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简化磁流体方程二阶向后差分伽辽金有限元方法

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摘 要

本文针对低磁雷诺数下的简化磁流体（**MHD**）方程，提出了一种高效的线性化二阶时步伽辽金有限元方法。该方法时间离散采用二阶向后差分格式（**BDF2**）保证时间精度，针对方程中的非线性对流项，设计了 **Adams-Bashforth** （**AB**）外推处理格式，将原非线性问题转化为线性求解模式，实现了每个时间步仅需求解一个线性方程组，避免了非线性迭代，大幅降低了计算复杂度。空间离散采用满足 **inf-sup** 条件的混合有限元方法，对速度、压力、电势进行协调逼近。本文严格证明了格式的无条件稳定性，推导得到了速度、压力、电势的最优二阶误差估计。

关键词

磁流体动力学，有限元方法，稳定性分析，误差分析

BDF2 Galerkin Finite Element Method for the Simplified Magnetohydrodynamic Flows

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Abstract

An efficient linearized second-order time-stepping Galerkin finite element method is developed for simplified magnetohydrodynamic (MHD) flows in the low magnetic Reynolds number regime. The method uses the second-order backward difference formula (BDF2) for temporal discretization and an Adams-Bashforth (AB) extrapolation technique for the nonlinear convective terms, which linearizes the problem and yields only one linear system to solve at each time step, thus eliminating nonlinear iterations and lowering computational cost substantially. A stable mixed finite element pair satisfying the inf-sup condition is used for the conforming spatial discretization of velocity, pressure and electric potential. The unconditional stability of the scheme is established, and optimal second-order error estimates for all three variables are obtained through rigorous mathematical analysis.

Keywords

Magnetohydrodynamics, Finite Element Method, Stability Analysis, Error Analysis

1. 引言

磁流体力学在工业与工程中有着重要应用, 简化MHD方程能有效描述低磁雷诺数下的导电流体运动. 目前针对这类方程的高效、稳定数值方法仍有待深入研究. 本文针对时变简化MHD方程, 构建易于计算的线性化数值格式, 在保证精度与稳定性的同时提升计算效率, 为相关流动问题的模拟提供实用方法.

磁雷诺数定义为$Re_m = \frac{UL}{\eta_m}$，其中U为特征速度、L为特征长度，$\eta_m = \frac{1}{\nu_{0\sigma}}$ 为磁扩散系数ν_0 为真空磁导率，σ为流体电导率。当$Re_m \ll 1$ 时，流体运动产生的感应磁场远小于外加恒定磁场，可忽略感应磁场对原磁场的反馈作用，计算复杂度大幅降低 [1]。

无量纲化简化磁流体方程具体描述如下：对于有界正域 $\Omega \subset \mathbb{R}^d$ (d=2或3), 给定时间 $T > 0$ 和

体积力 f, 求 $u:\Omega\times[0,T]\to\mathbb{R}^d$, $p:\Omega\times[0,T]\to\mathbb{R}$ 以及 $\phi:\Omega\times[0,T]\to\mathbb{R}$ 满足:

$$\begin{aligned}&N^{-1}(u_t+(u\cdot\nabla)u)=f+M^{-2}\Delta u-\nabla p+B\times\nabla\phi+(u\times B)\times B,\\&\nabla\cdot u=0,\\&-\triangle\phi+\nabla\cdot(u\times B)=0.\end{aligned}\tag{1.1}$$

其中B表示磁场, 假定其为已知量且满足 $\nabla\cdot B=0$, 此外, u代表流体速度, p为压力, ϕ 为电势. 除电势 ϕ 外, 电流密度j定义为 $j=-\nabla\phi+u\times B$. 这是 MHD 流动中另一个需要确定的重要电磁物理量. M 和 N 分别为哈特曼数与相互作用参数, 其定义为 $M=BL\sqrt{\frac{\sigma}{\rho\nu}}$, $N=\sigma B^2\frac{L}{\rho\nu}$ 其中 v , B , L 分别为特征速度,特征磁场与特征长度. 式中出现的其他参数分别为:密度 ρ , 运动粘性系数 ν , 以及电导率 σ , 均假定为常数. 该方程组满足如下初始边值条件: $u(x,t)=0,\ \forall(x,t)\in\partial\Omega\times[0,T]$; $\phi(x,t)=0,\ \forall(x,t)\in\partial\Omega\times[0,T]$; $u(x,0)=u_0(x),\ \forall x\in\Omega$. 本文提出线性化BDF2 伽辽金有限元方法, 采用BDF2 二阶时间离散保证时间精度 [2,3], 对非线性对流项采用显式AB2 格式线性化处理 [3], 每个时间步仅需求解一个线性方程组, 避免每一时间步进行非线性迭代 [1,4,5]。算法如下:

算法1.1. 已知 $u_h^{n-1},u_h^n\in X_h$, 求 $u_h^{n+1}\in X_h$, $p_h^{n+1}\in M_h$, $\phi_h^{n+1}\in S_h$ 使得对任意 $v_h\in X_h,q_h\in M_h,\psi_h\in S_h$ 分别满足

$$\begin{aligned}&N^{-1}\left(\frac{3u_h^{n+1}-4u_h^n+u_h^{n-1}}{2\Delta t},v_h\right)+N^{-1}b^*(2u_h^n-u_h^{n-1},u_h^{n+1},v_h)+M^{-2}(\nabla u_h^{n+1},\nabla v_h)\\&\quad-(p_h^{n+1},\nabla\cdot v_h)+(-\nabla\phi_h^{n+1}+u_h^{n+1}\times B^{n+1},v_h\times B^{n+1})=(f(t_{n+1}),v_h),\ \forall\, v_h\in X_h,\\&(\nabla\cdot u_h^{n+1},q_h)=0,\ \forall\, q_h\in M_h,\\&(-\nabla\phi_h^{n+1}+u_h^{n+1}\times B^{n+1},\nabla\psi_h)=0,\ \forall\,\psi_h\in S_h.\end{aligned}\tag{1.2}$$

2. 稳定性分析

算法 2.1的稳定性分析如下:

定理2.1. 假设 $f\in L^2(0,T;H^{-1}(\Omega))$, $u_0\in L^2(\Omega)$, 则数值格式是无条件稳定的. 对任意时间步长 Δt, 网格尺寸 h 以及 $K\geq1$, 该格式满足如下能量估计:

$$\begin{aligned}&\|u_h^K\|^2+\|2u_h^K-u_h^{K-1}\|^2+\sum_{n=1}^{K-1}\|u_h^{n+1}-2u_h^n+u_h^{n-1}\|^2\\&+2\Delta tNM^{-2}\sum_{n=1}^{K-1}\|\nabla u_h^{n+1}\|^2+4\Delta tN\sum_{n=1}^{K-1}\|j_h^{n+1}\|^2\\&\leq\|u_h^1\|^2+\|2u_h^1-u_h^0\|^2+2NM^2|||f|||_{2,-1}^2,\end{aligned}\tag{2.1}$$

并且 $\|\nabla\phi\|_{L^\infty(L^2)}^2\leq\bar{B}^2\big(\|u_h^1\|^2+\|2u_h^1-u_h^0\|^2+2NM^2|||f|||_{2,-1}^2\big)$, 其中 $\bar{B}=\|B\|_{L^\infty(L^\infty)}$.

证明. 在 (2.1) 的第一个方程中取 $v_h = u_h^{n+1}$,$a = u_h^{n+1}$, $b = u_h^n$, $c = u_h^{n-1}$ [6,7]可得

$$\begin{aligned}&\frac{N^{-1}}{4\Delta t}\left[\|u_h^{n+1}\|^2 + \|2u_h^{n+1} - u_h^n\|^2 - \|u_h^n\|^2 - \|2u_h^n - u_h^{n-1}\|^2 + \|u_h^{n+1} - 2u_h^n + u_h^{n-1}\|^2\right]\\&+ M^{-2}\|\nabla u_h^{n+1}\|^2 + (j_h^{n+1}, u_h^{n+1} \times B^{n+1}) = (f(t_{n+1}), u_h^{n+1}),\end{aligned} \tag{2.2}$$

其中 $j_h^{n+1} = -\nabla\phi_h^{n+1} + u_h^{n+1} \times B^{n+1}$. 在 (2.1) 的第三个方程中选取 $\psi_h = \phi_h^{n+1}$, 可得 $(j_h^{n+1}, -\nabla\phi_h^{n+1}) = 0$.

将式 (2.2) 与式 $(j_h^{n+1}, -\nabla\phi_h^{n+1}) = 0$.相加, 并对右端项运用 Cauchy-Schwarz 不等式与 Young's 不等式, 后将所得式两边同乘以 $4\Delta tN$, $n = 1$ to $K - 1$ 求和, 可得

$$\begin{aligned}&\|u_h^K\|^2 + \|2u_h^K - u_h^{K-1}\|^2 + \sum_{n=1}^{K-1}\|u_h^{n+1} - 2u_h^n + u_h^{n-1}\|^2\\&+ 2\Delta tNM^{-2}\sum_{n=1}^{K-1}\|\nabla u_h^{n+1}\|^2 + 4\Delta tN\sum_{n=1}^{K-1}\|j_h^{n+1}\|^2\\&\le \|u_h^1\|^2 + \|2u_h^1 - u_h^0\|^2 + 2\Delta tNM^2\sum_{n=1}^{K-1}\|f(t_{n+1})\|_{-1}^2,\end{aligned} \tag{2.3}$$

从而估计式 (2.1) 得证.

接下来, 在 (2.1) 的第三个方程中取 $\psi_h = \phi_h^{n+1}$, 应用 Cauchy-Schwarz 不等式, 可得

$$\|\nabla\phi_h^{n+1}\|^2 \le \frac{1}{2}\|u_h^{n+1} \times B^{n+1}\|^2 + \frac{1}{2}\|\nabla\phi_h^{n+1}\|^2. \tag{2.4}$$

于是, 得到

$$\|\nabla\phi_h^{n+1}\|^2 \le \|u_h^{n+1} \times B^{n+1}\|^2 \le \bar{B}^2\|u_h^{n+1}\|^2 \le \bar{B}^2\|u_h\|_{L^\infty(L^2)}^2. \tag{2.5}$$

综上所述, 证明完毕。 □

3. 误差分析

为建立算法 1.1 的最优渐近误差估计,我们假设问题真解满足如下正则性假设 [2,6] $u \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;H^{k+1}(\Omega)) \cap H^3(0,T;L^2(\Omega)) \cap H^2(0,T;H^1(\Omega)), p \in L^2(0,T;H^{s+1}(\Omega))$, $\phi \in H^1(0,T;H^{r+1}(\Omega))$. 将任意时间层 n 处的 $u(t_n)$, $\phi(t_n)$ 记为 u^n, ϕ^n. 并将有限元解与真解之间的误差记为 $e_u^n = u(t_n) - u_h^n$, 和 $e_\phi^n = \phi(t_n) - \phi_h^n$. 同时将误差分解为插值误差与逼近误差项, 即 $e_u^n = (u(t_n) - I_h^{V_h}u(t_n)) - (u_h^n - I_h^{V_h}u(t_n)) = \eta^n - \Gamma_h^n$, $e_\phi^n = (\phi(t_n) - I_h^{S_h}\phi(t_n)) - (\phi_h^n - I_h^{S_h}\phi(t_n)) = \xi^n - \Phi_h^n$. 其中 $I_h^{V_h}$ 和 $I_h^{S_h}$ 分别为映到空间 V_h 和 S_h, 上的插值算子 [5]. 同样的, 电流密度误差为 $e_j^n = -\nabla\xi^n + \eta^n \times B^{n+1} - (-\nabla\Phi_h^n + \Gamma^n \times B^{n+1}) = \chi^n - \mathcal{J}_h^n$.

定理3.1. 设 u,p,ϕ 满足弱形式，且 u_h^{n+1},ϕ_h^{n+1} 由算法 (2.1) 给出，则有如下误差估计:

$$
\begin{aligned}
&\|e_u^K\|^2+\|2e_u^K-e_u^{K-1}\|^2+\sum_{n=1}^{K-1}\|e_u^{n+1}-2e_u^n+e_u^{n-1}\|^2+2\Delta tM^{-2}N\|\nabla e_u^K\|^2\\
&+\Delta tM^{-2}N\|\nabla e_u^{K-1}\|^2+2\Delta tN\sum_{n=1}^{K-1}\|e_j^{n+1}\|^2+2\Delta tN\sum_{n=1}^{K-1}\|e_\phi^{n+1}\|^2\\
&\leq\exp\big(C(M^6N^{-3}+\bar{B}^2N)T\big)\Big[2M^{-2}\Delta tN\|\nabla e_u^1\|^2+M^{-2}\Delta tN\|\nabla e_u^0\|^2\\
&+\|e_u^1\|^2+\|2e_u^1-e_u^0\|^2+C\big(M^2N^{-1}h^{2k+2}\||u_t\||_{2,k+1}^2+M^{-2}Nh^{2k}\||u\||_{2,k+1}^2\\
&+Nh^{2r}\||\phi\||_{2,r+1}^2+\bar{B}^2h^{2k+2}\||u\||_{2,k+1}^2+M^2N^{-1}\|\nabla u\|_{\infty,0}^2h^{2k}\||u\||_{2,k+1}^2\\
&+M^2N^{-1}\Delta t^4\||\nabla u_{tt}\||_{2,0}^2+M^2Nh^{2s+2}\||p\||_{2,s+1}^2\\
&+M^2N^{-1}h^{2k}\||u\||_{2,k+1}^2+M^2N\Delta t^4\||u_{ttt}\||_{2,0}^2\big)\Big].
\end{aligned}
\tag{3.1}
$$

证明. 将连续变分形式在 $t=t_{n+1}$ 处重新写出. 对任意 $v_h\in V_h$, $\psi_h\in S_h$, 通过变形可得

$$
\begin{aligned}
&N^{-1}\left(\frac{3u^{n+1}-4u^n+u^{n-1}}{2\Delta t},v_h\right)+N^{-1}b^*(u^{n+1},u^{n+1},v_h)+M^{-2}(\nabla u^{n+1},\nabla v_h)\\
&-(p^{n+1},\nabla\cdot v_h)+(-\nabla\phi^{n+1}+u^{n+1}\times B^{n+1},v_h\times B^{n+1})\\
&=(f(t_{n+1}),v_h)+Intp(u^{n+1};v_h),\\
&(-\nabla\phi^{n+1}+u^{n+1}\times B^{n+1},\nabla\psi_h)=0,
\end{aligned}
\tag{3.2}
$$

其中 $Intp(u^{n+1},v_h)=\left(\frac{3u^{n+1}-4u^n+u^{n-1}}{2\Delta t}-u_t(t_{n+1}),v_h\right)$.

在 (2.1) 中选取 $v_h\in V_h,\psi_h\in S_h$ 消去压力项, 再用式 (3.2) 减去式 (2.1), 可得:

$$
\begin{aligned}
&N^{-1}\left(\frac{3e_u^{n+1}-4e_u^n+e_u^{n-1}}{2\Delta t},v_h\right)+N^{-1}\big[b^*(u^{n+1},u^{n+1},v_h)\\
&-b^*(2u_h^n-u_h^{n-1},u_h^{n+1},v_h)\big]+M^{-2}(\nabla e_u^{n+1},\nabla v_h)-(p^{n+1}-q_h,\nabla\cdot v_h)\\
&+(-\nabla e_\phi^{n+1}+e_u^{n+1}\times B^{n+1},v_h\times B^{n+1})=Intp(u^{n+1};v_h),\\
&(-\nabla e_\phi^{n+1}+e_u^{n+1}\times B^{n+1},\nabla\psi_h)=0
\end{aligned}
\tag{3.3}
$$

对任意的 $q_h\in M_h$ 成立. 注意到利用误差分解式, 式 (3.3) 可化为

$$
\begin{aligned}
&N^{-1}\left(\frac{3\Gamma_h^{n+1}-4\Gamma_h^n+\Gamma_h^{n-1}}{2\Delta t},v_h\right)+M^{-2}(\nabla\Gamma_h^{n+1},\nabla v_h)+(\mathcal{J}_h^{n+1},v_h\times B^{n+1})\\
&=N^{-1}\left(\frac{3\eta^{n+1}-4\eta^n+\eta^{n-1}}{2\Delta t},v_h\right)+M^{-2}(\nabla\eta^{n+1},\nabla v_h)+(\chi^{n+1},v_h\times B^{n+1})\\
&+N^{-1}\left[b^*(u^{n+1},u^{n+1},v_h)-b^*(2u_h^n-u_h^{n-1},u^{n+1},v_h)\right]-(p^{n+1}-q_h,\nabla\cdot v_h)\\
&+(-\nabla\xi^n+\eta^n\times B^{n+1},v_h\times B^{n+1})-Intp(u^{n+1};v_h),
\end{aligned}
\tag{3.4}
$$

$$
(-\nabla\Phi_h^{n+1}+\Gamma_h^{n+1}\times B^{n+1},\nabla\psi_h)=(-\nabla\xi^{n+1}+\eta^{n+1}\times B^{n+1},\nabla\psi_h). \tag{3.5}
$$

在3.5中取 $\psi_h=\Phi_h^{n+1}$, 可得 $\|\nabla\Phi_h^{n+1}\|^2=(\Gamma_h^{n+1}\times B^{n+1}-\chi^{n+1},\nabla\Phi_h^{n+1})$.

在 (3.4) 中取 $v_h=\Gamma_h^{n+1}$ 消去压力项, 再用 (3.5) 减去 (3.4), 整理可得:

$$
\begin{aligned}
&\frac{N^{-1}}{4\Delta t}\big[\|\Gamma_h^{n+1}\|^2+\|2\Gamma_h^{n+1}-\Gamma_h^n\|^2-\|\Gamma_h^n\|^2-\|2\Gamma_h^n-\Gamma_h^{n-1}\|^2\\
&+\|\Gamma_h^{n+1}-2\Gamma_h^n+\Gamma_h^{n-1}\|^2\big]+M^{-2}\|\nabla\Gamma_h^{n+1}\|^2+\|\mathcal{J}_h^{n+1}\|^2+\|\nabla\Phi_h^{n+1}\|^2\\
&=N^{-1}\left(\frac{3\eta^{n+1}-4\eta^n+\eta^{n-1}}{2\Delta t},\Gamma_h^{n+1}\right)+M^{-2}(\nabla\eta^{n+1},\nabla\Gamma_h^{n+1})+(\chi^{n+1},\mathcal{J}_h^{n+1})\\
&+N^{-1}\left[b^*(u^{n+1},u^{n+1},\Gamma_h^{n+1})-b^*(2u_h^n-u_h^{n-1},u^{n+1},\Gamma_h^{n+1})\right]\\
&-(p^{n+1}-q_h,\nabla\cdot\Gamma_h^{n+1})+(\Gamma_h^{n+1}\times B^{n+1}-\chi^{n+1},\nabla\Phi_h^{n+1})-Intp(u^{n+1};\Gamma_h^{n+1}).
\end{aligned}
\tag{3.6}
$$

对式 (3.6) 右端的剩余项逐项进行估计.

$$
\begin{aligned}
N^{-1}\left(\frac{3\eta^{n+1}-4\eta^n+\eta^{n-1}}{2\Delta t},\Gamma_h^{n+1}\right)&\le\frac{M^{-2}}{18}\|\nabla\Gamma_h^{n+1}\|^2+\frac{CM^2N^{-2}}{\Delta t}\int_{t_{n-1}}^{t_{n+1}}\|\eta_t\|^2dt\\
M^{-2}(\nabla\eta^{n+1},\nabla\Gamma_h^{n+1})&\le\frac{M^{-2}}{18}\|\nabla\Gamma_h^{n+1}\|^2+CM^{-2}\|\nabla\eta^{n+1}\|^2,\\
(\chi^{n+1},\mathcal{J}_h^{n+1})&\le\frac{1}{2}\|\mathcal{J}_h^{n+1}\|^2+\frac{1}{2}\|\chi^{n+1}\|^2.
\end{aligned}
\tag{3.7}
$$

对于式 (3.6) 中的非线性项, 通过适当加减辅助项, 有

$$
\begin{aligned}
&N^{-1}\left[b^*(u^{n+1},u^{n+1},\Gamma_h^{n+1})-b^*(2u_h^n-u_h^{n-1},u_h^{n+1},\Gamma_h^{n+1})\right]\\
&=N^{-1}b^*(u^{n+1},\eta^{n+1},\Gamma_h^{n+1})+N^{-1}b^*(u^{n+1}-2u^n+u^{n-1},u_h^{n+1},\Gamma_h^{n+1})\\
&+N^{-1}b^*(2\eta^n-\eta^{n-1},u_h^{n+1},\Gamma_h^{n+1})-2N^{-1}b^*(\Gamma_h^n,u_h^{n+1},\Gamma_h^{n+1})\\
&+N^{-1}b^*(\Gamma_h^{n-1},u_h^{n+1},\Gamma_h^{n+1}).
\end{aligned}
\tag{3.8}
$$

对上述非线性项进行如下估计:

$$N^{-1}b^*(u^{n+1},\eta^{n+1},\Gamma_h^{n+1})$$

$$\le \frac{M^{-2}}{18}\|\nabla\Gamma_h^{n+1}\|^2+CM^2N^{-2}\|\nabla u^{n+1}\|^2\|\nabla\eta^{n+1}\|^2.$$

$$N^{-1}b^*(u^{n+1}-2u^n+u^{n-1},u_h^{n+1},\Gamma_h^{n+1})$$

$$\le \frac{M^{-2}}{18}\|\nabla\Gamma_h^{n+1}\|^2+CM^2N^{-2}\Delta t^3\left(\int_{t_{n-1}}^{t_{n+1}}\|\nabla u_{tt}\|^2dt\right)\|\nabla u_h^{n+1}\|^2.$$

$$N^{-1}b^*(2\eta^n-\eta^{n-1},u_h^{n+1},\Gamma_h^{n+1}) \tag{3.9}$$

$$\le \frac{M^{-2}}{18}\|\nabla\Gamma_h^{n+1}\|^2+CM^2N^{-2}(\|\nabla\eta^n\|^2+\|\nabla\eta^{n-1}\|^2)\|\nabla u_h^{n+1}\|^2.$$

$$2N^{-1}b^*(\Gamma_h^n,u_h^{n+1},\Gamma_h^{n+1})$$

$$\le \frac{\epsilon}{2}\|\nabla\Gamma_h^{n+1}\|^2+\frac{N^{-2}}{2\epsilon}(\frac{\sigma}{2}\|\nabla\Gamma_h^n\|^2+\frac{C}{2\sigma}\|\nabla u_h^{n+1}\|^4\|\Gamma_h^n\|^2)$$

$$\le \frac{M^{-2}}{18}\|\nabla\Gamma_h^{n+1}\|^2+\frac{M^{-2}}{4}\|\nabla\Gamma_h^n\|^2+CM^6N^{-4}\|\nabla u_h^{n+1}\|^4\|\Gamma_h^n\|^2.$$

采用与上述相同的方法, 我们可得

$$N^{-1}b^*(\Gamma_h^{n-1},u_h^{n+1},\Gamma_h^{n+1})$$

$$\le \frac{M^{-2}}{18}\|\nabla\Gamma_h^{n+1}\|^2+\frac{M^{-2}}{4}\|\nabla\Gamma_h^{n-1}\|^2+CM^6N^{-4}\|\nabla u_h^{n+1}\|^4\|\Gamma_h^{n-1}\|^2.$$

$$(p^{n+1}-q_h,\nabla\cdot\Gamma_h^{n+1})$$

$$\le \frac{M^{-2}}{18}\|\nabla\Gamma_h^{n+1}\|^2+CM^2d\|p^{n+1}-q_h\|^2.$$

$$(\Gamma_h^{n+1}\times B^{n+1}-\chi^{n+1},\nabla\Phi_h^{n+1}) \tag{3.10}$$

$$\le \frac{1}{2}\|\nabla\Phi_h^{n+1}\|^2+\bar{B}^2\|\Gamma_h^{n+1}\|^2+\|\chi^{n+1}\|^2.$$

$$Intp(u^{n+1};\Gamma_h^{n+1})$$

$$=\left(\frac{3u^{n+1}-4u^n+u^{n-1}}{2\Delta t}-u_t(t_{n+1}),\Gamma_h^{n+1}\right)$$

$$\le \frac{M^{-2}}{18}\|\nabla\Gamma_h^{n+1}\|^2+CM^2\Delta t^3\int_{t_{n-1}}^{t_{n+1}}\|u_{ttt}\|^2dt.$$

将上述估计代入原方程, 把得到的结果两边同乘以 $4N\Delta t$, 并对 $n=1$ 到 $K-1$ 求和, 同时利用

离散 Gronwall 引理 [5], 整理可得

$$
\begin{aligned}
&\|\Gamma_h^K\|^2 + \|2\Gamma_h^K - \Gamma_h^{K-1}\|^2 + \sum_{n=1}^{K-1} \|\Gamma_h^{n+1} - 2\Gamma_h^n + \Gamma_h^{n-1}\|^2 + 2\Delta t M^{-2} N \|\nabla \Gamma_h^K\|^2 \\
&+ \Delta t M^{-2} N \|\nabla \Gamma_h^{K-1}\|^2 + 2\Delta t N \sum_{n=1}^{K-1} \|\mathcal{J}_h^{n+1}\|^2 + 2\Delta t N \sum_{n=1}^{K-1} \|\nabla \Phi_h^{n+1}\|^2 \\
&\leq \exp\left(C(M^6 N^{-3} + \bar{B}^2 N)T\right) \Big[2M^{-2}\Delta t N \|\nabla \Gamma_h^1\|^2 + M^{-2}\Delta t N \|\nabla \Gamma_h^0\|^2 \\
&+ \|\Gamma_h^1\|^2 + \|2\Gamma_h^1 - \Gamma_h^0\|^2 + C\big(M^2 N^{-1} h^{2k+2} |||u_t|||_{2,k+1}^2 + M^{-2} N h^{2k} |||u|||_{2,k+1}^2 \\
&+ N h^{2r} |||\phi|||_{2,r+1}^2 + \bar{B}^2 h^{2k+2} |||u|||_{2,k+1}^2 + M^2 N^{-1} \|\nabla u\|_{\infty,0}^2 h^{2k} |||u|||_{2,k+1}^2 \\
&+ M^2 N^{-1} \Delta t^4 |||\nabla u_{tt}|||_{2,0}^2 + M^2 N h^{2s+2} |||p|||_{2,s+1}^2 + M^2 N^{-1} h^{2k} |||u|||_{2,k+1}^2 \\
&+ M^2 N \Delta t^4 |||u_{ttt}|||_{2,0}^2 \big) \Big].
\end{aligned}
\tag{3.11}
$$

最后利用三角不等式, 我们得到误差估计式 (3.1), 从而完成证明. □

4. 结论

本文针对低磁雷诺数下简化磁流体方程提出了线性化二阶时步伽辽金有限元方法，并通过推导证明了其无条件性与稳定性。

参考文献

[1] Yuksel, G. and Ingram, R. (2013) Numerical Analysis of a Finite Element, Crank-Nicolson Discretization for MHD Flows at Small Magnetic Reynolds Numbers. *International Journal of Numerical Analysis and Modelling*, **10**, 74-98.

[2] Heywood, J.G. and Rannacher, R. (1990) Finite-Element Approximation of the Nonstationary Navier-Stokes Problem. Part IV: Error Analysis for Second-Order Time Discretization. *SIAM Journal on Numerical Analysis*, **27**, 353-384. https://doi.org/10.1137/0727022

[3] Layton, W., Mays, N., Neda, M. and Trenchea, C. (2014) Numerical Analysis of Modular Regularization Methods for the BDF2 Time Discretization of the Navier-Stokes Equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, **48**, 765-793. https://doi.org/10.1051/m2an/2013120

[4] Girault, V. and Raviart, P.A. (1986) Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms, Vol. 87. Springer.

[5] Brenner, S.C. and Scott, R. (2008) The Mathematical Theory of Finite Element Methods, Vol. 15. Springer.

[6] Gunzburger, M. (1989) Finite Element Methods for Incompressible Viscous Flows: A Guide to Theory, Practice and Algorithms. Academic Press.

[7] He, Y. (2014) Unconditional Convergence of the Euler Semi-Implicit Scheme for the Three-Dimensional Incompressible MHD Equations. *IMA Journal of Numerical Analysis*, **35**, 767-801. https://doi.org/10.1093/imanum/dru015