qquad\quad x\in(a,b),\end{aligned}\tag{2}$$

$u_0(x)$ 是光滑函数。

在此引入辅助函数 $q=u_x,p=q_x,r=p_x$，能够得到和方程(2)等价的一阶系统

$$\begin{aligned}&u_t+u_x-q_x+r_x=0,\\&q=u_x,\\&p=q_x,\\&r=p_x,\end{aligned}\tag{3}$$

给一阶系统两端同乘试验函数，对于 $t\in[0,T]$，任意试验函数 $\rho,\omega,\phi,\psi\in V_h$，使得数值解 $u_h,p_h,q_h,r_h\in V_h$ 满足

$$\begin{aligned}&\int_{I_j}(u_h)_t\rho\mathrm{d}x-\int_{I_j}u_h\rho_x\mathrm{d}x+u_h^{\theta}\rho^-\Big|_{j+\frac12}-u_h^{\theta}\rho^+\Big|_{j-\frac12}+\int_{I_j}q_h\rho_x\mathrm{d}x-q_h^{\tilde\theta}\rho^-\Big|_{j+\frac12}+q_h^{\tilde\theta}\rho^+\Big|_{j-\frac12}\\&\quad-\int_{I_j}r_h\rho_x\mathrm{d}x+r_h^{\tilde\theta}\rho^-\Big|_{j+\frac12}-r_h^{\tilde\theta}\rho^+\Big|_{j-\frac12}=0,\\&\int_{I_j}q_h\omega\mathrm{d}x+\int_{I_j}u_h\omega_x\mathrm{d}x-u_h^{\theta}\omega^-\Big|_{j+\frac12}+u_h^{\theta}\omega^+\Big|_{j-\frac12}=0,\\&\int_{I_j}p_h\phi\mathrm{d}x+\int_{I_j}q_h\phi_x\mathrm{d}x-q_h^{\tilde\theta}\phi^-\Big|_{j+\frac12}+q_h^{\tilde\theta}\phi^+\Big|_{j-\frac12}=0,\\&\int_{I_j}r_h\psi\mathrm{d}x+\int_{I_j}p_h\psi_x\mathrm{d}x-p_h^{\theta}\psi^-\Big|_{j+\frac12}+p_h^{\theta}\psi^+\Big|_{j-\frac12}=0,\end{aligned}\tag{4}$$

其中 $u_h^{\theta},q_h^{\tilde\theta},p_h^{\theta},r_h^{\tilde\theta}$ 为数值流通量，为了数值格式有更大的灵活度，所以选择广义交替数值流通量，其表达式如下：

$$u_h^{\theta}=\theta u_h^-+\tilde\theta u_h^+,\ q_h^{\tilde\theta}=\theta q_h^++\tilde\theta q_h^-,\ p_h^{\theta}=\theta p_h^-+\tilde\theta p_h^+,\ r_h^{\tilde\theta}=\theta r_h^++\tilde\theta r_h^-\tag{5}$$

这里 $\tilde\theta=1-\theta$。

为方便后面的分析引入离散算子，对于 $\forall u,v\in V_h$，

$$\begin{aligned}H^+(u,v)&=\int uv_x\mathrm{d}x-u^{\theta}v^-\Big|_{j+\frac12}+u_h^{\theta}v^+\Big|_{j-\frac12},\\H^-(u,v)&=\int uv_x\mathrm{d}x-u^{\tilde\theta}v^-\Big|_{j+\frac12}+u_h^{\tilde\theta}v^+\Big|_{j-\frac12}.\end{aligned}\tag{6}$$

因此 LDG 格式可写为：对于 $t\in[0,T]$，任意试验函数 $\rho,\omega,\phi,\psi\in V_h$，使得 $u_h,p_h,q_h,r_h\in V_h$ 满足

$$\big((u_h)_t,\rho\big)=H(\varphi_h,\rho)+H^-(r_h,\rho),\tag{7.1}$$

$$(q_h,\omega)=-H^+(u_h,\omega),\tag{7.2}$$

$$(p_h,\phi)=-H^-(q_h,\phi),\tag{7.3}$$

$$(r_h,\psi)=-H^+(p_h,\psi),\tag{7.4}$$

其中 $H(\varphi_h,\rho)=H^+(u_h,\rho)-H^-(q_h,\rho)$。

当数值流通量选择广义交替数值流通量，对于周期边界条件，离散算子有如下性质：

$$H^+(u,v)+H^-(v,u)=0,\tag{8}$$

$$H^+(v,v)=-(\theta-1/2)[\![v]\!]^2.\tag{9}$$

引理 2.1 [11]对于 $\forall w,v\in V_h$，独立于 h 的 μ 表示逆常数，离散算子满足如下不等式：

$$\left|H^{\pm}(w,v)\right|\le\left(\|w_x\|+\sqrt{\mu h^{-1}}[\![w]\!]\right)\|v\|,$$

$$\left|H^{\pm}(w,v)\right|\le\left(\|v_x\|+\sqrt{\mu h^{-1}}[\![v]\!]\right)\|w\|,$$

引理 2.2 [11]对于 $u,q,p\in V_h$ 且满足式(7.1)~(7.3)，存在一个与 h 无关但可能与 μ 有关的正常数 C_μ 满足

$$\|u_x\|+\sqrt{\mu h^{-1}}[\![u]\!]\le C_\mu\|q\|,$$

$$\|q_x\|+\sqrt{\mu h^{-1}}[\![q]\!]\le C_\mu\|p\|.$$

2.3. 四阶线性方程的全离散 LDG 格式

时间离散方式选择文献[16]里的三阶 IMEX-SSP(4,3,3)方案，则可得到全离散 LDG 格式如下：对于 $\forall\rho,\omega,\phi,\psi\in V_h$， $q_h^{n,i},p_h^{n,l},r_h^{n,l}\in V_h$ 满足

$$\begin{aligned}
&\left(u_h^{n,1},\rho\right)=\left(u_h^n,\rho\right)+\alpha\tau H^-\left(r_h^{n,1},\rho\right)\\
&\left(u_h^{n,2},\rho\right)=\left(u_h^n,\rho\right)-\alpha\tau H^-\left(r_h^{n,1},\rho\right)+\alpha\tau H^-\left(r_h^{n,2},\rho\right)\\
&\left(u_h^{n,3},\rho\right)=\left(u_h^n,\rho\right)+\tau H\left(\varphi_h^{n,2},\rho\right)+(1-\alpha)\tau H^-\left(r_h^{n,1},\rho\right)+\alpha\tau H^-\left(r_h^{n,2},\rho\right)\\
&\left(u_h^{n,4},\rho\right)=\left(u_h^n,\rho\right)+\frac{\tau}{4}\left[H\left(\varphi_h^{n,2},\rho\right)+H\left(\varphi_h^{n,3},\rho\right)\right]+\beta\tau H^-\left(r_h^{n,1},\rho\right)+\delta\tau H^-\left(r_h^{n,2},\rho\right)\\
&\qquad+\sigma\tau H^-\left(r_h^{n,3},\rho\right)+\alpha\tau H^-\left(r_h^{n,4},\rho\right)\\
&\left(u_h^{n+1},\rho\right)=\left(u_h^n,\rho\right)+\frac{\tau}{6}\left[H\left(\varphi_h^{n,2},\rho\right)+H\left(\varphi_h^{n,3},\rho\right)+4H\left(\varphi_h^{n,4},\rho\right)\right]\\
&\qquad+\frac{\tau}{6}\left[H^-\left(r_h^{n,2},\rho\right)+H^-\left(r_h^{n,3},\rho\right)+4H^-\left(r_h^{n,4},\rho\right)\right]\\
&\left(q_h^{n,l},\omega\right)=-H^+\left(u_h^{n,l},\omega\right)\\
&\left(p_h^{n,l},\phi\right)=-H^-\left(q_h^{n,l},\phi\right)\\
&\left(r_h^{n,l},\psi\right)=-H^+\left(p_h^{n,l},\psi\right)
\end{aligned}\tag{10}$$

其中 $u^{n,0}=u^n,l=0,1,2,3,4$ 。

3. 稳定性分析

定理 3.1：当 $\theta>\frac{1}{2}$，存在与 h 无关的正常数 τ_0，当 $\tau\le\tau_0$，一维四阶线性方程(2)的隐显式全离散 LDG 格式(10)基于广义交替数值流通量(5)有 $\|u_h^n\|\le c\|u_h^0\|$ 。

证明：由式(8)和式(10)可得

$$H^-\left(r_h^{n,k},u_h^{n,s}\right)=-\left(p_h^{n,s},p_h^{n,k}\right),\tag{11}$$

$$H^-\left(q_h^{n,k},u_h^{n,s}\right)=\left(q_h^{n,s},q_h^{n,k}\right).\tag{12}$$

由式(10)可得

$$\begin{aligned}
&\left(u_h^{n,1}-u_h^n,\rho_1\right)=\alpha\tau H^-\left(r_h^{n,1},\rho_1\right)\\
&\left(u_h^{n,2}-u_h^{n,1},\rho_2\right)=-2\alpha\tau H^-\left(r_h^{n,1},\rho_2\right)+\alpha\tau H^-\left(r_h^{n,2},\rho_2\right)\\
&\left(u_h^{n,3}-u_h^{n,2},\rho_3\right)=\tau H\left(\varphi_h^{n,2},\rho_3\right)+\alpha\tau H^-\left(r_h^{n,1},\rho_3\right)+(1-2\alpha)\tau H^-\left(r_h^{n,2},\rho_3\right)+\alpha\tau H^-\left(r_h^{n,3},\rho_3\right)\\
&\left(4u_h^{n,4}-3u_h^{n,2}-u_h^{n,3},\rho_4\right)=\tau H\left(\varphi_h^{n,3},\rho_4\right)+4\alpha\tau H^-\left(r_h^{n,1},\rho_4\right)-4\alpha\tau H^-\left(r_h^{n,2},\rho_4\right)\\
&\qquad+(1-4\alpha)\tau H^-\left(r_h^{n,3},\rho_4\right)+4\alpha\tau H^-\left(r_h^{n,4},\rho_4\right)\\
&\left(\frac{3}{2}u_h^{n+1}-3u_h^{n,4}-\frac{1}{2}u_h^{n,2},\rho_5\right)=\tau H\left(\varphi_h^{n,4},\rho_5\right)+\frac{\alpha}{4}\tau H^-\left(r_h^{n,1},\rho_5\right)+\frac{3\alpha}{4}\tau H^-\left(r_h^{n,3},\rho_5\right)\\
&\qquad+(1-\alpha)\tau H^-\left(r_h^{n,4},\rho_5\right)
\end{aligned}\tag{13}$$

令 $\rho_1=6u_h^{n,1},\rho_2=6u_h^{n,2},\rho_3=u_h^{n,2},\rho_4=u_h^{n,3},\rho_5=4u_h^{n,4}$ 代入式(13)相加并根据式(11)可得

$$3\left\|u_h^{n+1}\right\|^2-3\left\|u_h^{n}\right\|^2+3\left\|u_h^{n,1}-u_h^{n}\right\|^2+3\left\|u_h^{n,2}-u_h^{n,1}\right\|^2=T_1+T_2+T_3 \tag{14}$$

其中，

$$T_1=\tau H\left(\varphi_h^{n,2},u_h^{n,2}\right)+\tau H\left(\varphi_h^{n,3},u_h^{n,3}\right)+4\tau H\left(\varphi_h^{n,4},u_h^{n,4}\right) \tag{15}$$

$$T_2=-\tau\int P^nA_1\left(P^n\right)^{\mathrm{T}}\mathrm{d}x \tag{16}$$

$$T_3=-\left\|2u_h^{n,4}-u_h^{n,2}-u_h^{n,3}\right\|^2+T_4+T_5 \tag{17}$$

其中 $P^n=\left(p_h^{n,1},p_h^{n,2},p_h^{n,3},p_h^{n,4}\right)$，

$$A_1=\begin{pmatrix}6\alpha & -\frac{11}{2}\alpha & 2\alpha & \frac{1}{2}\alpha\\ -\frac{11}{2}\alpha & 1+4\alpha & -\frac{3}{2}\alpha & 0\\ 2\alpha & -\frac{3}{2}\alpha & 1-4\alpha & \frac{7}{2}\alpha\\ \frac{1}{2}\alpha & 0 & \frac{7}{2}\alpha & 4-4\alpha\end{pmatrix}$$

$$\begin{aligned}T_4=&2\left\|2u_h^{n,4}-u_h^{n,2}-u_h^{n,3}\right\|^2+3\left(u_h^{n+1}-2u_h^{n,4}+u_h^{n,2},u_h^{n,3}-u_h^{n,2}\right)\\&+3\left(u_h^{n+1}-2u_h^{n,4}+u_h^{n,2},2u_h^{n,4}-u_h^{n,2}-u_h^{n,3}\right)\end{aligned} \tag{18}$$

$$T_5=3\left\|u_h^{n+1}-2u_h^{n,4}+u_h^{n,2}\right\|^2 \tag{19}$$

由式(10)可得

$$\begin{aligned}\left(2u_h^{n,4}-u_h^{n,2}-u_h^{n,3},v\right)=&\frac{\tau}{2}\left[H\left(\varphi_h^{n,3},v\right)-H\left(\varphi_h^{n,2},v\right)\right]+\frac{\tau}{2}\alpha\tau H^-\left(r_h^{n,1},v\right)-\left(\frac{1}{2}+\alpha\right)\tau H^-\left(r_h^{n,2},v\right)\\&+\left(\frac{1}{2}-\frac{5\alpha}{2}\right)\tau H^-\left(r_h^{n,3},v\right)+2\alpha H^-\left(r_h^{n,4},v\right)\end{aligned} \tag{20}$$

$$\begin{aligned}\left(u_h^{n+1}-2u_h^{n,4}+u_h^{n,2},v\right)=&\frac{\tau}{3}\left[2H\left(\varphi_h^{n,4},v\right)-H\left(\varphi_h^{n,3},v\right)-H\left(\varphi_h^{n,2},v\right)\right]-\frac{3}{2}\alpha\tau H^-\left(r_h^{n,1},v\right)\\&+\left(2\alpha-\frac{1}{3}\right)\tau H^-\left(r_h^{n,2},v\right)+\left(\frac{3}{2}\alpha-\frac{1}{3}\right)\tau H^-\left(r_h^{n,3},v\right)+\left(\frac{2}{3}-2\alpha\right)H^-\left(r_h^{n,4},v\right)\end{aligned} \tag{21}$$

令 $v_1=2u_h^{n,4}-u_h^{n,2}-u_h^{n,3},v_2=u_h^{n,3}-u_h^{n,2},v_3=2u_h^{n,4}-u_h^{n,2}-u_h^{n,3}$，将 $v=2v_1$ 带入式(20)，$v=3v_2,v=3v_3$ 分别带入式(21)相加并由式(11)可得

$$T_4=T_6-\tau\int P^nA_2\left(P^n\right)^{\mathrm{T}}\mathrm{d}x \tag{22}$$

其中

$$A_2=\begin{pmatrix}0 & 3\alpha & -\frac{3}{2}\alpha & -\frac{3}{2}\alpha\\ 3\alpha & 3-10\alpha & 1-\alpha & 8\alpha-4\\ -\frac{3}{2}\alpha & 1-\alpha & 5\alpha-1 & -\frac{5}{2}\alpha\\ -\frac{3}{2}\alpha & 8\alpha-4 & -\frac{5}{2}\alpha & 4-4\alpha\end{pmatrix}$$

$$T_6=\tau\left[H\left(\varphi_h^{n,3},v_1\right)-H\left(\varphi_h^{n,2},v_1\right)\right]+\tau\left[2H\left(\varphi_h^{n,4},v_2\right)-H\left(\varphi_h^{n,3},v_2\right)-H\left(\varphi_h^{n,2},v_2\right)\right]$$
$$+\tau\left[2H\left(\varphi_h^{n,4},v_3\right)-H\left(\varphi_h^{n,3},v_3\right)-H\left(\varphi_h^{n,2},v_3\right)\right] \tag{23}$$

令 $v_4=-2u_h^{n,4}+u_h^{n,2}$，将 $v=4v_4$ 带入式(21)并由式(11)可得

$$T_5=\left(u_h^{n+1}-2u_h^{n,4}+u_h^{n,2},3u_h^{n+1}+2u_h^{n,4}-u_h^{n,2}\right)+T_7+6\alpha\tau\left(p_h^{n,1},p_h^{n,2}\right)-12\alpha\left(p_h^{n,1},p_h^{n,4}\right)$$
$$+\left(\frac{4}{3}-8\alpha\right)\left(p_h^{n,2},p_h^{n,2}\right)+\left(\frac{4}{3}-6\alpha\right)\left(p_h^{n,2},p_h^{n,3}\right)+\left(24\alpha-\frac{16}{3}\right)\left(p_h^{n,2},p_h^{n,4}\right) \tag{24}$$
$$+\left(12\alpha-\frac{8}{3}\right)\left(p_h^{n,3},p_h^{n,4}\right)+\left(\frac{16}{3}-16\alpha\right)\left(p_h^{n,4},p_h^{n,4}\right)$$

其中

$$T_7=\frac{4}{3}\tau\left[2H\left(\varphi_h^{n,4},-2u_h^{n,4}+u_h^{n,2}\right)-H\left(\varphi_h^{n,3},-2u_h^{n,4}+u_h^{n,2}\right)-H\left(\varphi_h^{n,2},-2u_h^{n,4}+u_h^{n,2}\right)\right] \tag{25}$$

由 Cauchy-Schwarz 不等式和 Young 不等式可得

$$\begin{aligned}\left(v_4,3u_h^{n+1}+2u_h^{n,4}-u_h^{n,2}\right)&=3\left\|u_h^{n+1}\right\|^2-4\left(u_h^{n+1},u_h^{n,4}\right)+4\left(u_h^{n,4},u_h^{n,2}\right)+2\left(u_h^{n+1},u_h^{n,2}\right)-4\left\|u_h^{n,4}\right\|^2-\left\|u_h^{n,2}\right\|^2\\&=\left\|\sqrt{2}u_h^{n,4}-\sqrt{2}u_h^{n+1}\right\|^2-\left\|u_h^{n,2}-u_h^{n+1}\right\|^2+4\left\|u_h^{n,4}\right\|\left\|u_h^{n,2}\right\|+2\left\|u_h^{n+1}\right\|^2-6\left\|u_h^{n,4}\right\|^2\\&\le\left\|\sqrt{2}u_h^{n,4}-u_h^{n,2}\right\|^2+\left(5-2\sqrt{2}\right)\left\|u_h^{n+1}\right\|^2-2\left\|u_h^{n,4}\right\|^2+4\left\|u_h^{n,2}\right\|^2\\&\le 2\left\|u_h^{n,4}\right\|^2+\left\|u_h^{n,2}\right\|^2+\left(5-2\sqrt{2}\right)\left\|u_h^{n+1}\right\|^2-2\left\|u_h^{n,4}\right\|^2+4\left\|u_h^{n,2}\right\|^2\\&=\left(5-2\sqrt{2}\right)\left\|u_h^{n+1}\right\|^2+5\left\|u_h^{n,2}\right\|^2\end{aligned} \tag{26}$$

将 $\rho=\left\|u_h^{n,2}\right\|^2$ 带入式(10)中的第二个式子并由式(11)可得

$$\frac{1}{2}\left\|u_h^{n,2}\right\|^2+\frac{1}{2}\left\|u_h^{n,2}-u_h^n\right\|^2=\frac{1}{2}\left\|u_h^n\right\|^2+\alpha\tau\left(p_h^{n,1},p_h^{n,2}\right)-\alpha\tau\left(p_h^{n,2},p_h^{n,2}\right) \tag{27}$$

将式(26)和式(27)带入式(24)可得

$$T_5\le T_7+\left(5-2\sqrt{2}\right)\left\|u_h^{n+1}\right\|^2+5\left\|u_h^n\right\|^2-\tau\int P^nA_3\left(P^n\right)^{\mathrm{T}}\mathrm{d}x \tag{28}$$

其中

$$A_3=\begin{pmatrix}0 & -\frac{11}{2}\alpha & 0 & 6\alpha\\ -\frac{11}{2}\alpha & 13\alpha-\frac{4}{3} & 3\alpha-\frac{2}{3} & \frac{8}{3}-12\alpha\\ 0 & 3\alpha-\frac{2}{3} & 0 & -6\alpha+\frac{4}{3}\\ 6\alpha & \frac{8}{3}-12\alpha & -6\alpha+\frac{4}{3} & -\frac{16}{3}+16\alpha\end{pmatrix}$$

将式(15)，式(23)，式(25)相加，根据式(9)，式(12)，引理 2.1，引理 2.2 和 Young 不等式可得

$$T_1+T_6+T_7\le -7\tau\left(\theta-\frac{1}{2}\right)\left[\!\left[u_h^{n,4}\right]\!\right]^2-\tau\left(\theta-\frac{1}{2}\right)\left[\!\left[u_h^{n,4}-u_h^{n,2}\right]\!\right]^2-\frac{4}{3}\tau\left(\theta-\frac{1}{2}\right)\left[\!\left[u_h^{n,2}\right]\!\right]^2$$
$$+\tau\left(\frac{16C_\mu^2}{3\varepsilon_1}+\frac{2C_\mu^2}{\varepsilon_2}+\frac{2C_\mu^2}{3\varepsilon_3}+\frac{2C_\mu^2}{\varepsilon_4}\right)\left\|2u_h^{n,4}-u_h^{n,2}-u_h^{n,3}\right\|^2-\tau\int Q^n\left(B_1\right)\left(Q^n\right)^{\mathrm{T}}\mathrm{d}x \tag{29}$$
$$+\left\|u_h^{n,2}-u_h^{n,1}\right\|^2+\left\|u_h^{n,1}-u_h^{n}\right\|^2+\left\|u_h^{n}\right\|^2+\frac{\varepsilon_4}{2}\tau\left\|p_h^{n,3}\right\|^2$$

其中 $Q^n=\left(q_h^{n,2},q_h^{n,3},q_h^{n,4}\right)$，

$$B_1=\begin{pmatrix}\frac{8}{3}-\frac{\varepsilon_2}{2}-\frac{\varepsilon_3}{6} & \frac{5}{6} & -\frac{5}{6}\\ \frac{5}{6} & 1 & \frac{1}{3}\\ -\frac{5}{6} & \frac{1}{3} & \frac{8}{3}-\frac{4\varepsilon_1}{3}-3C_\mu^2\tau\end{pmatrix}$$

取 $\varepsilon_1=\frac{3}{4},\varepsilon_2=\varepsilon_3=1$，当 $\tau\le\frac{1}{6C_\mu^2}$ 时，B_1 正定。

取 $\varepsilon_4=0.08$，$-\tau\int P^n\left(A_1+A_2+A_3\right)\left(P^n\right)^{\mathrm{T}}\mathrm{d}x+\frac{\varepsilon_4}{2}\tau\left\|p_h^{n,3}\right\|^2=-\tau\int P^nA_4\left(P^n\right)^{\mathrm{T}}\mathrm{d}x\le 0$，这里 A_4 正定，

$$A_4=\begin{pmatrix}6\alpha & -8\alpha & \frac{\alpha}{2} & 5\alpha\\ -8\alpha & 12\alpha+\frac{8}{3} & \frac{\alpha}{2}+\frac{1}{3} & -\frac{4}{3}-4\alpha\\ \frac{\alpha}{2} & \frac{\alpha}{2}+\frac{1}{3} & \alpha-\frac{\varepsilon_4}{2} & \frac{4}{3}-5\alpha\\ 5\alpha & -\frac{4}{3}-4\alpha & \frac{4}{3}-5\alpha & \frac{8}{3}+8\alpha\end{pmatrix}.$$

当 $\theta>\frac{1}{2}$，$-7\tau\left(\theta-\frac{1}{2}\right)\left[\!\left[u_h^{n,4}\right]\!\right]^2-\tau\left(\theta-\frac{1}{2}\right)\left[\!\left[u_h^{n,4}-u_h^{n,2}\right]\!\right]^2-\frac{4}{3}\tau\left(\theta-\frac{1}{2}\right)\left[\!\left[u_h^{n,2}\right]\!\right]^2\le 0$。

将式(14)，式(16)，式(17)，式(22)，式(28)和式(29)相加，综上分析可得 $\left(2\sqrt{2}-2\right)\left\|u_h^{n+1}\right\|^2-9\left\|u_h^{n}\right\|^2+2\left\|u_h^{n,1}-u_h^{n}\right\|^2+2\left\|u_h^{n,2}-u_h^{n,1}\right\|^2\le-\left(1-\frac{313C^2\tau}{9}\right)\left\|2u_h^{n,4}-u_h^{n,2}-u_h^{n,3}\right\|^2$，所以，当 $\tau\le\tau_0=\frac{9}{313C_\mu^2}$ 时，$\left(2\sqrt{2}-2\right)\left\|u_h^{n+1}\right\|^2\le 9\left\|u_h^{n}\right\|^2$，即可得到 $\left\|u_h^{n}\right\|\le c\left\|u_h^{0}\right\|$。

4. 误差分析

设 u 是方程(2)的精确解，为了进行误差估计，引入参考函数 $u^{(l)}(t),q^{(l)}(t),p^{(l)}(t),r^{(l)}(t),(l=0,1,2,3,4)$，$u^{(0)}(t)=u(t)$，满足：

$$\begin{aligned}&u^1=u^0-\alpha\tau r_x^1\\&u^2=u^0+\alpha\tau r_x^1-\alpha\tau r_x^2\\&u^3=u^0-\tau\varphi^2-(1-\alpha)\tau r_x^2-\alpha\tau r_x^3\\&u^4=u^0-\frac{\tau}{4}\left(\varphi^2+\varphi^3\right)-\beta\tau r_x^1-\delta\tau r_x^2-\sigma\tau r_x^3-\alpha\tau r_x^4\\&u^{n+1}=u^0-\frac{\tau}{6}\left(\varphi^2+\varphi^3+4\varphi^4\right)-\frac{\tau}{6}\left(r_x^2+r_x^3+4r_x^4\right)+\zeta^n\end{aligned}\tag{30}$$

其中 $\varphi^l = u^l - q^l, q^l = u_x^l, p^l = q_x^l, r^l = p_x^l$。$\varsigma^n$ 是估计的截断误差，满足

$$\left\|\varsigma^n\right\| \le C\tau^4$$

这里 C 依赖于 u_t, u_{tt}, u_{ttt} 和 u_{tttt}。

假设精确解满足如下的光滑度条件：

$$u \in L^\infty\left(0,T;H^{k+4}\right), D_t^1 u \in L^\infty\left(0,T;H^{k+3}\right), D_t^2 u \in L^\infty\left(0,T;H^{k+2}\right), D_t^3 u \in L^\infty\left(0,T;H^{k+1}\right), D_t^4 u \in L^\infty\left(0,T;L^2\right) \quad (31)$$

这里 $D_t^\gamma u$ 表示 u 对 t 的 γ 阶导数。

在时间层面上定义参考函数 $u^{n,l} = u^l\left(x,t^n\right), q^{n,l} = q^l\left(x,t^n\right), p^{n,l} = p^l\left(x,t^n\right), r^{n,l} = r^l\left(x,t^n\right)$

参考函数满足如下变分形式，对于 $\forall \rho,\omega,\phi,\psi \in V_h$ 有

$$\begin{aligned}
&\left(u^{n,1},\rho\right) = \left(u^n,\rho\right) + \alpha\tau H^-\left(r^{n,1},\rho\right)\\
&\left(u^{n,2},\rho\right) = \left(u^n,\rho\right) - \alpha\tau H^-\left(r^{n,1},\rho\right) + \alpha\tau H^-\left(r^{n,2},\rho\right)\\
&\left(u^{n,3},\rho\right) = \left(u^n,\rho\right) + \tau H\left(\varphi^{n,2},\rho\right) + (1-\alpha)\tau H^-\left(r^{n,1},\rho\right) + \alpha\tau H^-\left(r^{n,2},\rho\right)\\
&\left(u^{n,4},\rho\right) = \left(u^n,\rho\right) + \frac{\tau}{4}\left[H\left(\varphi^{n,2},\rho\right) + H\left(\varphi^{n,3},\rho\right)\right] + \beta\tau H^-\left(r^{n,1},\rho\right)\\
&\qquad + \delta\tau H^-\left(r^{n,2},\rho\right) + \sigma\tau H^-\left(r^{n,3},\rho\right) + \alpha\tau H^-\left(r^{n,4},\rho\right)\\
&\left(u^{n+1},\rho\right) = \left(u^n,\rho\right) + \frac{\tau}{6}\left[H\left(\varphi^{n,2},\rho\right) + H\left(\varphi^{n,3},\rho\right) + 4H\left(\varphi^{n,4},\rho\right)\right]\\
&\qquad + \frac{\tau}{6}\left[H^-\left(r^{n,2},\rho\right) + H^-\left(r^{n,3},\rho\right) + 4H^-\left(r^{n,4},\rho\right)\right] + \left(\varsigma^n,\rho\right)\\
&\left(q^{n,l},\omega\right) = -H^+\left(u^{n,l},\omega\right)\\
&\left(p^{n,l},\phi\right) = -H^-\left(q^{n,l},\phi\right)\\
&\left(r^{n,l},\psi\right) = -H^+\left(p^{n,l},\psi\right)
\end{aligned} \quad (32)$$

其中 $l = 0,1,2,3,4$。

定义精确解 u 和数值解 u_h 之间的误差为

$$\left(e_u^{n,l}, e_q^{n,l}, e_p^{n,l}, e_r^{n,l}\right) = \left(u^{n,l} - u_h^{n,l}, q^{n,l} - q_h^{n,l}, p^{n,l} - p_h^{n,l}, r^{n,l} - r_h^{n,l}\right), l = 0,1,2,3,4。$$

作为有限元分析中的标准处理，我们一般将误差分成 $e = \xi - \eta$，其中

$$\begin{aligned}
\eta &= \left(\eta_u,\eta_q,\eta_p,\eta_r\right) = \left(P_h u - u, P_h q - q, P_h p - p, P_h r - r\right),\\
\xi &= \left(\xi_u,\xi_q,\xi_p,\xi_r\right) = \left(P_h u - u_h, P_h q - q_h, P_h p - p_h, P_h r - r_h\right)
\end{aligned}$$

这里为了方便，上标 n,l 被省略了。

由投影的定义我们得到：对于 $\forall \rho,\omega,\phi,\psi \in V_h$，有

$H^+\left(\eta_u,\rho\right) = 0, H^-\left(\eta_q,\omega\right) = 0, H^+\left(\eta_p,\phi\right) = 0, H^-\left(\eta_r,\varphi\right) = 0$。

根据全局 Gauss-Radau 投影的性质，引理 1.1 和光滑性假设(31)可得

$$\left\|\eta_u^{n,l}\right\| + \left\|\eta_q^{n,l}\right\| + \left\|\eta_p^{n,l}\right\| + \left\|\eta_r^{n,l}\right\| + h^{1/2}\left\|\eta_u^{n,l}\right\|_{\Gamma_h} \le Ch^{k+1} \quad (33)$$

$$\begin{aligned}
&\left\|\eta_u^{n,1} - \eta_u^n\right\| \le Ch^{k+1}\tau, \left\|\eta_u^{n,2} - \eta_u^{n,1}\right\| \le Ch^{k+1}\tau, \left\|\eta_u^{n,3} - \eta_u^{n,2}\right\| \le Ch^{k+1}\tau\\
&\left\|4\eta_u^{n,4} - 3\eta_u^{n,2} - \eta_u^{n,3}\right\| \le Ch^{k+1}\tau, \left\|\frac{3}{2}\eta_u^{n+1} - 3\eta_u^{n,4} - \frac{1}{2}\eta_u^{n,2}\right\| \le Ch^{k+1}\tau
\end{aligned} \quad (34)$$

这里 $l=0,1,2,3,4$， C 的取值取决于精确解的光滑度。

接下来用式(32)减去式(10)，给出误差 ξ 的估计如下：

$$\begin{aligned}
&\left(\xi_u^{n,1},\rho\right)=\left(\xi_u^{n},\rho\right)+\alpha\tau H^-\left(\xi_r^{n,1},\rho\right)+\left(\eta_u^{n,1}-\eta_u^{n},\rho\right)\\
&\left(\xi_u^{n,2},\rho\right)=\left(\xi_u^{n},\rho\right)-\alpha\tau H^-\left(\xi_r^{n,1},\rho\right)+\alpha\tau H^-\left(\xi_r^{n,2},\rho\right)+\left(\eta_u^{n,2}-\eta_u^{n},\rho\right)\\
&\left(\xi_u^{n,3},\rho\right)=\left(\xi_u^{n},\rho\right)+\tau H\left(\xi_\varphi^{n,2},\rho\right)+(1-\alpha)\tau H^-\left(\xi_r^{n,1},\rho\right)+\alpha\tau H^-\left(\xi_r^{n,2},\rho\right)+\left(\eta_u^{n,3}-\eta_u^{n},\rho\right)\\
&\left(\xi_u^{n,4},\rho\right)=\left(\xi_u^{n},\rho\right)+\frac{\tau}{4}\left[H\left(\xi_\varphi^{n,2},\rho\right)+H\left(\xi_\varphi^{n,3},\rho\right)\right]+\beta\tau H^-\left(\xi_r^{n,1},\rho\right)+\delta\tau H^-\left(\xi_r^{n,2},\rho\right)\\
&\qquad+\sigma\tau H^-\left(\xi_r^{n,3},\rho\right)+\alpha\tau H^-\left(\xi_r^{n,4},\rho\right)+\left(\eta_u^{n,4}-\eta_u^{n},\rho\right)\\
&\left(\xi_u^{n+1},\rho\right)=\left(\xi_u^{n},\rho\right)+\frac{\tau}{6}\left[H\left(\xi_\varphi^{n,2},\rho\right)+H\left(\xi_\varphi^{n,3},\rho\right)+4H\left(\xi_\varphi^{n,4},\rho\right)\right]\\
&\qquad+\frac{\tau}{6}\left[H^-\left(\xi_r^{n,2},\rho\right)+H^-\left(\xi_r^{n,3},\rho\right)+4H^-\left(\xi_r^{n,4},\rho\right)\right]+\left(\eta_u^{n+1}-\eta_u^{n},\rho\right)+\left(\zeta^{n},\rho\right)\\
&\left(\xi_q^{n,l},\omega\right)=-H^+\left(\xi_u^{n,l},\omega\right)+\left(\eta_q^{n,l},\omega\right)\\
&\left(\xi_p^{n,l},\phi\right)=-H^-\left(\xi_q^{n,l},\phi\right)+\left(\eta_p^{n,l},\phi\right)\\
&\left(\xi_r^{n,l},\psi\right)=-H^+\left(\xi_p^{n,l},\psi\right)+\left(\eta_r^{n,l},\psi\right)
\end{aligned}\tag{35}$$

引理 4.1 [12]设 $\xi_u,\xi_q,\xi_p\in V_h$ 满足(35)，并且 $v\in V_h$，则存在正整数 C 满足如下不等式

$$\left|H\left(\xi_u,v\right)\right|\le C\left(\left\|\xi_q\right\|+h^{k+1}\right)\|v\|,\left|H\left(\xi_q,v\right)\right|\le C\left(\left\|\xi_p\right\|+h^{k+1}\right)\|v\|.$$

定理 4.1 u 是一维四阶线性方程(2)的精确解而且满足光滑性假设(31)，有限元空间是分段多项式空间 V_h，u_h 是隐显式 Runge-Kutta LDG 格式(11)的数值解，当 $\theta>\frac{1}{2}$，存在与 h 无关的正常数 τ_0，当 $\tau\le\tau_0$，有如下误差估计

$$\max_{n\tau\le T}\left\|u\left(t^n\right)-u_h^n\right\|\le C\left(h^{k+1}+\tau^3\right).$$

证明：由式(35)可得

$$\begin{aligned}
&\left(\xi_u^{n,1}-\xi_u^{n},\rho_1\right)=\alpha\tau H^-\left(\xi_r^{n,1},\rho_1\right)+\left(\eta_u^{n,1}-\eta_u^{n},\rho_1\right)\\
&\left(\xi_u^{n,2}-\xi_u^{n,1},\rho_2\right)=-2\alpha\tau H^-\left(\xi_r^{n,1},\rho_2\right)+\alpha\tau H^-\left(\xi_r^{n,2},\rho_2\right)+\left(\eta_u^{n,2}-\eta_u^{n,1},\rho_2\right)\\
&\left(\xi_u^{n,3}-\xi_u^{n,2},\rho_3\right)=\tau H\left(\xi_\varphi^{n,2},\rho_3\right)+\alpha\tau H^-\left(\xi_r^{n,1},\rho_3\right)+(1-2\alpha)\tau H^-\left(\xi_r^{n,2},\rho_3\right)+\alpha\tau H^-\left(\xi_r^{n,3},\rho_3\right)+\left(\eta_u^{n,3}-\eta_u^{n,2},\rho_3\right)\\
&\left(4\xi_u^{n,4}-3\xi_u^{n,2}-\xi_u^{n,3},\rho_4\right)=\tau H\left(\xi_\varphi^{n,3},\rho_4\right)+4\alpha\tau H^-\left(\xi_r^{n,1},\rho_4\right)-4\alpha\tau H^-\left(\xi_r^{n,2},\rho_4\right)+(1-4\alpha)\tau H^-\left(\xi_r^{n,3},\rho_4\right)\\
&\qquad+4\alpha\tau H^-\left(\xi_r^{n,4},\rho_4\right)+\left(4\eta_u^{n,4}-3\eta_u^{n,2}-\eta_u^{n,3},\rho_4\right)\\
&\left(\frac{3}{2}\xi_u^{n+1}-3\xi_u^{n,4}-\frac{1}{2}\xi_u^{n,2},\rho_5\right)=\tau H\left(\xi_\varphi^{n,4},\rho_5\right)+\frac{\alpha}{4}\tau H^-\left(\xi_r^{n,1},\rho_5\right)+\frac{3\alpha}{4}\tau H^-\left(\xi_r^{n,3},\rho_5\right)+(1-\alpha)\tau H^-\left(\xi_r^{n,4},\rho_5\right)\\
&\qquad+\left(\frac{3}{2}\eta_u^{n+1}-\eta\xi_u^{n,4}-\frac{1}{2}\eta_u^{n,2},\rho_5\right)+\frac{3}{2}\left(\zeta^{n},\rho_5\right)
\end{aligned}\tag{36}$$

由式(35)和式(8)可得

$$H^-\left(\xi_r^{n,k},\xi_u^{n,s}\right)=\left(\eta_p^{n,s},\xi_p^{n,k}\right)-\left(\xi_p^{n,s},\xi_p^{n,k}\right)\tag{37}$$

$$H^-\left(\xi_q^{n,k},\xi_u^{n,s}\right)=\left(\xi_q^{n,s},\xi_q^{n,k}\right)-\left(\eta_q^{n,s},\xi_q^{n,k}\right)\tag{38}$$

令 $\rho_1=6\xi_u^{n,1},\rho_2=6\xi_u^{n,2},\rho_3=\xi_u^{n,2},\rho_4=\xi_u^{n,3},\rho_5=4\xi_u^{n,4}$ 带入式(36)可得

$$3\left\|\xi_u^{n+1}\right\|^2-3\left\|\xi_u^{n}\right\|^2+3\left\|\xi_u^{n,1}-\xi_u^{n}\right\|^2+3\left\|\xi_u^{n,2}-\xi_u^{n,1}\right\|^2=R_1+R_2+R_3+R_4 \tag{39}$$

其中，

$$R_1=\tau H\left(\xi_\varphi^{n,2},\xi_u^{n,2}\right)+\tau H\left(\xi_\varphi^{n,3},\xi_u^{n,3}\right)+4\tau H\left(\xi_\varphi^{n,4},\xi_u^{n,4}\right) \tag{40}$$

$$R_2=-\tau\int\xi_p^nA_1\left(\xi_p^n\right)^{\mathrm{T}}\mathrm{d}x+\tau\int\eta_p^nA_1\left(\eta_p^n\right)^{\mathrm{T}}\mathrm{d}x \tag{41}$$

$$R_3=-\left\|2\xi_u^{n,4}-\xi_u^{n,2}-\xi_u^{n,3}\right\|^2+R_5+R_6 \tag{42}$$

$$\begin{aligned}R_4=&6\left(\eta_u^{n,1}-\eta_u^{n},\xi_u^{n,1}\right)+6\left(\eta_u^{n,2}-\eta_u^{n,1},\xi_u^{n,2}\right)+\left(\eta_u^{n,3}-\eta_u^{n,2},\xi_u^{n,2}\right)+\left(4\eta_u^{n,4}-3\eta_u^{n,2}-\eta_u^{n,3},\xi_u^{n,3}\right)\\&+\left(6\eta_u^{n,4}-4\eta_u^{n,2}-2\eta_u^{n,3},\xi_u^{n,4}\right)+6\left(\zeta^n,\xi_u^{n,4}\right)\end{aligned} \tag{43}$$

其中 $\xi_p^n=\left(\xi_p^{n,1},\xi_p^{n,2},\xi_p^{n,3},\xi_p^{n,4}\right),\eta_p^n=\left(\eta_p^{n,1},\eta_p^{n,2},\eta_p^{n,3},\eta_p^{n,4}\right)$，

$$R_5=2\left\|2\xi_u^{n,4}-\xi_u^{n,2}-\xi_u^{n,3}\right\|^2+3\left(\xi_u^{n+1}-2\xi_u^{n,4}+\xi_u^{n,3},\xi_u^{n,3}-\xi_u^{n,2}\right)+3\left(\xi_u^{n+1}-2\xi_u^{n,4}+\xi_u^{n,3},2\xi_u^{n,4}-\xi_u^{n,2}-\xi_u^{n,3}\right) \tag{44}$$

$$R_6=3\left\|\xi_u^{n+1}-2\xi_u^{n,4}+\xi_u^{n,3}\right\|^2 \tag{45}$$

由式(36)可得

$$\begin{aligned}\left(2\xi_u^{n,4}-\xi_u^{n,2}-\xi_u^{n,3},v\right)=&\frac{\tau}{2}\left[H\left(\xi_\varphi^{n,3},v\right)-H\left(\xi_\varphi^{n,2},v\right)\right]+\frac{\tau}{2}\alpha\tau H^-\left(\xi_r^{n,1},v\right)-\left(\frac{1}{2}+\alpha\right)\tau H^-\left(\xi_r^{n,2},v\right)\\&+\left(\frac{1}{2}-\frac{5}{2}\alpha\right)\tau H^-\left(\xi_r^{n,3},v\right)+2\alpha H^-\left(\xi_r^{n,4},v\right)+\left(2\eta_u^{n,4}-\eta_u^{n,2}-\eta_u^{n,3},v\right)\end{aligned} \tag{46}$$

$$\begin{aligned}\left(\xi_u^{n+1}-2\xi_u^{n,4}+\xi_u^{n,2},v\right)=&\frac{\tau}{3}\left[2H\left(\xi_\varphi^{n,4},v\right)-H\left(\xi_\varphi^{n,3},v\right)-H\left(\xi_\varphi^{n,2},v\right)\right]-\frac{3}{2}\alpha\tau H^-\left(\xi_r^{n,1},v\right)\\&+\left(2\alpha-\frac{1}{3}\right)\tau H^-\left(\xi_r^{n,2},v\right)+\left(\frac{3}{2}\alpha-\frac{1}{3}\right)\tau H^-\left(\xi_r^{n,3},v\right)\\&+\left(\frac{2}{3}-2\alpha\right)H^-\left(\xi_r^{n,4},v\right)+\left(\eta_u^{n+1}-2\eta_u^{n,4}+\eta_u^{n,2},v\right)+\left(\zeta^n,v\right)\end{aligned} \tag{47}$$

令 $v_5=2\xi_u^{n,4}-\xi_u^{n,2}-\xi_u^{n,3},v_6=\xi_u^{n,3}-\xi_u^{n,2},v_7=2\xi_u^{n,4}-\xi_u^{n,2}-\xi_u^{n,3}$，将 $v=2v_5,v=3v_6$ 带入式(46)，将 $v=3v_6$ $v=3v_7$ 带入式(47)可得

$$R_5=R_7-\tau\int\xi_p^nA_2\left(\xi_p^n\right)^{\mathrm{T}}\mathrm{d}x+\tau\int\eta_p^nA_2\left(\eta_p^n\right)^{\mathrm{T}}\mathrm{d}x+R_8 \tag{48}$$

其中

$$\begin{aligned}R_7=&\tau H\left(\xi_\varphi^{n,3},v_5\right)-\tau H\left(\xi_\varphi^{n,2},v_5\right)+2\tau H\left(\xi_\varphi^{n,4},v_6\right)-\tau H\left(\xi_\varphi^{n,3},v_6\right)\\&-\tau H\left(\xi_\varphi^{n,2},v_6\right)+2\tau H\left(\xi_\varphi^{n,4},v_7\right)-\tau H\left(\xi_\varphi^{n,3},v_7\right)-\tau H\left(\xi_\varphi^{n,2},v_7\right)\end{aligned} \tag{49}$$

$$R_8=\left(2\eta_u^{n,4}-\eta_u^{n,2}-\eta_u^{n,3},v_5\right)+\left(\eta_u^{n+1}-2\eta_u^{n,4}+\eta_u^{n,2},v_6\right)+\left(\zeta^n,v_6\right)+\left(\eta_u^{n+1}-2\eta_u^{n,4}+\eta_u^{n,2},v_7\right)+\left(\zeta^n,v_7\right) \tag{50}$$

由式(26)得

$$\left(\xi_u^{n+1}-2\xi_u^{n,4}+\xi_u^{n,2},3\xi_u^{n+1}+2\xi_u^{n,4}-\xi_u^{n,2}\right)\le\left(5-2\sqrt{2}\right)\left\|\xi_u^{n+1}\right\|^2+5\left\|\xi_u^{n,2}\right\|^2 \tag{51}$$

令 $\rho=2\xi_u^{n,1}$ 带入式(35)的第二个式子可得

$$\left\|\xi_u^{n,2}\right\|^2+\left\|\xi_u^{n,2}-\xi_u^n\right\|^2-\left\|\xi_u^n\right\|^2=-2\alpha\tau\left(\eta_p^{n,2},\xi_p^{n,1}\right)+2\alpha\tau\left(\xi_p^{n,2},\xi_p^{n,1}\right)+2\alpha\tau\left(\eta_p^{n,2},\xi_p^{n,2}\right) \\ -2\alpha\tau\left(\xi_p^{n,2},\xi_p^{n,2}\right)+2\left(\eta_u^{n,2}-\eta_u^n,\xi_u^{n,2}\right) \tag{52}$$

$$5\left\|\xi_u^{n,2}\right\|^2\le 5\left\|\xi_u^n\right\|^2-10\alpha\tau\left(\eta_p^{n,2},\xi_p^{n,1}\right)+10\alpha\tau\left(\xi_p^{n,2},\xi_p^{n,1}\right)+10\alpha\tau\left(\eta_p^{n,2},\xi_p^{n,2}\right) \\ -10\alpha\tau\left(\xi_p^{n,2},\xi_p^{n,2}\right)+10\left(\eta_u^{n,2}-\eta_u^n,\xi_u^{n,2}\right) \tag{53}$$

令 $v_8=2\xi_u^{n,4}+\xi_u^{n,2}$，将 $v=4v_4$ 入式(47)并由式(51)和式(53)可得

$$R_6\le\left(5-2\sqrt{2}\right)\left\|\xi_u^{n+1}\right\|^2+5\left\|\xi_u^n\right\|^2+R_9-\tau\int\xi_p^nA_3\left(\xi_p^n\right)^{\mathrm{T}}\mathrm{d}x+\tau\int\eta_p^nA_3\left(\eta_p^n\right)^{\mathrm{T}}\mathrm{d}x+R_{10} \tag{54}$$

其中，

$$R_9=\frac{4\tau}{3}\left[2H\left(\xi_\varphi^{n,4},-2\xi_u^{n,4}+\xi_u^{n,2}\right)-H\left(\xi_\varphi^{n,3},-2\xi_u^{n,4}+\xi_u^{n,2}\right)-H\left(\xi_\varphi^{n,2},-2\xi_u^{n,4}+\xi_u^{n,2}\right)\right] \tag{55}$$

$$R_{10}=\left(\eta_u^{n+1}-2\eta_u^{n,4}+\eta_u^{n,2},4v_8\right)+\left(\zeta^n,4v_8\right)+10\left(\eta_u^{n,2}-\eta_u^n,\xi_u^{n,2}\right) \tag{56}$$

将式(40)，式(49)，式(55)相加，根据式(9)，式(33)，式(34)，式(38)，引理 4.1 和 Young 不等式可得

$$R_1+R_7+R_9\le-7\tau\left(\theta-\frac{1}{2}\right)[\![\xi_u^{n,4}]\!]^2-\tau\left(\theta-\frac{1}{2}\right)[\![\xi_u^{n,4}-\xi_u^{n,2}]\!]^2-\frac{4}{3}\tau\left(\theta-\frac{1}{2}\right)[\![\xi_u^{n,2}]\!]^2 \\ +\tau\left(\frac{16C^2}{3\varepsilon_1}+\frac{2C^2}{\varepsilon_2}+\frac{2C^2}{3\varepsilon_3}+\frac{2C^2}{\varepsilon_4}\right)\left\|2\xi_u^{n,4}-\xi_u^{n,2}-\xi_u^{n,3}\right\|^2+\frac{\varepsilon_4}{2}\tau\left\|\xi_p^{n,4}\right\|^2 \\ +Ch^{2k+2}\tau-\tau\int\xi_q^nB_1\left(\xi_q^n\right)^Tdx+\tau\int\eta_q^nB_1\left(\xi_q^n\right)^Tdx+3C_\mu^2\tau^2\left\|\xi_q^{n,4}\right\|^2 \\ +\left\|\xi_u^{n,2}-\xi_u^{n,1}\right\|^2+\left\|\xi_u^{n,1}-\xi_u^n\right\|^2+\left\|\xi_u^n\right\|^2 \tag{57}$$

其中 $\xi_q^n=\left(\xi_q^{n,2},\xi_q^{n,3},\xi_q^{n,4}\right),\eta_p^n=\left(\eta_q^{n,2},\eta_q^{n,3},\eta_q^{n,4}\right)$。

将式(43)，式(50)，式(56)相加，根据式(33)，式(34)，Cauchy-Schwarz 不等式和 Young 不等式可得

$$R_4+R_8+R_{10}\le C\left(h^{2k+2}\tau+\tau^7\right)+\frac{\varepsilon}{2}\left(\left\|\xi_u^{n,1}\right\|^2+\left\|\xi_u^{n,2}\right\|^2+\left\|\xi_u^{n,3}\right\|^2+\left\|\xi_u^{n,4}\right\|^2\right) \tag{58}$$

令 $\rho=2\xi_u^{n,1}$ 带入式(33)的第一个式子并由式(37)和 Young 不等式可得

$$\left\|\xi_u^{n,1}\right\|^2+\left\|\xi_u^{n,1}-\xi_u^n\right\|^2-\left\|\xi_u^n\right\|^2=2\alpha\tau\left(\eta_p^{n,1},\xi_p^{n,1}\right)-2\alpha\tau\left(\xi_p^{n,1},\xi_p^{n,1}\right)+2\left(\eta_u^{n,1}-\eta_u^n,\xi_u^{n,1}\right) \\ \le\left(\frac{\tilde{\varepsilon}_1}{2}-2\alpha\right)\tau\left\|\xi_p^{n,1}\right\|^2+\frac{1}{2}\left\|\xi_u^{n,1}\right\|^2+Ch^{2k+2}\tau$$

由式(52)可得

$$\left\|\xi_u^{n,2}\right\|^2+\left\|\xi_u^{n,2}-\xi_u^n\right\|^2-\left\|\xi_u^n\right\|^2\le\frac{\tilde{\varepsilon}_2}{2}\tau\left\|\xi_p^{n,1}\right\|^2+2\alpha\tau\left(\xi_p^{n,2},\xi_p^{n,1}\right)-\left(2\alpha-\frac{\tilde{\varepsilon}_2}{2}\right)\tau\left\|\xi_p^{n,2}\right\|^2+\frac{1}{2}\left\|\xi_u^{n,2}\right\|^2+Ch^{2k+2}\tau$$

则有 $\left\|\xi_u^{n,2}\right\|^2\le 2\left\|\xi_u^n\right\|^2+\tilde{\varepsilon}_2\tau\left\|\xi_p^{n,1}\right\|^2+4\alpha\tau\left(\xi_p^{n,2},\xi_p^{n,1}\right)-\left(4\alpha-\tilde{\varepsilon}_2\right)\tau\left\|\xi_p^{n,2}\right\|^2+Ch^{2k+2}\tau$ 。

令 $\rho=2\xi_u^{n,3}$ 带入式(33)的第三个式子并由式(37)，式(38)和 Young 不等式可得

$$\left\|\xi_u^{n,3}\right\|^2+\left\|\xi_u^{n,3}-\xi_u^n\right\|^2-\left\|\xi_u^n\right\|^2 \\ \le C\tau^2\left\|\xi_q^{n,2}\right\|^2+\frac{1}{2}\left\|\xi_u^{n,3}\right\|^2+Ch^{2k+2}\tau+2\tau\left(\eta_q^{n,3},\xi_q^{n,2}\right)-2\tau\left(\xi_q^{n,3},\xi_q^{n,2}\right) \\ +2(1-\alpha)\tau\left(\eta_p^{n,3},\xi_p^{n,2}\right)-2(1-\alpha)\tau\left(\xi_p^{n,3},\xi_p^{n,2}\right)+2\alpha\tau\left(\eta_p^{n,3},\xi_p^{n,3}\right)-2\alpha\tau\left(\xi_p^{n,3},\xi_p^{n,3}\right)$$

则有

$$\left\|\xi_u^{n,3}\right\|^2 \le 2\left\|\xi_u^n\right\|^2 + C\tau^2\left\|\xi_q^{n,2}\right\|^2 + Ch^{2k+2}\tau + 4\tau\left(\eta_q^{n,3},\xi_q^{n,2}\right) - 4\tau\left(\xi_q^{n,3},\xi_q^{n,2}\right) + 4(1-\alpha)\tau\left(\eta_p^{n,3},\xi_p^{n,2}\right)$$
$$-4(1-\alpha)\tau\left(\xi_p^{n,3},\xi_p^{n,2}\right) + 4\alpha\tau\left(\eta_p^{n,3},\xi_p^{n,3}\right) - 4\alpha\tau\left(\xi_p^{n,3},\xi_p^{n,3}\right)$$

令 $\rho = 2\xi_u^{n,4}$ 带入式(33)的第四个式子并由式(37)，式(38)和 Young 不等式可得

$$\begin{aligned}
&\left\|\xi_u^{n,4}\right\|^2 + \left\|\xi_u^{n,4} - \xi_u^n\right\|^2 - \left\|\xi_u^n\right\|^2 \\
&\le C\tau^2\left\|\xi_q^{n,2}\right\|^2 + C\tau^2\left\|\xi_q^{n,3}\right\|^2 + \frac{\tau}{2}\left(\eta_q^{n,4},\xi_q^{n,2}\right) - \frac{\tau}{2}\left(\xi_q^{n,4},\xi_q^{n,2}\right) + \frac{\tau}{2}\left(\eta_q^{n,4},\xi_q^{n,3}\right) - \frac{\tau}{2}\left(\xi_q^{n,4},\xi_q^{n,3}\right) \\
&\quad + 2\beta\tau\left(\eta_p^{n,4},\xi_p^{n,1}\right) - 2\beta\tau\left(\xi_p^{n,4},\xi_p^{n,1}\right) + 2\delta\tau\left(\eta_p^{n,4},\xi_p^{n,2}\right) - 2\delta\tau\left(\xi_p^{n,4},\xi_p^{n,2}\right) + 2\sigma\tau\left(\eta_p^{n,4},\xi_p^{n,3}\right) \\
&\quad - 2\sigma\tau\left(\xi_p^{n,4},\xi_p^{n,2}\right) + 2\alpha\tau\left(\eta_p^{n,4},\xi_p^{n,4}\right) - 2\alpha\tau\left(\xi_p^{n,4},\xi_p^{n,4}\right) + \frac{1}{2}\left\|\xi_u^{n,3}\right\|^2 + Ch^{2k+2}\tau
\end{aligned}$$

则有

$$\begin{aligned}
\left\|\xi_u^{n,4}\right\|^2 &\le 2\left\|\xi_u^n\right\|^2 + C\tau^2\left\|\xi_q^{n,2}\right\|^2 + C\tau^2\left\|\xi_q^{n,3}\right\|^2 + \tau\left(\eta_q^{n,4},\xi_q^{n,2}\right) - \tau\left(\xi_q^{n,4},\xi_q^{n,2}\right) + \tau\left(\eta_q^{n,4},\xi_q^{n,3}\right) - \tau\left(\xi_q^{n,4},\xi_q^{n,3}\right) \\
&\quad + 4\beta\tau\left(\eta_p^{n,4},\xi_p^{n,1}\right) - 4\beta\tau\left(\xi_p^{n,4},\xi_p^{n,1}\right) + 4\delta\tau\left(\eta_p^{n,4},\xi_p^{n,2}\right) - 4\delta\tau\left(\xi_p^{n,4},\xi_p^{n,2}\right) + 4\sigma\tau\left(\eta_p^{n,4},\xi_p^{n,3}\right) \\
&\quad - 4\sigma\tau\left(\xi_p^{n,4},\xi_p^{n,3}\right) + 4\alpha\tau\left(\eta_p^{n,4},\xi_p^{n,4}\right) - 4\alpha\tau\left(\xi_p^{n,4},\xi_p^{n,4}\right) + Ch^{2k+2}\tau
\end{aligned}$$

综上分析可得，

$$\begin{aligned}
&\left\|\xi_u^{n,1}\right\|^2 + \left\|\xi_u^{n,2}\right\|^2 + \left\|\xi_u^{n,3}\right\|^2 + \left\|\xi_u^{n,4}\right\|^2 \\
&\le C\left\|\xi_u^n\right\|^2 - \tau\int \xi_p^n A_5\left(\xi_p^n\right)^{\mathrm{T}}\mathrm{d}x + \tau\int \eta_p^n A_5\left(\eta_p^n\right)^{\mathrm{T}}\mathrm{d}x - \tau\int \xi_q^n B_2\left(\xi_q^n\right)^{\mathrm{T}}\mathrm{d}x \\
&\quad + \tau\int \eta_q^n B_2\left(\eta_q^n\right)^{\mathrm{T}}dx + Ch^{2k+2}\tau + 3\alpha\tau\int \xi_p^n\left(\xi_p^n\right)^{\mathrm{T}}\mathrm{d}x - 3\alpha\tau\int \eta_p^n\left(\eta_p^n\right)^{\mathrm{T}}\mathrm{d}x \\
&\quad + (3+C\tau)\tau\int \xi_q^n\left(\xi_q^n\right)^{\mathrm{T}}\mathrm{d}x - (3+C\tau)\tau\int \eta_q^n\left(\eta_q^n\right)^{\mathrm{T}}\mathrm{d}x
\end{aligned}$$

其中，

$$A_5 = \begin{pmatrix} 5\alpha - \tilde{\varepsilon}_1 - \tilde{\varepsilon}_2 & -2\alpha & 0 & \frac{\alpha}{2} \\ -2\alpha & 7\alpha - \tilde{\varepsilon}_2 & 2\alpha - 2\alpha & \frac{1}{2} - \alpha \\ 0 & 2\alpha - 2\alpha & 7\alpha & \frac{1}{2} - \frac{3\alpha}{2} \\ \frac{\alpha}{2} & \frac{1}{2} - \alpha & \frac{1}{2} - \frac{3\alpha}{2} & 7\alpha \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 3 & 2 & \frac{1}{2} \\ 2 & 3 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 3 \end{pmatrix}$$

当 $\tilde{\varepsilon}_1 = 0.01, \tilde{\varepsilon}_2 = 0.01$ 时，且存在 $\tilde{\varepsilon}_3, \tilde{\varepsilon}_4 \le 0.05$ 使得 $A_5 - \tilde{\varepsilon}_3 I$ 正定， $B_2 - \tilde{\varepsilon}_4 I$ 正定，则有

$$\left\|\xi_u^{n,1}\right\|^2 + \left\|\xi_u^{n,2}\right\|^2 + \left\|\xi_u^{n,3}\right\|^2 + \left\|\xi_u^{n,4}\right\|^2 \le C\left\|\xi_u^n\right\|^2 + Ch^{2k+2}\tau + 3\alpha\tau\int \xi_p^n\left(\xi_p^n\right)^{\mathrm{T}}\mathrm{d}x - 3\alpha\tau\int \eta_p^n\left(\eta_p^n\right)^{\mathrm{T}}\mathrm{d}x$$
$$+\left(3+C\tau\right)\tau\int \xi_q^n\left(\xi_q^n\right)^{\mathrm{T}}\mathrm{d}x - \left(3+C\tau\right)\tau\int \eta_q^n\left(\eta_q^n\right)^{\mathrm{T}}\mathrm{d}x$$

综上所述，所以有

$$\begin{aligned} R_4 + R_8 + R_{10} \le{}& C\left(h^{2k+2}\tau + \tau^7\right) + C\left\|\xi_u^n\right\|^2 + \frac{3}{2}\alpha\varepsilon\tau\int \xi_p^n\left(\xi_p^n\right)^{\mathrm{T}}\mathrm{d}x - \frac{3}{2}\alpha\varepsilon\tau\int \eta_p^n\left(\eta_p^n\right)^{\mathrm{T}}\mathrm{d}x \\ &+ \frac{3+C\tau}{2}\varepsilon\tau\int \xi_q^n\left(\xi_q^n\right)^{\mathrm{T}}\mathrm{d}x - \frac{3+C\tau}{2}\varepsilon\tau\int \eta_q^n\left(\eta_q^n\right)^{\mathrm{T}}\mathrm{d}x \end{aligned} \tag{59}$$

取 $\varepsilon_1 = \frac{3}{4}, \varepsilon_2 = \varepsilon_3 = 1, \varepsilon_4 = 0.08$，当 $\theta > \frac{1}{2}$ 时，将式(39)，式(41)，式(42)，式(48)，式(54)，式(57)和式(59)相加可得

$$\begin{aligned} &\left(2\sqrt{2}-2\right)\left\|\xi_u^{n+1}\right\|^2 - 9\left\|\xi_u^n\right\|^2 \\ \le{}& C\left\|\xi_u^n\right\|^2 - \left(1-\frac{313C^2\tau}{9}\right)\left\|2\xi_u^{n,4} - \xi_u^{n,2} - \xi_u^{n,3}\right\|^2 - \tau\int \xi_p^n\left(A_4 - \frac{3}{2}\alpha\varepsilon I\right)\left(\xi_p^n\right)^{\mathrm{T}}\mathrm{d}x \\ &+ \tau\int \eta_p^n\left(A_4 - \frac{3}{2}\alpha\varepsilon I\right)\left(\xi_p^n\right)^{\mathrm{T}}\mathrm{d}x - \tau\int \xi_q^n\left(B_1 - \frac{3+C\tau}{2}\varepsilon I\right)\left(\xi_q^n\right)^{\mathrm{T}}\mathrm{d}x \\ &+ \tau\int \eta_q^n\left(B_1 - \frac{3+C\tau}{2}\varepsilon I\right)\left(\xi_q^n\right)^{\mathrm{T}}\mathrm{d}x + C\left(h^{2k+2}\tau + \tau^7\right) \\ \le{}& C\left\|\xi_u^n\right\|^2 - \left(1-\frac{313C^2\tau}{9}\right)\left\|2\xi_u^{n,4} - \xi_u^{n,2} - \xi_u^{n,3}\right\|^2 - \tau\int \xi_p^n\left(A_4 - \frac{3}{2}\alpha\varepsilon I - \hat{\varepsilon}_1 I\right)\left(\xi_p^n\right)^{\mathrm{T}}\mathrm{d}x \\ &- \tau\int \xi_q^n\left(B_1 - \frac{3+C\tau}{2}\varepsilon I - \hat{\varepsilon}_2 I\right)\left(\xi_q^n\right)^{\mathrm{T}}\mathrm{d}x + C\left(h^{2k+2}\tau + \tau^7\right) \end{aligned}$$

取 $\varepsilon = \min\left\{\frac{1}{300\alpha}, \frac{1}{300+C\tau}\right\}$， $\hat{\varepsilon}_1, \hat{\varepsilon}_2 \le 0.005$， $A_4 - \frac{3}{2}\alpha\varepsilon I - \hat{\varepsilon}_1 I$ 正定， $B_1 - \frac{3+C\tau}{2}\varepsilon I - \hat{\varepsilon}_2 I$ 正定，当 $\tau \le \tau_0 = \frac{9}{313C^2}$，即可得到 $\left\|\xi_u^{n+1}\right\|^2 \le C\left\|\xi_u^n\right\|^2 + C\left(h^{2k+2}\tau + \tau^7\right)$，由于 $\left\|\xi_u^0\right\| \le Ch^{k+1}$，再由 Gronwall's 不等式可得 $\left\|\xi_u^n\right\| \le C\left(h^{k+1} + \tau^3\right)$。

注意这里证明过程中 C 的值不相同。

5. 数值算例

例 1 考虑如下一维四阶线性方程，边界条件是周期边界条件：

$$u_t + u_x + u_{xxxx} = u_{xx}, \qquad (x,t) \in (0,2\pi)\times(0,T],$$
$$u(x,0) = \sin x, \qquad x \in (0,2\pi).$$

方程的精确解为 $u(x,t) = \mathrm{e}^{-2t}\sin(x-t)$，在 P^0, P^1, P^2 有限元空间上数值误差和收敛阶。

在表 1 中，时间步长选择 $\tau = h$，在表 2 中，时间步长选择 $\tau = 0.01\times h^2$，在表 3 中，时间步长选择 $\tau = 0.001\times h^4$，时刻都是 $T = 0.01$，从表格中可以看到选取不同数值流通量，隐显式全离散 LDG 方法可以达到 $k+1$ 阶收敛精度，数值结果与理论结果相符。

Table 1. L^2 Error and convergence order for different θ $(k=0)$

表 1. 在不同 θ 下 L^2 误差与收敛阶 $(k=0)$

N	$\theta=0.75$		$\theta=1$		$\theta=1.25$	
	L^2 误差	阶数	L^2 误差	阶数	L^2 误差	阶数
20	5.1544e−1	—	5.1192e−1	—	5.1174e−1	—
40	2.5679e−1	—	2.5528e−1	—	2.5423e−1	—
60	1.6815e−1	1.01	1.6715e−1	1.00	1.6628e−1	1.01
80	1.2417e−1	1.05	1.2341e−1	1.05	1.2270e−1	1.05
120	8.1538e−2	1.04	8.102 e−2	1.04	8.0519e−2	1.05
160	6.1737e−2	1.01	6.1352e−2	1.01	6.0972e−2	1.01

Table 2. L^2 Error and convergence order for different θ $(k=1)$

表 2. 在不同 θ 下 L^2 误差与收敛阶 $(k=1)$

N	$\theta=0.75$		$\theta=1$		$\theta=1.25$	
	L^2 误差	阶数	L^2 误差	阶数	L^2 误差	阶数
20	4.7597e−2	—	6.0818e−2	—	8.6066e−2	—
40	1.2333e−2	—	1.5929e−2	—	2.2979e−2	—
60	5.5595e−3	1.95	7.2039e−3	1.93	1.0455e−2	1.91
80	3.1536e−3	1.97	4.0920e−3	1.96	5.9550e−3	1.95
120	1.4185e−3	1.97	1.8418e−3	1.97	2.6856e−3	1.96
160	8.0672e−4	1.97	1.0469e−3	1.97	1.5262e−3	1.96

Table 3. L^2 Error and convergence order for different θ $(k=2)$

表 3. 在不同 θ 下 L^2 误差与收敛阶 $(k=2)$

N	$\theta=0.75$		$\theta=1$		$\theta=1.25$	
	L^2 误差	阶数	L^2 误差	阶数	L^2 误差	阶数
20	3.8863e−2	—	3.8994e−2	—	3.9221e−2	—
40	4.6709e−3	—	4.6713e−3	—	4.6744e−3	—
60	1.3774e−3	3.06	1.3773e−3	3.06	1.3776e−3	3.07
80	5.8034e−4	3.01	5.8030e−4	3.01	5.8034e−4	3.01
120	1.7182e−4	3.00	1.7181e−4	3.00	1.7181e−4	3.00
160	7.2469e−5	3.00	7.1466e−5	3.00	7.2467e−5	3.00

6. 总结

针对一类四阶线性方程，将 LDG 方法和三阶隐显式 Runge-Kutta 时间离散相结合，数值流通量采用广义交通数值流通量，从而得到全离散 LDG 格式，分析了该格式的稳定性和最优误差估计。通过选择不同数值流通量进行数值实验，表明该方法在 P^0, P^1, P^2 有限元空间上的精度与理论结果保持一致。

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