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有界区间上一维非线性方程组的粘性消失极限

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摘　要

对于定义在有界区域上的一维非线性粘性抛物方程组，本文主要探讨其与相应无粘方程组之间解的渐近等价性。具体而言，首先利用匹配渐近展开的方法构建粘性守恒律方程组的近似解，再借助能量估计的方法进行稳定性分析，证明在远离边界层的区域中，抛物方程组的解一致收敛于相应无粘方程组的解。

关键词

边界层，匹配渐近展开，能量估计

The Vanishing Viscosity Limit for a One-Dimensional Nonlinear System of Equations on a Bounded Interval

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Abstract

For a system of one-dimensional nonlinear viscous parabolic equations defined on a bounded domain, this paper primarily investigates the asymptotic equivalence between its solutions and those of the corresponding inviscid system. Specifically, the method of matched asymptotic expansions is first employed to construct approximate solutions for the system of viscous conservation laws; subsequently, energy estimates are utilized to conduct a stability analysis, thereby demonstrating that, in regions away from the boundary layers, the solutions of the parabolic system converge uniformly to the solutions of the corresponding inviscid system.

Keywords

Boundary Layer, Matched Asymptotic Expansions, Energy Estimates

1. 前言

1.1. 引言

在粘性流体力学中，探究粘性抛物方程组与相应无粘双曲方程组解之间的渐近等价性（即粘性消失极限）始终是一个核心课题。为了透彻理解边界层效应与激波微观构造等物理现象，探明流体方程的耗散极限尤为关键，即在耗散系数 $\varepsilon \to 0^+$ 的前提下，对应于粘性流体运动的抛物型方程解 u^ε 是如何实现向无粘理想流态的双曲型方程组的解 u^0 逼近的。

关于粘性消失极限的研究已有丰富成果。例如，经典的可压缩 Navier-Stokes 方程与 Euler 方程之间的极限关系被广泛研究 [1–7]。对于粘性矩阵为单位阵的情形， K. T. Joserph [8] 在有界区间上证明了粘性解一致趋近于无粘解。 J. Goodman 和 Z. P. Xin [9] 在一维拟线性方程组中，严格证明了在存在满足熵条件的激波时，粘性解在 L^∞ 意义下收敛到无粘解。关于激波存在时的渐近等价性，有大量深入研究 [10–17]。进一步地， J. Wang [18] 研究了当粘性矩阵为更一般的正定阵时单个边界层的稳定性。

在 Goodman 与 Xin [9] 工作的启发下，本文考虑如下定义在有界区间 $x \in (0,1)$ 上的一维非线性粘性守恒律方程组的初边值问题：

$$\partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = \varepsilon \partial_x (B(u^\varepsilon)\partial_x u^\varepsilon), \quad t \in [0,T], \tag{1}$$

$$u^\varepsilon(0,t) = u_l(t), \quad u^\varepsilon(1,t) = u_r(t), \tag{2}$$

$$u^\varepsilon(x,0) = u_0^\varepsilon(x), \tag{3}$$

及其对应的无粘双曲方程组初值问题：

$$\partial_t u + \partial_x f(u) = 0, \quad t \in [0,T], \tag{4}$$

$$u(x,0) = u_0^0(x). \tag{5}$$

其中 $u \in \mathbb{R}^n$, f 光滑，且存在 g 使得 $\nabla_u g = B$。本文研究在两个不相互作用的弱边界层共存的情况下，方程组 (1)-(3) 的解与方程组 (4)-(5) 的解之间的渐近等价性。

证明分为两部分：在第二章，利用匹配渐近展开法构造近似解 v^ε，通过中心流形定理 [19] 分析边界层主导项性质，并用截断函数拼接不同区域的展开。第三章的核心工作是利用能量法对稳定性进行分析，得到扰动误差的 H^1 估计，从而证明在远离边界层的区域，粘性方程的解在 L^∞ 意义下收敛到无粘解。

为保证问题的适定性，我们引入以下结构性假设 (I)-(IV)。

(I) 设 u_0^ε 是光滑函数，满足

$$\left\| u_0^\varepsilon - m_1 \sum_{i=0}^{2} \varepsilon^i u_b^i\left(\frac{x}{\varepsilon},0\right) - m_2 \sum_{i=0}^{2} \varepsilon^i \bar{u}_b^i\left(\frac{x-1}{\varepsilon},0\right) - (1-m_1-m_2)\sum_{i=0}^{2}\varepsilon^i u_0^i(x) \right\|_{H^2(0,1)} \le C\varepsilon^2,$$

其中 $u_b^i\left(\frac{x}{\varepsilon},0\right), \bar{u}_b^i\left(\frac{x-1}{\varepsilon},0\right), u_0^i(x) \in H^2(0,1)$, $i = 0,1,2$，并且 $m_1 = m\left(\frac{x}{\varepsilon^\gamma}\right), m_2 = m\left(\frac{x-1}{\varepsilon^\gamma}\right)$，这里 $x \in (0,1), \gamma \in \left(\frac{17}{20},1\right)$。其中截断函数 $m(x) \in C_0^\infty(\mathbb{R})$，并满足 $0 \le m(x) \le 1$，

$$m(x) = \begin{cases} 1, & |x| \le 1, \\ h(x), & 1 < |x| < 2, \\ 0, & |x| \ge 2, \end{cases}$$

这里 $h(x)$ 是一个光滑函数；

(II) 设 $A(u) = f'(u)$，$A(u)$ 可对角化，即存在正定矩阵 $R(u)$ 和 $L(u)$ 使得

$$A = R\Lambda L, \quad RL = I,$$

其中 $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$，这里 $\lambda_i (i = 1,\ldots,n)$ 是矩阵 A 的 n 个不同的特征值，同时 LBR 是正定的；

(III) 存在 $\eta_1 > 0$，使得对任意 $x \in [0, \eta_1)$，有

$$\lambda_1(u_0^0) < \cdots < \lambda_n(u_0^0) < 0;$$

存在 $\eta_2 > 0$，使得对任意 $x \in (1-\eta_2, 1]$，有

$$0 < \lambda_1(u_0^0) < \cdots < \lambda_n(u_0^0);$$

(IV) 存在正定矩阵 $S(u)$，使得 SA 对称，SB 对称正定，即存在 $c_0 > 0$，使得对于任意 $\xi \in \mathcal{U}$，有

$$(SB\xi, \xi) \geq c_0|\xi|^2.$$

其中，$A(u) = Df(u)$ 可对角化，存在正定矩阵 $R(u), L(u)$ 使 $A = R\Lambda L$，且 LBR 正定。边界附近特征值符号条件确保边界层的存在性与稳定性。此外，存在正定矩阵 $S(u)$ 使得 SA 对称，SB 对称正定，这为能量估计提供了关键结构。

1.2. 模型与假设

为保证问题的适定性与收敛性证明的可行性，我们引入了结构性假设 (I)-(IV)。本小节旨在阐明这些关键假设的数学与物理意义、作用及其合理性。

假设 (I) 规定了初值 u_0^ε 需与构造的近似解在 H^2 范数下仅相差 $O(\varepsilon^2)$。这种设定是为了消除初始时刻可能产生的“初始层”扰动。通过要求初始条件与匹配渐近展开式在高阶项上保持一致，我们能够更专注于研究只由边界效应引起的边界层稳定性。

假设 (III) 是界定本文问题范围的核心边界条件。它要求在全特征边界 $x = 0$ 处所有特征值为负，而在 $x = 1$ 处所有特征值为正。在双曲系统理论中，这分别对应严格的“流入”边界和“流出”边界。这是许多物理场景（如管道流动、边界层理论）中的典型设定：在流入边界，外部条件主导，信息流入计算域；在流出边界，内部流场的信息自由传出，不受外部反射干扰。从数学角度看，此条件与假设 (II)（$A(u)$ 可对角化且 LBR 正定）相结合，确保了边界层方程 (18) 和 (44) 是适定的椭圆型边值问题，其解（u_b^0 和 $\bar{u}_b^0$）具有指数衰减性（引理 2.2, 2.3）。这一衰减性质是后续进行分区能量估计并控制边界项影响的理论基石。

最后，在稳定性分析中，我们将扰动误差设为 $\varepsilon^{5/8}\phi$。这一特定幂次的选取是基于近似解的余项 R^ε 的阶（$O(\varepsilon^{5\gamma/2})$，见 2.4 节）与通过能量估计所能得到的 ϕ 及其导数的衰减阶之间精细权衡的结果。具体而言，在 ϕ 的方程线性化后，此尺度使得非线性项、余项 R^ε 的扰动效应与线性主部提供的耗散效应达到一个平衡，从而能够应用 Gronwall 不等式闭合先验估计（引理 3.2-3.4）。这是一种在奇异摄动问题稳定性分析中常用的尺度匹配技巧。

1.3. 主要结果

本文主要结论如下：

定理 1.1. *假设初边值条件 (2)，(3) 满足任意阶相容性，且无粘解* $u^0 \in C^2([0,T]; H^6(0,1))$。*进一*

步假设边界数据与无粘解在边界值的偏差充分小，即存在 $\delta_0>0$ 使得

$$\sup_{0\le t\le T}|u_l(t)-u^0(0,t)|\le\delta_0,$$
$$\sup_{0\le t\le T}|u_r(t)-u^0(1,t)|\le\delta_0.$$

则存在 $\varepsilon_1, T>0$，使得对任意 $0<\varepsilon\le\varepsilon_1$，粘性初边值问题 *(1)*-*(3)* 存在唯一光滑解 $u^\varepsilon\in C^1([0,T];H^2(0,1))$。此外，存在构造的近似解 v^ε，使得以下估计成立：

$$\sup_{0\le t\le T}\|u^\varepsilon(\cdot,t)-v^\varepsilon(\cdot,t)\|_{L^\infty(0,1)}\le C\varepsilon, \tag{6}$$
$$\sup_{0\le t\le T}\|u^\varepsilon(\cdot,t)-u^0(\cdot,t)\|_{L^2(0,1)}\le C\varepsilon, \tag{7}$$
$$\sup_{\substack{0\le t\le T\\ h\le x\le 1-h}}|u^\varepsilon(x,t)-u^0(x,t)|\le C\varepsilon,\quad (h>0). \tag{8}$$

估计 *(7)* 和 *(8)* 表明，在 L^2 范数及远离边界层的内部区域一致范数下，粘性解 u^ε 以 $O(\varepsilon)$ 的速率收敛到无粘解 u^0。

2. 近似解的构造

2.1. 远离边界的展开

在远离边界 $\{x=0\}$ 和 $\{x=1\}$ 的区域中，方程组 (1) 的解可近似为

$$u^\varepsilon(x,t)\sim u^0(x,t)+\varepsilon u^1(x,t)+\varepsilon^2u^2(x,t)+\cdots. \tag{9}$$

将其代入 (1.1)，推导出各阶主导项为

$$O(1):\quad \partial_t u^0+\partial_x f\left(u^0\right)=0, \tag{10}$$
$$O(\varepsilon):\quad \partial_t u^1+\partial_x\left(f'\left(u^0\right)u^1\right)=\partial_x\left(B\left(u^0\right)\partial_x u^0\right), \tag{11}$$
$$O\left(\varepsilon^2\right):\quad \partial_t u^2+\partial_x\left(f'\left(u^0\right)u^2\right)=\partial_x^2\left(B\left(u^0\right)u^1\right)-\frac{1}{2}\partial_x\left(f''\left(u^0\right)\left(u^1\right)^2\right). \tag{12}$$

同时得到初始条件

$$u^0(x,t=0)=u_0^0(x), \tag{13}$$
$$u^1(x,t=0)=u_0^1(x), \tag{14}$$
$$u^2(x,t=0)=u_0^2(x), \tag{15}$$

基于上述推导，零阶主导项 u^0 恰为理想无粘双曲初值问题 (4) 与 (5) 的唯一解析解。该解具备如下正则性：

$$u^0(x,t)\in C^2\left([0,T];H^6(0,1)\right).$$

对于一阶修正 (11)，通过线性对称双曲方程的经典适定性理论，可得其存在唯一解 $u^1(x,t)$ 满足

$$u^1(x,t) \in C^2\left([0,T]; H^4(0,1)\right).$$

同理，方程组 (12) 存在唯一解 $u^2(x,t)$ 满足

$$u^2(x,t) \in C^2\left([0,T]; H^2(0,1)\right).$$

2.2. 边界 $\{x=0\}$ 附近的展开

在边界 $\{x=0\}$ 附近，粘性解 u^ε 可近似为

$$u^\varepsilon(x,t) \sim u_b^0(y,t) + \varepsilon u_b^1(y,t) + \varepsilon^2 u_b^2(y,t) + \cdots, \tag{16}$$

其中

$$y = \frac{x}{\varepsilon}. \tag{17}$$

将 (16) 代入方程组 (1) 得到

$$O\left(\frac{1}{\varepsilon}\right): \quad \partial_y f\left(u_b^0\right) - \partial_y^2 g\left(u_b^0\right) = 0, \tag{18}$$

$$O(1): \quad \partial_t u_b^0 + \partial_y\left(f'\left(u_b^0\right) u_b^1\right) = \partial_y^2\left(B\left(u_b^0\right) u_b^1\right), \tag{19}$$

$$\begin{aligned} O(\varepsilon): \quad & \partial_t u_b^1 + \partial_y\left(f'\left(u_b^0\right) u_b^2\right) + \frac{1}{2}\partial_y\left(f''\left(u_b^0\right)\left(u_b^1, u_b^1\right)\right) \\ & = \partial_y^2\left(B\left(u_b^0\right) u_b^2\right) + \frac{1}{2}\partial_y^2\left(B'\left(u_b^0\right)\left(u_b^1, u_b^1\right)\right). \end{aligned} \tag{20}$$

当 $y \to +\infty$ 时，得到如下匹配条件

$$u_b^0(y,t) = u^0(0,t) + o(1), \tag{21}$$

$$u_b^1(y,t) = u^1(0,t) + y\partial_x u^0(0,t) + o(1), \tag{22}$$

$$u_b^2(y,t) = u^2(0,t) + y\partial_x u^1(0,t) + \frac{1}{2}y^2\partial_x^2 u^0(0,t) + o(1). \tag{23}$$

同时，在边界 $\{y=0\}$ 处有

$$u_b^0(y=0,t) = u_l, \tag{24}$$

$$u_b^1(y=0,t) = 0, \tag{25}$$

$$u_b^2(y=0,t) = 0. \tag{26}$$

记 $\widetilde{u}_b^0(y,t)=u_b^0(y,t)-u^0(0,t)$，那么 $\widetilde{u}_b^0(y,t)$ 满足

$$\partial_y f\left(\widetilde{u}_b^0(y,t)+u^0(0,t)\right)=\partial_y^2 g\left(\widetilde{u}_b^0(y,t)+u^0(0,t)\right), \tag{27}$$

$$\widetilde{u}_b^0(y=0,t)=u, \tag{28}$$

$$\widetilde{u}_b^0(y\to+\infty,t)\to 0. \tag{29}$$

其中 $u=u_l-u^0(0,t)$。将 $\widetilde{u}_b^0(y,t)$ 满足的方程组关于空间变量在 $(y,+\infty)$ 上积分得到

$$\partial_y\widetilde{u}_b^0(y,t)=B^{-1}\left(u^0(0,t)\right)A\left(u^0(0,t)\right)\widetilde{u}_b^0(y,t)+O(1)\left(\widetilde{u}_b^0(y,t)\right)^2. \tag{30}$$

设 $\mathcal{C}=\{u\in R^n\mid(28)-(30)$ 有解 $\widetilde{u}_b^0\}$，则关于 $\mathcal{C}$ 有以下引理成立。

引理 2.1. *在假设 (I)-(IV) 成立的前提下，存在一个 0 的小邻域 $\mathcal{V}$，其中 $0\in R^n,\mathcal{V}\in\mathcal{U}$，使得*

(i) $\mathcal{C}$ 是 $\mathcal{V}$ 中经过 0 的光滑流形, $0\in\mathcal{V}$;

(ii) $T_0\mathcal{C}=N(0),N(0)$ 是 $B^{-1}A$ 负特征值对应特征向量生成的不变子空间;

(iii) $\mathcal{O}$ 为 0 的小邻域, 当 $\widetilde{u}_b^0\in\mathcal{O}$ 时, 存在常数 $C,\alpha>0$, 有

$$\left|\partial_u\partial_y\widetilde{u}_b^0\right|\le C\exp(-\alpha\xi).$$

引理 2.2. *边值问题 (18)， (21)，和 (24) 存在唯一光滑解 $u_b^0(y,t)$，并且存在 $\alpha_1,\delta_1>0$，使得*

$$\left|\partial_y u_b^0(y,t)\right|\le C\delta_1 e^{-\alpha_1 y},\forall y>0. \tag{31}$$

由引理 2.1 的结论，即得引理 2.2。

对于展开的第二项 $u_b^1(y,t)$ 满足

$$\partial_t u_b^0+\partial_y\left(f'\left(u_b^0\right)u_b^1\right)=\partial_y^2\left(B\left(u_b^0\right)u_b^1\right), \tag{32}$$

$$u_b^1(y=0,t)=0, \tag{33}$$

$$u_b^1(y\to+\infty,t)\to u^1(0,t)+y\partial_x u^0(0,t). \tag{34}$$

令 $u_b^1=V_1+D_1$，其中 $D_1(y,t)=u^1(0,t)+y\partial_x u^0(0,t)$，代入上述方程组有

$$\begin{aligned}&\partial_y^2\left[B\left(u_b^0(y,t)\right)V_1(y,t)\right]-\partial_y\left[f'\left(u_b^0\right)V_1(y,t)\right]\\=&\partial_y\left[f'\left(u_b^0\right)D_1(y,t)\right]-\partial_y^2\left[B\left(u_b^0(y,t)\right)D_1(y,t)\right]+\partial_t u_b^0(y,t).\end{aligned} \tag{35}$$

将上述方程组右边记为 $G(y,t)$，那么有 $|G(y,t)|\le Ce^{-\alpha_1 y}$。将上述方程组关于空间变量在 $(y,+\infty)$

上积分得到：

$$\partial_y \left[B\left(u_b^0(y,t)\right)V_1(y,t)\right]=f'\left(u_b^0\right)V_1(y,t)+\int_y^{+\infty}G(\tau,t)d\tau, \tag{36}$$

$$V_1(y=0,t)=0, \tag{37}$$

$$V_1(y\to+\infty,t)\to 0. \tag{38}$$

再令 $\widetilde{V}_1(y,t)=B\left(u_b^0(y,t)\right)V_1(y,t)$，得到：

$$\partial_y\widetilde{V}_1(y,t)=f'\left(u_b^0\right)B^{-1}\left(u_b^0(y,t)\right)\widetilde{V}_1(y,t)+\int_y^{+\infty}G(\tau,t)d\tau, \tag{39}$$

$$\widetilde{V}_1(y=0,t)=0, \tag{40}$$

$$\widetilde{V}_1(y\to+\infty,t)\to 0. \tag{41}$$

由引理 2.1，可得 $\widetilde{V}_1(y,t)$ 的存在性，即可以证明 $u_b^1(y,t)$ 的存在性。类似地可以证明 $u_b^2(y,t)$ 的存在性。

2.3. 边界 $\{x=1\}$ 附近的展开

类似地，在边界 $\{x=1\}$ 附近，将 u^ε 近似为

$$u^\varepsilon(x,t)\sim\bar{u}_b^0(\xi,t)+\varepsilon\bar{u}_b^1(\xi,t)+\varepsilon^2\bar{u}_b^2(\xi,t)+\varepsilon^3\bar{u}_b^3(\xi,t)+\cdots, \tag{42}$$

其中

$$\xi=\frac{x-1}{\varepsilon}. \tag{43}$$

将 (42) 代入方程组 (1) 得到

$$O\left(\frac{1}{\varepsilon}\right):\quad \partial_\xi f\left(\bar{u}_b^0\right)-\partial_\xi^2 g\left(\bar{u}_b^0\right)=0, \tag{44}$$

$$O(1):\quad \partial_t\bar{u}_b^0-\partial_\xi\left(f'\left(\bar{u}_b^0\right)u_b^1\right)=\partial_\xi^2\left(B\left(\bar{u}_b^0\right)\bar{u}_b^1\right), \tag{45}$$

$$O(\varepsilon):\quad \partial_t\bar{u}_b^1+\partial_\xi\left(f'\left(\bar{u}_b^0\right)\bar{u}_b^2\right)+\frac{1}{2}\partial_\xi\left(f''\left(\bar{u}_b^0\right)\left(\bar{u}_b^1,\bar{u}_b^1\right)\right) \tag{46}$$

$$=\partial_\xi^2\left(B\left(\bar{u}_b^0\right)\bar{u}_b^2\right)+\frac{1}{2}\partial_\xi^2\left(B'\left(\bar{u}_b^0\right)\left(\bar{u}_b^1,\bar{u}_b^1\right)\right). \tag{47}$$

同样地，在匹配区域中，当 $\xi\to-\infty$ 时，有如下匹配条件

$$\bar{u}_b^0(\xi,t)=u^0(1,t)+o(1), \tag{48}$$

$$\bar{u}_b^1(\xi,t)=u^1(1,t)+\xi\partial_x u^0(1,t)+o(1), \tag{49}$$

$$\bar{u}_b^2(\xi,t)=u^2(1,t)+\xi\partial_x u^1(1,t)+\frac{1}{2}\xi^2\partial_x^2u^0(1,t)+o(1). \tag{50}$$

同时在边界 $\{\xi = 0\}$ 处有

$$\bar{u}_b^0(\xi = 0, t) = u_r, \tag{51}$$

$$\bar{u}_b^1(\xi = 0, t) = 0, \tag{52}$$

$$\bar{u}_b^2(\xi = 0, t) = 0. \tag{53}$$

引理 2.3. 边值问题 *(44)*，*(48)*，和 *(51)* 存在唯一光滑解 $\bar{u}_b^0(\xi, t)$，并且存在 $\alpha_2, \delta_2 > 0$，使得

$$\left|\partial_\xi \bar{u}_b^0(\xi, t)\right| \le C\delta_2 e^{\alpha_2 \xi}, \forall \xi < 0. \tag{54}$$

证明. 已知 $\bar{u}_b^0$ 满足

$$\partial_\xi f(\bar{u}_b^0) = \partial_\xi^2 g(\bar{u}_b^0),$$

$$\bar{u}_b^0(\xi = 0, t) = u_r,$$

$$\bar{u}_b^0(\xi \to -\infty, t) \to u^0(1, t).$$

令 $s = -\xi$，设 $\hat{u}_b^0(s, t) = \bar{u}_b^0(\xi, t) - u^0(1, t)$，则 $\hat{u}_b^0(s, t)$ 满足

$$\partial_s f\left(\hat{u}_b^0(s, t) + u^0(1, t)\right) = -\partial_s^2 g\left(\hat{u}_b^0(s, t) + u^0(1, t)\right),$$

$$\hat{u}_b^0(s = 0, t) = u_r - u^0(1, t),$$

$$\hat{u}_b^0(s \to +\infty, t) \to 0.$$

注意到该问题类似于问题 (27)–(29)，同样的方法完成引理 2.3 的证明。同理可证 $\bar{u}_b^1$ 和 $\bar{u}_b^2$。 □

2.4. 近似解

我们先将远离边界以及边界附近的解依次定义为

$$O(x, t) = u^0(x, t) + \varepsilon u^1(x, t) + \varepsilon^2 u^2(x, t), \tag{55}$$

$$I_1(x, t) = u_b^0\left(\frac{x}{\varepsilon}, t\right) + \varepsilon u_b^1\left(\frac{x}{\varepsilon}, t\right) + \varepsilon^2 u_b^2\left(\frac{x}{\varepsilon}, t\right), \tag{56}$$

$$I_2(x, t) = \bar{u}_b^0\left(\frac{x-1}{\varepsilon}, t\right) + \varepsilon \bar{u}_b^1\left(\frac{x-1}{\varepsilon}, t\right) + \varepsilon^2 \bar{u}_b^2\left(\frac{x-1}{\varepsilon}, t\right), \tag{57}$$

其中 $u_i, u_b^i, \bar{u}_b^i (i = 0, 1, 2)$ 的构造已经在上一章节中给出。由此，定义方程组 (1) 的近似解为

$$v^\varepsilon(x, t) = m_1 I_1 + m_2 I_2 + (1 - m_1 - m_2)\, O. \tag{58}$$

结合粘性解满足的方程组 (1) 可以得到

$$\begin{aligned}
&\partial_t v^\varepsilon + \partial_x f\left(v^\varepsilon\right) - \varepsilon \partial_x^2 g\left(v^\varepsilon\right) = R^\varepsilon, && (59)\\
&v^\varepsilon(x=0,t) = u_l(t), && (60)\\
&v^\varepsilon(x=1,t) = u_r(t), && (61)\\
&v^\varepsilon|_{t=0} = m_1 \sum_{i=0}^{2} \varepsilon^i u_b^i\left(\frac{x}{\varepsilon},0\right) + m_2 \sum_{i=0}^{2} \varepsilon^i \bar{u}_b^i\left(\frac{x-1}{\varepsilon},0\right)\\
&\qquad + (1-m_1-m_2)\sum_{i=0}^{2}\varepsilon^i u_0^i(x). && (62)
\end{aligned}$$

即 $v^\varepsilon(x,t)$ 是以上初边值问题 (59)-(61) 的解，其中 $R^\varepsilon = \sum_{i=1}^{4} q_i(x,t)$ 是光滑函数，其定义如下

$$\begin{aligned}
q_1(x,t) = {}& (1-m_1-m_2)\Big\{ f(0) - f(u^0) - \varepsilon f'(u^0)u^1 - \varepsilon^2 f'(u^0)u^2\\
& - \frac{\varepsilon^2}{2} f''(u^0)(u^1,u^1) - \varepsilon^3 \Big[B(u^0)u_x^2 + B'(u^0)u^2(\varepsilon u_x^1 + \varepsilon^2 u_x^2)\\
& + \frac{1}{2}B''(u^0)(u^1+\varepsilon u^2)^2(u_x^0 + \varepsilon u_x^1 + \varepsilon^2 u_x^2)\Big]_x \Big\},\\
q_2(x,t) = {}& m_1\Big\{ \Big[f\left(I_1\right) - f\left(u_b^0\right) - \varepsilon f'\left(u_b^0\right)u_b^1 - \varepsilon^2 f'\left(u_b^0\right)u_b^2 - \frac{\varepsilon^2}{2} f''\left(u_b^0\right)\left(u_b^1,u_b^1\right)\Big]_x\\
& + \varepsilon^2 \partial_t u_b^2 - \varepsilon^4 \Big[B'\left(u_b^0\right)u_b^2\partial_x u_b^1 + B'\left(u_b^0\right)u_b^1\partial_x u_b^2 + \varepsilon B'\left(u_b^0\right)u_b^2\partial_x u_b^2\\
& + \frac{1}{2}B''\left(u^0\right)\left(2u_b^1u_b^2 + \varepsilon u_b^2\right)\left(\partial_x u_b^0 + \varepsilon\partial_x u_b^1 + \varepsilon^2\partial_x u_b^2\right)\Big]_x\Big\},\\
q_3(x,t) = {}& m_2\Big\{ \Big[f\left(I_2\right) - f\left(\bar{u}_b^0\right) - \varepsilon f'\left(\bar{u}_b^0\right)\bar{u}_b^1 - \varepsilon^2 f'\left(\bar{u}_b^0\right)\bar{u}_b^2 - \frac{\varepsilon^2}{2} f''\left(\bar{u}_b^0\right)\left(\bar{u}_b^1,\bar{u}_b^1\right)\Big]_x\\
& + \varepsilon^2 \partial_t \bar{u}_b^2 - \varepsilon^4 \Big[B'\left(\bar{u}_b^0\right)\bar{u}_b^2\partial_x \bar{u}_b^1 + B'\left(\bar{u}_b^0\right)\bar{u}_b^1\partial_x \bar{u}_b^2 + \varepsilon B'\left(\bar{u}_b^0\right)\bar{u}_b^2\partial_x \bar{u}_b^2\\
& + \frac{1}{2}B''\left(\bar{u}^0\right)\left(2\bar{u}_b^1\bar{u}_b^2 + \varepsilon \bar{u}_b^2\right)\left(\partial_x \bar{u}_b^0 + \varepsilon\partial_x \bar{u}_b^1 + \varepsilon^2\partial_x \bar{u}_b^2\right)\Big]_x\Big\},\\
q_4(x,t) = {}& \partial_x f\left(m_1I_1 + m_2I_2 + (1-m_1-m_2)O\right)\\
& - \left\{m_1 f\left(I_1\right) + m_2 f\left(I_2\right) + (1-m_1-m_2)f(O)\right\}_x + \varepsilon\left(B(O)O_x\right)_x\\
& + \partial_x m_1\left(f\left(I_1\right) - f(O)\right) + \partial_x m_2\left(f\left(I_2\right) - f(O)\right)\\
& - \varepsilon\left\{g\left(m_1I_1 + m_2I_2 + (1-m_1-m_2)O\right)\right\}_{xx}\\
& + \varepsilon\left\{m_1\left(B\left(I_1\right) - B(O)\right)\left(I_1 - O\right)_x + m_2\left(B\left(I_2\right) - B(O)\right)\left(I_2 - O\right)_x\right\}_x\\
& - \varepsilon\partial_x m_1\left(B\left(I_1\right) - B(O)\right)\left(I_1 - O\right)_x - \varepsilon\partial_x m_2\left(B\left(I_2\right) - B(O)\right)\left(I_2 - O\right)_x\\
& + \varepsilon\left\{m_1B(O)\left(I_1 - O\right)_x + m_2B(O)\left(I_2 - O\right)_x\right\}_x
\end{aligned}$$

$$
\begin{aligned}
&-\varepsilon\partial_x m_1 B(O)\left(I_1-O\right)_x-\varepsilon\partial_x m_2 B(O)\left(I_2-O\right)_x\\
&+\varepsilon\left\{m_1\left(B\left(I_1\right)-B(O)\right)O_x+m_2\left(B\left(I_2\right)-B(O)\right)O_x\right\}_x\\
&-\varepsilon\partial_x m_1\left(B\left(I_1\right)-B(O)\right)O_x-\varepsilon\partial_x m_2\left(B\left(I_2\right)-B(O)\right)O_x.
\end{aligned}
$$

根据 q_i 的结构，可以得到

$$
\begin{aligned}
\text{(i)}\quad & \operatorname{supp} q_1 \subseteq \left\{(x,t): \varepsilon^\gamma \le x \le 1-\varepsilon^\gamma, 0\le t\le T\right\},\\
& \partial_x^l q_1(x,t)=O(1)\varepsilon^{3-l\gamma}, l=0,1,2, \qquad (63)\\
& \left(\int_0^T \left\|\partial_x^l q_1(\cdot,t)\right\|^2 dt\right)^{\frac{1}{2}} = O(1)\varepsilon^{3-(l-1/2)\gamma}, l=0,1,2.\\
\text{(ii)}\quad & \operatorname{supp} q_2 \subseteq \left\{(x,t): 0\le x\le 2\varepsilon^\gamma, 0\le t\le T\right\},\\
& \partial_x^l q_2(x,t)=O(1)\varepsilon^{(2-l)\gamma}, l=0,1,2. \qquad (64)\\
\text{(iii)}\quad & \operatorname{supp} q_3 \subseteq \left\{(x,t): 1-2\varepsilon^\gamma\le x\le 1, 0\le t\le T\right\},\\
& \partial_x^l q_3(x,t)=O(1)\varepsilon^{(2-l)\gamma}, l=0,1,2. \qquad (65)\\
\text{(iv)}\quad & \operatorname{supp} q_4 \subseteq \left\{(x,t): \varepsilon^\gamma\le x\le 2\varepsilon^\gamma, 1-2\varepsilon^\gamma\le x\le 1-\varepsilon^\gamma, 0\le t\le T\right\},\\
& \partial_x^l q_4(x,t)=O(1)\varepsilon^{(3-l)\gamma}, l=0,1,2. \qquad (66)
\end{aligned}
$$

这里的计算过程中用到了如下结论

$$
\partial_x^l\left(I_1-O\right)=O(1)\varepsilon^{(3-l)\gamma}, \left\{(x,t): \varepsilon^\gamma\le x\le 2\varepsilon^\gamma, t\in[0,T]\right\}, l=0,1,2, \tag{67}
$$

$$
\partial_x^l\left(I_2-O\right)=O(1)\varepsilon^{(3-l)\gamma}, \left\{(x,t): 1-2\varepsilon^\gamma\le x\le 1-\varepsilon^\gamma, t\in[0,T]\right\}, l=0,1,2. \tag{68}
$$

这是由匹配条件 (21)-(23)，(48)-(50) 以及 $o(1)=\exp\{-\alpha_0\mid\xi\mid\}$ 得到的。对于 (59) 中的 $R^\varepsilon=\sum_{i=1}^4 q_i(x,t)$，有 $\|R^\varepsilon\|_{L^2(0,1)}^2=O(1)\varepsilon^{5\gamma}$ 以及 $\|\partial_t R^\varepsilon\|_{L^2(0,1)}^2=O(1)\varepsilon^{5\gamma}$。

引理 2.4. *对于 (2.49) 定义的近似解* $v^\varepsilon(x,t)$，*有*

$$
v^\varepsilon(x,t)=\begin{cases}
u_b^0(y,t)+O(1)\varepsilon^\gamma, & 0\le x\le\varepsilon^\gamma,\\
u^0(x,t)+O(1)\varepsilon, & \varepsilon^\gamma\le x\le 1-\varepsilon^\gamma,\\
\overline{u}_b^0(\xi,t)+O(1)\varepsilon^\gamma, & 1-\varepsilon^\gamma\le x\le 1.
\end{cases} \tag{69}
$$

证明. 通过近似解的构造过程可知

$$
v^\varepsilon(x,t)=\begin{cases}
I_1, & 0\le x\le\varepsilon^\gamma,\\
O+m_1\left(I_1-O\right), & \varepsilon^\gamma\le x\le 2\varepsilon^\gamma,\\
O, & 2\varepsilon^\gamma\le x\le 1-2\varepsilon^\gamma,\\
O+m_2\left(I_2-O\right), & 1-2\varepsilon^\gamma\le x\le 1-\varepsilon^\gamma,\\
I_2, & 1-\varepsilon^\gamma\le x\le 1.
\end{cases}
$$

当 $0\le x\le\varepsilon^\gamma$ 时，$I_1(x,t)=u_b^0(y,t)+O(1)\varepsilon^\gamma$，当 $\varepsilon^\gamma\le x\le 1-\varepsilon^\gamma$ 时，$O(x,t)=u^0(x,t)+O(1)\varepsilon$，当 $1-\varepsilon^\gamma\le x\le 1$ 时，$I_2(x,t)=\bar{u}_b^0(\xi,t)+O(1)\varepsilon^\gamma$，于是有 (2.60). □

3. 稳定性分析

在上一节中，我们构造出了方程 (1)-(3) 的近似解 $v^\varepsilon(x,t)$，接下来进行稳定性分析。

3.1. 误差方程组

假设 $u^\varepsilon(x,t)$ 是方程组 (1) 的真实解，令

$$u^\varepsilon(x,t)=v^\varepsilon(x,t)+\varepsilon^{5/8}\phi(x,t),\qquad x\in(0,1),\qquad t\in[0,T], \tag{70}$$

代入近似解满足的初边值问题 (49) 得到

$$\partial_t\phi+\varepsilon^{-5/8}\partial_x\left[f\left(u^\varepsilon\right)-f\left(v^\varepsilon\right)\right]-\varepsilon^{3/8}\partial_x^2\left[g\left(u^\varepsilon\right)-g\left(v^\varepsilon\right)\right]=\varepsilon^{-5/8}R^\varepsilon, \tag{71}$$

$$\phi(x=0,t)=0,\phi(x=1,t)=0, \tag{72}$$

$$\phi(x,t=0)=O\left(\varepsilon^2\right). \tag{73}$$

3.2. 能量估计

命题 3.1. *(先验估计) 对于初边值问题 (71)-(73)，存在 $\varepsilon_1>0$，使得 $\forall\varepsilon$，满足 $0\le\varepsilon\le\varepsilon_1$，(71)-(73) 有唯一解 $\phi(x,t)\in C^1\left([0,T];H^2(0,1)\right)$，并且满足*

$$\sup_{0\le t\le T}\|\phi(\cdot,t)\|_{L^\infty(0,1)}\le C\varepsilon^{5\gamma/2-1}, \tag{74}$$

其中 $\gamma\in\left(\frac{17}{20},1\right)$。

由命题 3.1，我们能够得到问题 (1)-(3) 的解 u^ε 与问题 (4)-(7) 的解 u^0 之间有如下估计，即

$$\sup_{0\le t\le T}\|u^\varepsilon(\cdot,t)-v^\varepsilon(\cdot,t)\|_{L^\infty(0,1)}\le C\varepsilon^{5\gamma/2-3/8}. \tag{75}$$

引理 3.2. *设 $\phi(x,t)\in C^1\left([0,T];H^2(0,1)\right)$ 是误差方程组初边值问题 (71)-(73) 的解，假设存在常数 $C>0$ 使得下式*

$$\sup_{0\le t\le T}\|\phi(x,t)\|_{L^\infty(0,1)}\le C, \tag{76}$$

成立，那么有

$$\sup_{0\le t\le T}\int_0^1\phi^2dx+C\varepsilon\int_0^T\int_0^1\left|\partial_x\phi\right|^2dxdt\le C\varepsilon^{5\gamma-5/4}. \tag{77}$$

证明. 将 (71) 式左右两边同时乘以 ϕ，在 $(0,1)$ 上积分，整理得到

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1 \phi^2 dx+\varepsilon^{-5/8}\int_0^1 \partial_x\left[f\left(u^\varepsilon\right)-f\left(v^\varepsilon\right)\right]\phi dx-\varepsilon^{3/8}\int_0^1 \partial_x^2\left[g\left(u^\varepsilon\right)-g\left(v^\varepsilon\right)\right]\phi dx\\ =&\varepsilon^{-5/8}\int_0^1 R^\varepsilon \phi dx.\end{aligned}\tag{78}$$

首先考虑 (78) 的第二项，有

$$\begin{aligned}&\varepsilon^{-5/8}\int_0^1 \partial_x\left[f\left(u^\varepsilon\right)-f\left(v^\varepsilon\right)\right]\phi dx\\ =&-\varepsilon^{-5/8}\int_0^1\left[f'\left(v^\varepsilon\right)\left(\varepsilon^{5/8}\phi\right)+\frac{1}{2}f''\left(v_\theta\right)\left(\varepsilon^{5/8}\phi\right)^2\right]\partial_x\phi dx\\ \le&-\frac{1}{2}\int_0^1 f'\left(v^\varepsilon\right)\partial_x\phi^2 dx+C\varepsilon^{5/8}\int_0^1\left|\phi\cdot\partial_x\phi\right|dx\\ \le&\frac{1}{2}\int_0^1\partial_x\partial_u f\left(v^\varepsilon\right)\phi^2 dx+C\varepsilon^{5/4}\int_0^1\left|\partial_x\phi\right|^2 dx+C\int_0^1\phi^2 dx,\end{aligned}$$

其中 v_θ 位于 v^ε 与 u^ε 之间。令 $J=\frac{1}{2}\int_0^1\partial_x\partial_u f\left(v^\varepsilon\right)\phi^2 dx$。分区域对 J 做能量估计得到

$$\begin{aligned}J=&\frac{1}{2}\int_0^1\partial_x\partial_u f\left(v^\varepsilon\right)\phi^2 dx\\ =&\frac{1}{2}\int_0^{\varepsilon^\gamma}\partial_x\partial_u f\left(v^\varepsilon\right)\phi^2 dx+\frac{1}{2}\int_{\varepsilon^\gamma}^{2\varepsilon^\gamma}\partial_x\partial_u f\left(v^\varepsilon\right)\phi^2 dx\\ &+\frac{1}{2}\int_{2\varepsilon^\gamma}^{1-2\varepsilon^\gamma}\partial_x\partial_u f\left(v^\varepsilon\right)\phi^2 dx+\frac{1}{2}\int_{1-2\varepsilon^\gamma}^{1-\varepsilon^\gamma}\partial_x\partial_u f\left(v^\varepsilon\right)\phi^2 dx\\ &+\frac{1}{2}\int_{1-\varepsilon^\gamma}^{1}\partial_x\partial_u f\left(v^\varepsilon\right)\phi^2 dx=\sum_{i=1}^{5}J_i.\end{aligned}$$

首先估计第一项 J_1，有

$$\begin{aligned}J_1=&\frac{1}{2}\int_0^{\varepsilon^\gamma}\partial_u^2 f\left(v^\varepsilon\right)\partial_x v^\varepsilon\phi^2 dx\\ =&\frac{1}{2\varepsilon}\int_0^{\varepsilon^\gamma}\partial_u^2 f\left(v^\varepsilon\right)\partial_y\left(u_b^0+\varepsilon u_b^1+\varepsilon^2 u_b^2\right)\phi^2 dx\\ \le&\frac{1}{2\varepsilon}\int_0^{\varepsilon^\gamma}\left|\partial_u^2 f\left(v^\varepsilon\right)\partial_y u_b^0\right|\phi^2 dx+C\int_0^1\phi^2 dx\\ \le&\frac{1}{\varepsilon}C\int_0^{\varepsilon^\gamma}\left|\partial_y u_b^0\right|\cdot\left(\int_0^x\partial_\eta\phi(\eta,t)\cdot 1d\eta\right)^2 dx+C\int_0^1\phi^2 dx\\ \le&\frac{1}{\varepsilon}C\int_0^{\varepsilon^\gamma}\left|\partial_y u_b^0\right|\cdot xdx\cdot\int_0^1\left|\partial_x\phi\right|^2 dx+C\int_0^1\phi^2 dx\\ =&\varepsilon C\int_0^{+\infty}\left|\partial_y u_b^0\right|\cdot ydy\cdot\int_0^1\left|\partial_x\phi\right|^2 dx+C\int_0^1\phi^2 dx\\ \le&\varepsilon\delta_1 C\int_0^{+\infty}y\cdot e^{-\alpha_1 y}dy\cdot\int_0^1\left|\partial_x\phi\right|^2 dx+C\int_0^1\phi^2 dx\end{aligned}$$

$$\leq C\delta_1\varepsilon\int_0^1 |\partial_x\phi|^2\,dx + C\int_0^1 \phi^2 dx,$$

这里利用了近似解 v^ε 的结构性质，以及引理 2.2。接下来利用在匹配区域中得出的结论：

$$\partial_x^l(I_1 - O) = O(1)\varepsilon^{(3-l)\gamma},\ l = 0,\ 1,\ 2,$$

得到 J_2 的估计：

$$\begin{aligned}
J_2 &= \frac{1}{2}\int_{\varepsilon^\gamma}^{2\varepsilon^\gamma} \partial_u^2 f(v^\varepsilon)\partial_x v^\varepsilon\phi^2 dx \\
&= \frac{1}{2}\int_{\varepsilon^\gamma}^{2\varepsilon^\gamma} \partial_u^2 f(v^\varepsilon)\partial_x(O + m_1(I_1 - O))\phi^2 dx \\
&\leq C\int_0^1 \phi^2 dx.
\end{aligned}$$

同理可得

$$\begin{aligned}
J_3 &= \frac{1}{2}\int_{2\varepsilon^\gamma}^{1-2\varepsilon^\gamma} \partial_x\partial_u f(v^\varepsilon)\phi^2 dx \leq C\int_0^1 \phi^2 dx, \\
J_4 &= \frac{1}{2}\int_{1-2\varepsilon^\gamma}^{1-\varepsilon^\gamma} \partial_x\partial_u f(v^\varepsilon)\phi^2 dx \leq C\int_0^1 \phi^2 dx.
\end{aligned}$$

关于 J 的最后一项 J_5 的估计，类似于前面对 J_1 的估计，有

$$\begin{aligned}
J_5 &= \frac{1}{2}\int_{1-\varepsilon^\gamma}^{1} \partial_u^2 f(v^\varepsilon)\partial_x v^\varepsilon\phi^2 dx \\
&= \frac{1}{2\varepsilon}\int_{1-\varepsilon^\gamma}^{1} \partial_u^2 f(v^\varepsilon)\partial_\xi(\overline{u}_b^0 + \varepsilon\overline{u}_b^0 + \varepsilon^2\overline{u}_b^0)\phi^2 dx \\
&\leq \frac{1}{\varepsilon}C\int_{1-\varepsilon^\gamma}^{1} \mid \partial_\xi(\overline{u}_b^0) \mid \phi^2 dx + C\int_0^1 \phi^2 dx \\
&\leq \frac{1}{\varepsilon}C\int_{1-\varepsilon^\gamma}^{1} \mid \partial_\xi\overline{u}_b^0 \mid \cdot \left(-\int_x^1 \partial_\eta\phi(\eta,t)\cdot 1d\eta\right)^2 dx + C\int_0^1 \phi^2 dx \\
&\leq \frac{1}{\varepsilon}C\int_{1-\varepsilon^\gamma}^{1} \mid \partial_\xi\overline{u}_b^0 \mid \cdot(1-x)dx \cdot \int_0^1 \mid \partial_x\phi \mid^2 dx + C\int_0^1 \phi^2 dx \\
&\leq C\varepsilon\delta_2\int_{-\infty}^{0} \xi\cdot e^{\alpha_2\xi}d\xi \cdot \int_0^1 \mid \partial_x\phi \mid^2 dx + C\int_0^1 \phi^2 dx \\
&\leq C\delta_2\varepsilon\int_0^1 \mid \partial_x\phi \mid^2 dx + C\int_0^1 \phi^2 dx,
\end{aligned}$$

这里利用了引理 2.3。接下来考虑 (78) 的第三项

$$-\varepsilon^{3/8}\int_0^1 \partial_x^2[g(u^\varepsilon) - g(v^\varepsilon)]\phi dx$$

$$
\begin{aligned}
&= \varepsilon^{3/8} \int_0^1 \partial_x[g'(\widetilde{v}_\theta)\varepsilon^{5/8}\phi)]\partial_x\phi dx \\
&= \varepsilon \int_0^1 \partial_u^2 g(\widetilde{v}_\theta)\partial_x v^\varepsilon \phi \partial_x \phi dx + \theta\varepsilon^{13/8} \int_0^1 \partial_u^2 g(\widetilde{v}_\theta)(\partial_x\phi)^2\phi dx + \varepsilon \int_0^1 \partial_u g(\widetilde{v}_\theta)(\partial_x\phi)^2 dx,
\end{aligned}
$$

其中 $\widetilde{v}_\theta$ 位于 v^ε 与 u^ε 之间，并且满足 $\widetilde{v}_\theta = v^\varepsilon + \theta\varepsilon^{5/8}\phi$。我们把它们分别记为 M_i $(i = 1, 2, 3)$。

先估计第一项

$$
\begin{aligned}
M_1 &= \varepsilon \int_0^1 \partial_u^2 g(\widetilde{v}_\theta)\partial_x v^\varepsilon \cdot \phi \cdot \partial_x\phi dx \\
&= \varepsilon \int_0^{\varepsilon^\gamma} \partial_u^2 g(\widetilde{v}_\theta)\partial_x v^\varepsilon \cdot \phi \cdot \partial_x\phi dx + \varepsilon \int_{\varepsilon^\gamma}^{2\varepsilon^\gamma} \partial_u^2 g(\widetilde{v}_\theta)\partial_x v^\varepsilon \cdot \phi \cdot \partial_x\phi dx \\
&\quad + \varepsilon \int_{2\varepsilon^\gamma}^{1-2\varepsilon^\gamma} \partial_u^2 g(\widetilde{v}_\theta)\partial_x v^\varepsilon \cdot \phi \cdot \partial_x\phi dx + \varepsilon \int_{1-2\varepsilon^\gamma}^{1-\varepsilon^\gamma} \partial_u^2 g(\widetilde{v}_\theta)\partial_x v^\varepsilon \cdot \phi \cdot \partial_x\phi dx \\
&\quad + \varepsilon \int_{1-\varepsilon^\gamma}^{1} \partial_u^2 g(\widetilde{v}_\theta)\partial_x v^\varepsilon \cdot \phi \cdot \partial_x\phi dx = \sum_{i=1}^{5} K_i,
\end{aligned}
$$

逐项估计有

$$
\begin{aligned}
K_1 &= \varepsilon \int_0^{\varepsilon^\gamma} \partial_u^2 g(\widetilde{v}_\theta)\partial_x(u_b^0 + \varepsilon u_b^1 + \varepsilon^2 u_b^2)\phi \cdot \partial_x\phi dx \\
&= \int_0^{\varepsilon^\gamma} \partial_u^2 g(\widetilde{v}_\theta)\partial_y u_b^0 \phi \cdot \partial_x\phi dx + C\varepsilon^2 \int_0^1 |\partial_x\phi|^2 dx + C\int_0^1 \phi^2 dx \\
&\le C\delta_3\varepsilon \int_0^{\varepsilon^\gamma} |\partial_x\phi|^2 dx + C\frac{1}{\varepsilon}\int_0^{\varepsilon^\gamma} |\partial_y u_b^0|^2 \left(\int_0^x \partial_\eta\phi(\eta,t)\cdot 1 d\eta\right)^2 dx \\
&\quad + C\varepsilon^2 \int_0^1 |\partial_x\phi|^2 dx + C\int_0^1 \phi^2 dx \\
&\le C\delta_3\varepsilon \int_0^{\varepsilon^\gamma} |\partial_x\phi|^2 dx + C\frac{1}{\varepsilon}\int_0^{\varepsilon^\gamma} |\partial_y u_b^0|^2 \cdot x dx \cdot \int_0^1 |\partial_x\phi|^2 dx \\
&\quad + C\varepsilon^2 \int_0^1 |\partial_x\phi|^2 dx + C\int_0^1 \phi^2 dx \\
&\le C(\delta_1 + \delta_3)\varepsilon \int_0^1 |\partial_x\phi|^2 dx + C\int_0^1 \phi^2 dx,
\end{aligned}
$$

这里先利用了 Cauchy 不等式和近似解的构造原理，并再次利用 $\partial_y u_b^0$ 满足的不等式 (31)。对于 K_2，有

$$
\begin{aligned}
K_2 &= \varepsilon \int_{\varepsilon^\gamma}^{2\varepsilon^\gamma} \partial_u^2 g(\widetilde{v}_\theta)\partial_x(O + m_1(I_1 - O)) \cdot \phi \cdot \partial_x\phi dx \\
&\le C\varepsilon^2 \int_0^1 |\partial_x\phi|^2 dx + C\int_0^1 \phi^2 dx,
\end{aligned}
$$

这里再次利用了近似解的构造原理以及 (54)。同理，再结合 (55)，得到 K_3 和 K_4 的估计如下：

$$K_3=\varepsilon\int_{2\varepsilon^\gamma}^{1-2\varepsilon^\gamma}\partial_u^2 g(\widetilde{v}_\theta)\partial_x v^\varepsilon\cdot\phi\cdot\partial_x\phi dx\le C\varepsilon^2\int_0^1|\partial_x\phi|^2dx+C\int_0^1\phi^2dx,$$
$$K_4=\varepsilon\int_{1-2\varepsilon^\gamma}^{1-\varepsilon^\gamma}\partial_u^2 g(\widetilde{v}_\theta)\partial_x v^\varepsilon\cdot\phi\cdot\partial_x\phi dx\le C\varepsilon^2\int_0^1|\partial_x\phi|^2dx+C\int_0^1\phi^2dx.$$

K_5 的估计和 K_1 的估计方法类似，得到

$$\begin{aligned}K_5&=\varepsilon\int_{1-\varepsilon^\gamma}^{1}\partial_u^2 g(\widetilde{v}_\theta)\partial_x(\overline{u}_b^0+\varepsilon\overline{u}_b^1+\varepsilon^2\overline{u}_b^2)\phi\cdot\partial_x\phi dx\\&=\int_{1-\varepsilon^\gamma}^{1}\partial_u^2 g(\widetilde{v}_\theta)\partial_\xi\overline{u}_b^0\phi\cdot\partial_x\phi dx+C\varepsilon^2\int_0^1|\partial_x\phi|^2dx+C\int_0^1\phi^2dx\\&\le C\delta_4\varepsilon\int_{1-\varepsilon^\gamma}^{1}|\partial_x\phi|^2dx+C\frac{1}{\varepsilon}\int_{1-\varepsilon^\gamma}^{1}|\partial_\xi\overline{u}_b^0|^2\cdot\left(-\int_x^1\partial_\eta\phi(\eta,t)\cdot 1d\eta\right)^2dx\\&\quad+C\varepsilon^2\int_0^1|\partial_x\phi|^2dx+C\int_0^1\phi^2dx\\&\le C\delta_4\varepsilon\int_{1-\varepsilon^\gamma}^{1}|\partial_x\phi|^2dx+C\frac{1}{\varepsilon}\int_{1-\varepsilon^\gamma}^{1}|\partial_\xi\overline{u}_b^0|^2(1-x)dx\cdot\int_0^1|\partial_x\phi|^2dx\\&\quad+C\varepsilon^2\int_0^1|\partial_x\phi|^2dx+C\int_0^1\phi^2dx\\&\le C(\delta_2+\delta_3)\varepsilon\int_0^1|\partial_x\phi|^2dx+C\int_0^1\phi^2dx.\end{aligned}$$

将上述 $K_i(i=1,\cdots,5)$ 的估计进行整理得到 M_1 的估计，以及通过简单的计算得到的 M_2 的估计如下：

$$M_1=\varepsilon\int_0^1\partial_u^2 g(\widetilde{v}_\theta)\partial_x v^\varepsilon\cdot\phi\cdot\partial_x\phi dx\le C(\delta_1+\delta_2+\delta_3)\varepsilon\int_0^1|\partial_x\phi|^2dx+C\int_0^1\phi^2dx,$$
$$M_2=\theta\varepsilon^{13/8}\int_0^1\partial_u^2 g(\widetilde{v}_\theta)(\partial_x\phi)^2\cdot\phi dx\le C\varepsilon^{13/8}\int_0^1|\partial_x\phi|^2dx.$$

对于 M_3，由矩阵 B 满足的条件假设可知，存在 $c_0>0$，有

$$M_3=\varepsilon\int_0^1\partial_u g(\widetilde{v}_\theta)(\partial_x\phi)^2dx\ge c_0\varepsilon\int_0^1|\partial_x\phi|^2dx.$$

接下来，对于 (78) 的最后一项，有

$$\begin{aligned}\varepsilon^{-5/8}\int_0^1 R^\varepsilon\phi dx&\le\varepsilon^{-5/4}\int_0^1|R^\varepsilon|^2dx+\int_0^1\phi^2dx\\&\le C\varepsilon^{5\gamma-5/4}+\int_0^1\phi^2dx.\end{aligned}$$

综合以上所有估计，整理得到

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \phi^2 dx + (c_0-\delta)\varepsilon\int_0^1 |\partial_x\phi|^2 dx \le C\int_0^1 \phi^2 dx + C\varepsilon^{5\gamma-5/4}.$$

其中 $\delta=\sum_{i=1}^4 \delta_i$，选取适当小的 $\delta_i,\ i=1,2,3,4$，使得 $c_0-\delta>0$，再利用 Gronwall 不等式，有

$$\sup_{0\le t\le T}\int_0^1 \phi^2 dx + C\varepsilon\int_0^T\int_0^1 |\partial_x\phi|^2 dxdt \le C\varepsilon^{5\gamma-5/4}.$$

□

引理 3.3. *设引理 3.1 的假设条件依然成立，则存在常数 $C>0$，有*

$$\sup_{0\le t\le T}\int_0^1 |\partial_t\phi|^2 dx + C\varepsilon\int_0^T\int_0^1 |\partial_t\partial_x\phi|^2 dxdt \le C\varepsilon^{5\gamma-9/4}. \tag{79}$$

证明. 设 $v(x,t)=\partial_t\phi(x,t)$，由 (71) 有

$$v(x,t)+\varepsilon^{-5/8}\partial_x\left[f\left(u^{\varepsilon}\right)-f\left(v^{\varepsilon}\right)\right]-\varepsilon^{3/8}\partial_x^2\left[g\left(u^{\varepsilon}\right)-g\left(v^{\varepsilon}\right)\right]=\varepsilon^{-5/8}R^{\varepsilon}. \tag{80}$$

对 (80) 两边关于 t 求导得到

$$\partial_t v(x,t)+\varepsilon^{-5/8}\partial_x\partial_t\left[f\left(u^{\varepsilon}\right)-f\left(v^{\varepsilon}\right)\right]-\varepsilon^{3/8}\partial_x^2\partial_t\left[g\left(u^{\varepsilon}\right)-g\left(v^{\varepsilon}\right)\right]=\varepsilon^{-5/8}\partial_t R^{\varepsilon}, \tag{81}$$

以及初边值条件

$$v(x=0,t)=0, \tag{82}$$

$$v(x=1,t)=0, \tag{83}$$

$$v(x,t=0)=O\left(\varepsilon^2\right). \tag{84}$$

对 (81) 两边同时乘以 $v(x,t)$，并在区间 (0,1) 上积分得到

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1 \phi^2 dx+\varepsilon^{-5/8}\int_0^1 \partial_x\left[f\left(u^{\varepsilon}\right)-f\left(v^{\varepsilon}\right)\right]\phi dx-\varepsilon^{3/8}\int_0^1 \partial_x^2\left[g\left(u^{\varepsilon}\right)-g\left(v^{\varepsilon}\right)\right]\phi dx\\&=\varepsilon^{-5/8}\int_0^1 R^{\varepsilon}\phi dx.\end{aligned} \tag{85}$$

首先考虑 (85) 的第二项，

$$\begin{aligned}&\varepsilon^{-5/8}\int_0^1 \partial_x\partial_t[f(u^{\varepsilon})-f(v^{\varepsilon})]v\,dx\\&=-\varepsilon^{-5/8}\int_0^1 \partial_t\left[f'(v^{\varepsilon})(\varepsilon^{5/8}\phi)+\frac{1}{2}f''(v_\theta)(\varepsilon^{5/8}\phi)^2\right]\partial_x v\,dx\\&=-\varepsilon^{5/8}\int_0^1 f''(v_\theta)\phi v\partial_x v\,dx-\frac{1}{2}\varepsilon^{5/8}\int_0^1 \partial_t f''(v_\theta)\phi^2\partial_x v\,dx\end{aligned}$$

$$-\int_0^1 f'(v^\varepsilon)v\partial_x v\,dx-\int_0^1 \partial_t f'(v^\varepsilon)\phi\partial_x v\,dx=\sum_{i=1}^4 Q_i.$$

其中 v_θ 位于 v^ε 与 u^ε 之间。对于 Q_1 和 Q_2，由假设 (76) 以及 Cauchy 不等式，有

$$\begin{aligned}
Q_1&=-\varepsilon^{5/8}\int_0^1 f''(v_\theta)\,\phi v\partial_x v dx\le C\varepsilon^{5/4}\int_0^1 |\partial_x v|^2\,dx+C\int_0^1 v^2dx.\\
Q_2&=-\frac{1}{2}\varepsilon^{5/8}\int_0^1 \partial_t f''(v_\theta)\,\phi^2\partial_x v dx\le C\varepsilon^{5/4}\int_0^1 |\partial_x v|^2\,dx+C\int_0^1 \phi^2dx.
\end{aligned}$$

对 Q_3 进行分部积分，由边界条件 (82) 和 (83) 有

$$\begin{aligned}
Q_3&=-\int_0^1 f'(v^\varepsilon)\,v\partial_x v dx=-\frac{1}{2}\int_0^1 f'(v^\varepsilon)\,\partial_x v^2dx=\frac{1}{2}\int_0^1 \partial_x f'(v^\varepsilon)\,v^2dx\\
&=\frac{1}{2}\int_0^{\varepsilon^\gamma}\partial_x\partial_u f(v^\varepsilon)v^2dx+\frac{1}{2}\int_{\varepsilon^\gamma}^{2\varepsilon^\gamma}\partial_x\partial_u f(v^\varepsilon)v^2dx+\frac{1}{2}\int_{2\varepsilon^\gamma}^{1-2\varepsilon^\gamma}\partial_x\partial_u f(v^\varepsilon)v^2dx\\
&\quad+\frac{1}{2}\int_{1-2\varepsilon^\gamma}^{1-\varepsilon^\gamma}\partial_x\partial_u f(v^\varepsilon)v^2dx+\frac{1}{2}\int_{1-\varepsilon^\gamma}^{1}\partial_x\partial_u f(v^\varepsilon)v^2dx=\sum_{i=1}^5 R_i.
\end{aligned}$$

逐项估计，有

$$\begin{aligned}
R_1&=\frac{1}{2}\int_0^{\varepsilon^\gamma}\partial_u^2 f(v^\varepsilon)\partial_x v^\varepsilon v^2dx\\
&=\frac{1}{2\varepsilon}\int_0^{\varepsilon^\gamma}\partial_u^2 f(v^\varepsilon)\partial_y(u_b^0+\varepsilon u_b^1+\varepsilon^2u_b^2)v^2dx\\
&\le\frac{1}{2\varepsilon}\int_0^{\varepsilon^\gamma}|\partial_u^2 f(v^\varepsilon)\partial_y u_b^0|v^2dx+C\int_0^1 v^2dx\\
&\le\frac{1}{\varepsilon}C\int_0^{\varepsilon^\gamma}|\partial_y u_b^0|\cdot\Big(\int_0^x\partial_\eta v(\eta,t)\cdot 1d\eta\Big)^2dx+C\int_0^1 v^2dx\\
&\le\frac{1}{\varepsilon}C\int_0^{\varepsilon^\gamma}|\partial_y u_b^0|\cdot xdx\cdot\int_0^1|\partial_x v|^2dx+C\int_0^1 v^2dx\\
&\le C\delta_1\varepsilon\int_0^1|\partial_x v|^2dx+C\int_0^1 v^2dx,
\end{aligned}$$

这里利用了 u_b^0 的指数衰减性。接下来通过简单计算得到 $R_i\,(1=2,\,3,\,4)$ 的估计如下，

$$\begin{aligned}
R_2&=\frac{1}{2}\int_{2\varepsilon^\gamma}^{1-2\varepsilon^\gamma}\partial_x\partial_u f(v^\varepsilon)v^2dx\le C\int_0^1 v^2dx,\\
R_3&=\frac{1}{2}\int_{2\varepsilon^\gamma}^{1-2\varepsilon^\gamma}\partial_x\partial_u f(v^\varepsilon)v^2dx\le C\int_0^1 v^2dx,\\
R_4&=\frac{1}{2}\int_{1-2\varepsilon^\gamma}^{1-\varepsilon^\gamma}\partial_x\partial_u f(v^\varepsilon)v^2dx\le C\int_0^1 v^2dx.
\end{aligned}$$

对于 R_5，类似于 R_1 的估计，有

$$
\begin{aligned}
R_5 &= \frac{1}{2}\int_{1-\varepsilon^\gamma}^{1} \partial_u^2 f(v^\varepsilon)\partial_x v^\varepsilon v^2 dx \\
&= \frac{1}{2\varepsilon}\int_{1-\varepsilon^\gamma}^{1} \partial_u^2 f(v^\varepsilon)\partial_\xi(\overline{u}_b^0 + \varepsilon\overline{u}_b^0 + \varepsilon^2\overline{u}_b^0)v^2 dx \\
&\le \frac{1}{\varepsilon}C\int_{1-\varepsilon^\gamma}^{1} |\partial_\xi(\overline{u}_b^0)|v^2 dx + C\int_0^1 v^2 dx \\
&\le \frac{1}{\varepsilon}C\int_{1-\varepsilon^\gamma}^{1} |\partial_\xi\overline{u}_b^0|\cdot(-\int_x^1 \partial_\eta v(\eta,t)\cdot 1 d\eta)^2 dx + C\int_0^1 v^2 dx \\
&\le \frac{1}{\varepsilon}C\int_{1-\varepsilon^\gamma}^{1} |\partial_\xi\overline{u}_b^0|\cdot(1-x)dx\cdot\int_0^1 |\partial_x v|^2 dx + C\int_0^1 v^2 dx \\
&\le C\delta_2\varepsilon\int_0^1 |\partial_x v|^2 dx + C\int_0^1 v^2 dx,
\end{aligned}
$$

这里利用了 $\overline{u}_b^0$ 的指数衰减性。将上述 $R_i(i=1,\cdots,5)$ 的估计进行整理，可以得到 Q_3 的估计如下，

$$
Q_3 = -\int_0^1 f'(v^\varepsilon)v\partial_x v dx \le \delta\varepsilon\int_0^1 |\partial_x v|^2 dx + C\int_0^1 v^2 dx.
$$

对于 Q_4，有

$$
\begin{aligned}
Q_4 = -\int_0^1 \partial_t f'(v^\varepsilon)\phi\partial_x v dx &= -\frac{1}{\sqrt{\varepsilon}}\sqrt{\varepsilon}\int_0^1 \partial_t f'(v^\varepsilon)\phi\partial_x v dx \\
&\le \delta\varepsilon\int_0^1 |\partial_x v|^2 dx + C\frac{1}{\varepsilon}\int_0^1 \phi^2 dx \\
&\le \delta\varepsilon\int_0^1 |\partial_x v|^2 dx + C\varepsilon^{-1}\cdot\varepsilon^{5\gamma-5/4} \\
&= \delta\varepsilon\int_0^1 |\partial_x v|^2 dx + C\varepsilon^{5\gamma-9/4},
\end{aligned}
$$

这里利用了 Cauchy 不等式和引理 3.1 的结论。

接下来考虑 (85) 的第三项

$$
\begin{aligned}
&-\varepsilon^{\frac{3}{8}}\int_0^1 \partial_x^2\partial_t[g(u^\varepsilon) - g(v^\varepsilon)]v dx = \varepsilon^{\frac{3}{8}}\int_0^1 \partial_x\partial_t\left[g'(\widetilde{v}_\theta)\left(\varepsilon^{\frac{5}{8}}\phi\right)\right]\partial_x v dx \\
=&\varepsilon\int_0^1 \partial_x g'(\widetilde{v}_\theta)v\partial_x v dx + \varepsilon\int_0^1 g'(\widetilde{v}_\theta)(\partial_x v)^2 dx \\
&+\varepsilon\int_0^1 \partial_x\partial_t g'(\widetilde{v}_\theta)\phi\partial_x v dx + \varepsilon\int_0^1 \partial_t g'(\widetilde{v}_\theta)\partial_x\phi\partial_x v dx = \sum_{i=1}^{4} I_i,
\end{aligned}
$$

其中 $\widetilde{v}_\theta$ 位于 v^ε 与 u^ε 之间，并且满足 $\widetilde{v}_\theta = v^\varepsilon + \theta\varepsilon^{5/8}\phi$。

$$\begin{aligned}I_1 &= \varepsilon \int_0^1 \partial_x g'\left(\widetilde{v}_\theta\right) v \partial_x v dx \\ &= \varepsilon \int_0^1 g''\left(\widetilde{v}_\theta\right) \partial_x v^\varepsilon v \partial_x v dx + \theta \varepsilon^{13/8} \int_0^1 g''\left(\widetilde{v}_\theta\right) \partial_x \phi v \partial_x v dx.\end{aligned}$$

首先利用估计 M_1 的方法来估计 I_1 的第一项，有

$$g''\left(\widetilde{v}_\theta\right) \partial_x v^\varepsilon v \partial_x v dx \le C \int_0^1 v^2 dx + C\delta\varepsilon \int_0^1 \left|\partial_x v\right|^2 dx.$$

对于 I_1 的第二项，有

$$\begin{aligned}\theta \varepsilon^{13/8} \int_0^1 g''\left(\widetilde{v}_\theta\right) \partial_x \phi v \partial_x v dx &\le C\varepsilon \cdot \varepsilon^{5/8} \int_0^1 \partial_x \phi \partial_x v dx \\ &\le C\varepsilon \int_0^1 \left|\partial_x \phi\right|^2 dx + C\varepsilon^2 \int_0^1 \left|\partial_x v\right|^2 dx.\end{aligned}$$

接下来，对于 I_2，由矩阵 B 满足的条件假设可知，存在 $c_0 > 0$，有

$$g'\left(\widetilde{v}_\theta\right) \left|\partial_x v\right|^2 dx \ge c_0 \varepsilon \int_0^1 \left|\partial_x v\right|^2 dx.$$

紧接着，对于 I_3，有

$$\begin{aligned}I_3 &= \varepsilon \int_0^1 \partial_u \partial_t g'(\widetilde{v}_\theta)\, \partial_x \widetilde{v}_\theta\, \phi\, \partial_x v dx \\ &= \varepsilon^{\frac{13}{8}} \theta \int_0^1 \partial_t \partial_u^2 g(\widetilde{v}_\theta)\, \partial_x \phi\, \phi\, \partial_x v dx + \varepsilon \int_0^1 \partial_t \partial_u^2 g(\widetilde{v}_\theta)\, \partial_x v^\varepsilon\, \phi\, \partial_x v dx,\end{aligned}$$

其中 $\widetilde{v}_\theta = v^\varepsilon + \theta \varepsilon^{5/8} \phi$。

对 I_3 的第一项作用 Cauchy 不等式得到

$$\begin{aligned}\varepsilon^{13/8} \theta \int_0^1 \partial_t \partial_u^2 g(\widetilde{v}_\theta)\, \partial_x \phi\, \phi\, \partial_x v dx &\le C\varepsilon \cdot \varepsilon^{5/8} \int_0^1 \partial_x \phi\, \partial_x v dx \\ &\le C\varepsilon \int_0^1 |\partial_x \phi|^2 dx + \varepsilon^2 \int_0^1 |\partial_x v|^2 dx,\end{aligned}$$

这里也利用了 ϕ 的有界性。接下来对 I_3 的第二项分区域进行估计有

$$\begin{aligned}\varepsilon \int_0^1 \partial_t \partial_u^2 g(\widetilde{v}_\theta)\, \partial_x v^\varepsilon\, \phi\, \partial_x v dx &\le C\varepsilon \int_0^1 \partial_x v^\varepsilon\, \phi\, \partial_x v dx \\ &= C\varepsilon \int_0^{\varepsilon^\gamma} \partial_x v^\varepsilon\, \phi\, \partial_x v dx + C\varepsilon \int_{\varepsilon^\gamma}^{2\varepsilon^\gamma} \partial_x v^\varepsilon\, \phi\, \partial_x v dx \\ &\quad + C\varepsilon \int_{2\varepsilon^\gamma}^{1-2\varepsilon^\gamma} \partial_x v^\varepsilon\, \phi\, \partial_x v dx + C\varepsilon \int_{1-2\varepsilon^\gamma}^{1-\varepsilon^\gamma} \partial_x v^\varepsilon\, \phi\, \partial_x v dx\end{aligned}$$

$$+C\varepsilon\int_{1-\varepsilon^\gamma}^{1}\partial_x v^\varepsilon\,\phi\,\partial_x v dx=\sum_{i=1}^{5}P_i.$$

将估计 M_1 的方法来估计 P_1，有

$$\begin{aligned}
P_1 &= C\varepsilon\int_0^{\varepsilon^\gamma}\partial_x v^\varepsilon\phi\partial_x v dx\\
&= C\varepsilon\int_0^{\varepsilon^\gamma}\partial_x(u_b^0+\varepsilon u_b^1+\varepsilon^2 u_b^2)\,\phi\,\partial_x v dx\\
&\le C\varepsilon\int_0^{\varepsilon^\gamma}\partial_x u_b^0\,\phi\,\partial_x v dx+C\varepsilon^2\int_0^1|\partial_x v|^2dx+C\int_0^1\phi^2dx\\
&\le \delta\varepsilon\int_0^1|\partial_x v|^2dx+C\varepsilon\int_0^1|\partial_x\phi|^2dx+C\varepsilon^{5\gamma-5/4}.
\end{aligned}$$

这里也利用了引理 3.1 的结论。对于 P_2 有

$$\begin{aligned}
P_2 &= C\varepsilon\int_{\varepsilon^\gamma}^{2\varepsilon^\gamma}\partial_x v^\varepsilon\,\phi\,\partial_x v dx\\
&= C\varepsilon\int_0^{\varepsilon^\gamma}\partial_x(O+m_1(I_1-O))\,\phi\,\partial_x v dx\\
&\le \varepsilon^2\int_0^1|\partial_x v|^2dx+C\int_0^1\phi^2dx\\
&\le \varepsilon^2\int_0^1|\partial_x v|^2dx+C\varepsilon^{5\gamma-5/4}.
\end{aligned}$$

同理，

$$\begin{aligned}
P_3 &= C\varepsilon\int_{2\varepsilon^\gamma}^{1-2\varepsilon^\gamma}\partial_x v^\varepsilon\,\phi\,\partial_x v dx\le\varepsilon^2\int_0^1|\partial_x v|^2dx+C\varepsilon^{5\gamma-5/4},\\
P_4 &= C\varepsilon\int_{1-2\varepsilon^\gamma}^{1-\varepsilon^\gamma}\partial_x v^\varepsilon\,\phi\,\partial_x v dx\le\varepsilon^2\int_0^1|\partial_x v|^2dx+C\varepsilon^{5\gamma-5/4}.
\end{aligned}$$

与 P_1 的估计方法类似，得到 P_5 的估计如下，

$$\begin{aligned}
P_5 &= C\varepsilon\int_{1-\varepsilon^\gamma}^{1}\partial_x v^\varepsilon\phi\partial_x v dx\\
&= C\varepsilon\int_{1-\varepsilon^\gamma}^{1}\partial_x(\overline{u}_b^0+\varepsilon\overline{u}_b^0+\varepsilon^2\overline{u}_b^0)\phi\partial_x v dx\\
&\le C\varepsilon\int_0^{\varepsilon^\gamma}\partial_x\overline{u}_b^0\phi\partial_x v dx+C\varepsilon^2\int_0^1|\partial_x v|^2dx+C\int_0^1\phi^2dx\\
&\le C\varepsilon\int_0^{\varepsilon^\gamma}\partial_x\overline{u}_b^0\phi\,\partial_x v dx+C\varepsilon^2\int_0^1|\partial_x v|^2dx+C\varepsilon^{5\gamma-5/4}\\
&\le \delta\varepsilon\int_0^1|\partial_x v|^2dx+C\varepsilon\int_0^1|\partial_x\phi|^2dx+C\varepsilon^{5\gamma-5/4}.
\end{aligned}$$

将上述 $P_i(i=1,\cdots,5)$ 的估计进行整理得到 I_3 的估计，即

$$\begin{aligned} I_3 &= \varepsilon\int_0^1 \partial_x\partial_t g'(\widetilde{v}_\theta)\,\phi\,\partial_x v dx \\ &\leq \delta\varepsilon\int_0^1 |\partial_x v|^2 dx + C\varepsilon\int_0^1 |\partial_x\phi|^2 dx + C\varepsilon^{5\gamma-5/4}. \end{aligned}$$

对于 I_4，通过简单计算有

$$\begin{aligned} I_4 &= \varepsilon\int_0^1 \partial_t g'(\widetilde{v}_\theta)\partial_x\phi\partial_x v dx \leq C\sqrt{\varepsilon}\cdot\sqrt{\varepsilon}\int_0^1 \partial_x\phi\partial_x v dx \\ &\qquad\qquad\qquad\qquad \leq C\varepsilon\int_0^1 |\partial_x\phi|^2 dx + \delta\varepsilon\int_0^1 |\partial_x v|^2 dx. \end{aligned}$$

对于 (85) 的最后一项，我们有

$$\begin{aligned} \varepsilon^{-5/8}\int_0^1 \partial_t R^\varepsilon v dx &\leq \varepsilon^{-\frac{5}{4}}\int_0^1 |\partial_t R^\varepsilon|^2 dx + \int_0^1 v^2 dx \\ &\leq \int_0^1 v^2 dx + C\varepsilon^{5\gamma-5/4}. \end{aligned}$$

综合以上所有估计，整理得到

$$\frac{d}{dt}\int_0^1 v^2 dx + (c_0-\delta)\varepsilon\int_0^1 |\partial_x v|^2 dx \leq C\varepsilon\int_0^1 |\partial_x\phi|^2 dx + \int_0^1 v^2 dx + C\varepsilon^{5\gamma-9/4}.$$

选取适当小的 δ，使得 $c_0-\delta>0$，再利用 Gronwall 不等式得到

$$\sup_{0\leq t\leq T}\int_0^1 |\partial_t\phi|^2 dx + C\varepsilon\int_0^T\int_0^1 |\partial_t\partial_x\phi|^2 dxdt \leq C\varepsilon^{5\gamma-9/4}.$$

□

引理 3.4. 设引理 *3.1* 的假设条件依然成立，对充分小的 $\varepsilon>0$，则存在常数 $C>0$，有

$$\sup_{0\leq t\leq T}\int_0^1 |\partial_x\phi|^2\, dx \leq C\varepsilon^{5\gamma-11/4}. \tag{86}$$

证明. 由 (77)，我们有

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \phi^2 dx + \varepsilon\int_0^T\int_0^1 |\partial_x\phi|^2 dxdt \leq C\varepsilon^{5\gamma-5/4},$$

可得

$$\varepsilon\int_0^1 |\partial_x\phi|^2 dx \leq -\frac{1}{2}\frac{d}{dt}\int_0^1 \phi^2 dx + C\varepsilon^{5\gamma-5/4}$$

$$
\begin{aligned}
&\le -\int_0^1 \partial_t\phi\cdot\phi dx + C\varepsilon^{5\gamma-5/4}\\
&\le C\|\partial_t\phi\|_{L^2(0,1)}\cdot\|\phi\|_{L^2(0,1)} + C\varepsilon^{5\gamma-5/4}\\
&\le C\varepsilon^{5\gamma/2-9/8}\cdot\varepsilon^{5\gamma/2-5/8} + C\varepsilon^{5\gamma-5/4}\\
&\le C\varepsilon^{5\gamma-7/4} + C\varepsilon^{5\gamma-5/4}\\
&\le C\varepsilon^{5\gamma-7/4},
\end{aligned}
$$

其中我们利用了 Hölder 不等式，以及引理 3.1 和引理 3.2 的结论。进而有

$$\int_0^1 |\partial_x\phi|^2\, dx \le C\varepsilon^{5\gamma-11/4}.$$

□

3.3. 命题 3.1 的证明

命题 3.1 的证明. 要证明 (71) 存在解 $\phi(x,t)$，实际上，通过 (77)， (86) 以及 Sobolev 不等式可得：

$$
\begin{aligned}
\sup_{0\le t\le T}\|\phi(\cdot,t)\|_{L^\infty(0,1)} &\le \sqrt{2}\sup_{0\le t\le T}\|\phi(\cdot,t)\|^{1/2}_{L^2(0,1)}\cdot\sup_{0\le t\le T}\|\partial_x\phi(\cdot,t)\|^{1/2}_{L^2(0,1)}\\
&\le C\varepsilon^{5\gamma/4-5/16}\cdot\varepsilon^{5\gamma/4-11/16}\\
&\le C\varepsilon^{5\gamma/2-1},
\end{aligned}
$$

其中 $\gamma\in\left(\frac{17}{20},1\right)$，从而有

$$
\begin{aligned}
\sup_{0\le t\le T}\|u^\varepsilon(\cdot,t)-v^\varepsilon(\cdot,t)\|_{L^\infty(0,1)} &\le \sup_{0\le t\le T}\left\|\varepsilon^{5/8}\phi(\cdot,t)\right\|_{L^\infty(0,1)}\\
&\le C\varepsilon^{5/8}\cdot\varepsilon^{5\gamma/2-1}\\
&\le C\varepsilon^{5\gamma/2-3/8}.
\end{aligned}
$$

所以 (75) 成立，即命题 1 得证。 □

3.4. 定理 1.1 的证明

定理 1.1 的证明. 选择合适的 $\gamma\in(\frac{17}{20},1)$ 再结合 (56) 和 (77) 以及三角不等式，得到 (7) 的证明如下：

$$
\begin{aligned}
\sup_{0\le t\le T}\|u^\varepsilon(\cdot,t)-u^0\|_{L^2(0,1)} &\le \sup_{0\le t\le T}\|u^\varepsilon(\cdot,t)-v^\varepsilon(\cdot,t)\|_{L^2(0,1)} + \sup_{0\le t\le T}\|v^\varepsilon(\cdot,t)-u^0\|_{L^2(0,1)}\\
&\le C\varepsilon^{5/8}\cdot\varepsilon^{5\gamma/2-5/8}\\
&\le C\varepsilon.
\end{aligned}
$$

接下来，利用 (56) 和 (75)，对 $h>0, \forall x\in(h,1-h)$, 有

$$\begin{aligned}\sup_{0\le t\le T}|u^{\varepsilon}(x,t)-u^{0}(x,t)| &\le C\varepsilon^{5\gamma/2-3/8}+C\varepsilon\\ &\le C\varepsilon.\end{aligned}$$

其中 $\gamma\in(\frac{17}{20},1)$，即 (8) 得证，则定理 1.1 得证。 □

4. 结论与讨论

本文研究了有界区间上一维非线性粘性守恒律方程组解的粘性消失极限问题。我们在两个不相互作用的弱边界层共存的设定下，通过匹配渐近展开法构造了高阶近似解，并利用基于熵结构的能量估计方法，严格证明了在远离边界的内部区域，粘性解 u^{ε} 以 $O(\varepsilon)$ 的速率一致收敛到相应的无粘解 u^{0}（定理 1.1）。这一结果将有界区域上单个方程 [8] 或单个边界层 [17] 的相关结论推广到了方程组及多边界层的情形。

在收敛阶方面，本文在内部区域获得的 L^{∞} 意义下的 $O(\varepsilon)$ 收敛率，与 Goodman 和 Xin [9] 在全空间上对具有激波的解所证明的收敛率是一致的。在边界层理论中，这一阶数通常被认为是最优的。原因在于，即使在最简单的线性模型下，边界层本身的修正量在 L^{∞} 范数下是 $O(1)$ 的，其影响范围厚度为 $O(\varepsilon)$。因此，在包含边界层的全局区域上，一致收敛阶无法达到 $O(\varepsilon)$。本文定理 1.1 的结论表明，尽管全局一致收敛被边界层阻挡，但在内部区域我们仍然能够恢复出与无界区域问题相同的最优收敛率，这清晰地分离了边界层效应与内部收敛行为。

本文工作仍存在一些局限性，这为未来的研究指明了方向。假设 (III) 要求严格的流入/流出边界条件，这在数学上显著简化了边界层分析。一个自然的推广是考虑更具挑战性的特征边界条件，或是混合边界条件（一端为特征边界，另一端为非特征边界）。此时，边界层方程的类型可能发生变化，分析将更为复杂。本文假设无粘解 u^{0} 是光滑的，未考虑其内部存在激波（间断）的情形。在实际物理问题中，无粘解通常包含激波。当粘性解在激波附近产生内部层（激波层），并且激波层可能与边界层共存甚至相互作用时，粘性消失极限的分析将变得极其复杂。这是该领域一个备受关注且尚未完全解决的公开问题。定理 1.1 中对无粘解 u^{0} 及初值的高阶 Sobolev 空间正则性要求（如 H^{6}），源于证明中高阶近似解的构造及繁杂的能量估计。探索能否降低这些正则性假设，或发展更精巧的估计方法以适应更低正则性的解，是一个有理论价值的课题。本文工作局限于空间一维情形。物理上更相关的高维空间中的粘性消失极限，特别是边界层与激波相互作用的二维或三维问题，其数学描述和分析和将面临更大的困难，是极具前景的研究方向。

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