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分数阶微分方程无穷多点边值问题的正解

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摘 要

本文研究了带有积分边界条件的分数阶微分方程无穷多点边值问题的正解。该边值问题正解的存在性是通过Green函数的性质和不动点定理获得的。首先求出非线性系统对于线性系统的Green函数，之后给出Green函数的性质并构造合适的积分算子，然后通过使用不动点定理得到边值问题正解的存在性结果。最后，本文给出具体实例来说明得到定理的实用性。

关键词

分数阶微分方程，无穷多点，不动点定理，正解

Positive Solutions for Infinite-Point Boundary Value Problems of Fractional Differential Equations

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Abstract

This paper studies the positive solutions of the infinite-point boundary value problem for fractional differential equations with integral boundary conditions. The existence of positive solutions for boundary value problems is obtained through the properties of Green's function and the fixed point theorem. Firstly, Green's function for the nonlinear system relative to the linear system is derived. Then, the properties of the Green's function are presented and a suitable integral operator is constructed. Finally, the existence results of positive solutions for the boundary value problem are

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obtained by using the fixed point theorem. At the end, specific example is provided to illustrate the practicality of the obtained theorems.

Keywords

Fractional Differential Equation, Infinite Multipoint, Fixed Point Theorem, Positive Solution

1. 引言

近年来，分数阶微分方程在信号处理与控制、反常扩散和流体力学等领域都有着广泛的应用[1]。目前，对于非线性系统不含有导数项的分数阶微分方程的研究，主要的研究成果有分数阶泛函微分方程[2] [3]、分数阶脉冲微分方程[4] [5]、分数阶模糊微分方程[6]、分数阶随机微分方程[7] [8]和q阶分数阶微分方程[9]-[12]。近年来，许多学者在分数阶微分方程的理论研究中取得了一些成果，在这些研究的成果中，分数阶微分方程边值问题解和正解的存在性是研究的重点[13]-[15]。许多研究成果是将边值问题通过积分变换或拉普拉斯变换转化为等价的积分方程，之后通过应用不动点定理、拓扑度理论和上下解等方法得到边值问题解和正解的存在性[16] [17]。

在2011年，Gao和Han [18]讨论了下列具有非局部边界条件的分数阶微分方程边值问题

$$\begin{cases} D_{0+}^{\alpha}u(t)+a(t)f(t,u(t))=0, t\in(0,1), \\ u(0)=0, u(1)=\sum_{i=1}^{\infty}\alpha_i u(\xi_i) \end{cases}$$

正解的存在性，其中 D_{0+}^{α} 表示Riemann-Liouville分数阶导数，$1<\alpha\le 2$，$\xi_i\in(0,1)$，$\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}<1$，$\alpha_i\in[0,+\infty)$，$a(t)\in C([0,1],[0,+\infty))$ 并且在 $[a,b]\subset(0,1)$ 上，$a(t)\ne 0$，$f(t,u)\in C([0,1]\times[0,+\infty),[0,+\infty))$。他们利用了Guo-krasnoselskii不动点定理和Leggett-Williams不动点定理。

在2020年，通过使用Guo-Krasnoselskii不动点定理，Shen和Zhou [19]研究了下列具有积分边界条件的分数阶微分方程无穷多点边值问题

$$\begin{cases} D_{0+}^{\alpha}u(t)+h(t,u(t))=0, t\in[0,1], \\ u^{(i)}(0)=0, i=0,1,\cdots,n-2, \\ D_{0+}^{\beta}u(1)=\sum_{i=1}^{\infty}\beta_i\int_0^{\eta_i}u(s)\mathrm{d}s+\sum_{i=1}^{\infty}\gamma_i u(\eta_i) \end{cases}$$

正解的存在性，其中 D_{0+}^{α} 表示Riemann-Liouville分数阶导数，$n\in N^{+}$，$n\ge 3$，$n-1<\alpha\le n$，$1\le\beta\le\alpha-1$，$0<\eta_1<\eta_2<\cdots<\eta_i<\cdots<1$，$\xi=\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}-\frac{1}{\alpha}\sum_{i=1}^{\infty}\beta_i\eta_i^{\alpha}-\sum_{i=1}^{\infty}\gamma_i\eta_i^{\alpha-1}>0$，$\beta_i,\gamma_i>0$，$(i=1,2,\cdots)$，$h\in C([0,1]\times[0,+\infty),[0,+\infty))$。

在上述工作的启发下，本文研究了下列非线性项中含有未知函数导数项的分数阶微分方程边值问题

$$\begin{cases} D_{0+}^{\alpha}u(t)+f\left(t,u(t),D_{0+}^{p}u(t)\right)=0, t\in[0,1], \\ u^{(i)}(0)=0, i=0,1,\cdots,n-2, \\ D_{0+}^{\beta}u(1)=\sum_{i=1}^{\infty}\beta_i\int_0^{\eta_i}u(s)\mathrm{d}s+\sum_{i=1}^{\infty}\gamma_i u(\xi_i) \end{cases} \tag{1}$$

正解的存在性。其中 D_{0+}^{α} 表示Riemann-Liouville分数阶导数，$n\in N^{+}$，$n\ge 3$，$n-1<\alpha\le n$，$1\le\beta\le\alpha-1$，$0\le p\le\beta-1$， $0<\eta_1<\eta_2<\cdots<\eta_i<\cdots<1$， $0<\xi_1<\xi_2<\cdots<\xi_i<\cdots<1$， $\beta_i,\gamma_i>0\,(i=1,2,\cdots)$，$\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}-\frac{1}{\alpha}\sum_{i=1}^{\infty}\beta_i\eta_i^{\alpha}-\sum_{i=1}^{\infty}\gamma_i\xi_i^{\alpha-1}>0$， $f\in C\left([0,1]\times[0,+\infty)\times[0,+\infty),[0,+\infty)\right)$。

2. 预备知识

首先给出与分数阶微分方程相关的一些基本定义和引理。假设 $\mu>0$ 是一个常数并且 $[\mu]$ 表示 μ 的整数部分。

2.1. 定义 1 [1]

$[0,1]$ 上的 μ 阶 Riemann-Liouville 分数阶积分 $I_{0+}^{\mu}u$ 和 $I_{1-}^{\mu}u$ 分别为

$$\left(I_{0+}^{\mu}u\right)(t)=\frac{1}{\Gamma(\mu)}\int_0^t\frac{u(s)\mathrm{d}s}{(t-s)^{1-\mu}}$$

和

$$\left(I_{1-}^{\mu}u\right)(t)=\frac{1}{\Gamma(\mu)}\int_t^1\frac{u(s)\mathrm{d}s}{(s-t)^{1-\mu}}.$$

2.2. 定义 2 [1]

$[0,1]$ 上的 μ 阶 Riemann-Liouville 分数阶导数 $D_{0+}^{\mu}u$ 和 $D_{1-}^{\mu}u$ 分别定义为

$$\begin{aligned}\left(D_{0+}^{\mu}u\right)(t)&=\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m}\left(I_{0+}^{m-\mu}u\right)(t)\\&=\frac{1}{\Gamma(m-\mu)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m}\int_0^t\frac{u(s)\mathrm{d}s}{(t-s)^{\mu-m+1}}\end{aligned}$$

和

$$\begin{aligned}\left(D_{1-}^{\mu}u\right)(t)&=\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m}\left(I_{1-}^{m-\mu}u\right)(t)\\&=\frac{1}{\Gamma(m-\mu)}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m}\int_t^1\frac{u(s)\mathrm{d}s}{(s-t)^{\mu-m+1}},\end{aligned}$$

其中 $m=[\mu]+1$。

2.3. 引理 1 [1]

设 $u\in C[0,1]\cap[0,1]$ 具有 μ 阶导数且 $D_{0+}^{\mu}u\in C[0,1]\cap L[0,1]$，则有

$$I_{0+}^{\mu}D_{0+}^{\mu}u(t)=u(t)+C_1't^{\mu-1}+C_2't^{\mu-2}+\cdots+C_N't^{\mu-N},$$

其中， $C_i'\in R, i=1,2,\cdots,N$，

$$N=\begin{cases}[\mu]+1, & \mu\notin N^{+},\\ \mu, & \mu\in N^{+}.\end{cases}$$

2.4. 引理 2 [1]

设 $\mu>0$，$\rho>0$，$u\in C[0,1]\cap L[0,1]$，则有 $D_{0+}^{\mu}I_{0+}^{\rho}u(t)=I_{0+}^{\rho-\mu}x(t)$ 成立。

2.5. 引理 3 [20]

设 E 是 Banach 空间，P 是 E 中的锥，Ω_1 和 Ω_2 是 E 中的有界开子集，并且 $0\in\Omega_1$，$\bar{\Omega}_1\subset\Omega_2$。若全连续算子 $T:P\cap(\bar{\Omega}_2\setminus\Omega_1)\to P$ 满足下列条件之一：

(1) $\|Tx\|\le\|x\|$，$\forall x\in P\cap\partial\Omega_1$ 且 $\|Tx\|\ge\|x\|$，$\forall x\in P\cap\partial\Omega_2$；

(2) $\|Tx\|\ge\|x\|$，$\forall x\in P\cap\partial\Omega_1$ 且 $\|Tx\|\le\|x\|$，$\forall x\in P\cap\partial\Omega_2$，那么 T 在 $P\cap(\bar{\Omega}_2\setminus\Omega_1)$ 中有一个不动点。

2.6. 引理 4 [21]

设 P 是 Banach 空间 E 上的锥，$\alpha,\gamma:P\to R^{+}$ 是单调递增且非负的连续函数，$\theta:P\to R^{+}$ 是非负的连续函数，存在 N,a_3 使得

(1) 对于所有的 $x\in\bar{P}(\gamma,a_3)$，有 $\theta(0)=0$，$\gamma(x)\le\theta(x)\le\alpha(x)$，$\|x\|\le N\gamma(x)$；

(2) 存在 $0<a_1<a_2<a_3$，对于 $\forall\lambda\in[0,1]$，$x\in\partial P(\theta,a_2)$，有 $\theta(\lambda x)\le\lambda\theta(x)$；

(3) $T:\bar{P}(\gamma,a_3)\to P$ 是一个完全连续算子；

(4) 对于 $x\in\partial P(\gamma,a_3)$，有 $\gamma(Tx)>a_3$，对于 $x\in\partial P(\theta,a_2)$，有 $\theta(Tx)<a_2$，对于 $x\in\partial P(\alpha,a_1)$，有 $P(\alpha,a_1)\ne\phi$ 并且 $\alpha(Tx)>a_1$，其中 $P(\alpha,a_1)=\{x\in P:\alpha(x)<a_1\}$。

那么 T 至少有两个不动点 $x_1,x_2\in P(\gamma,a_3)$ 使得 $a_1<\alpha(x_1)$，$\theta(x_1)<a_2<\theta(x_2)$，$\gamma(x_2)<a_3$。

2.7. 引理 5

设 $h\in L^{1}[0,1]$，则下面边值问题

$$\begin{cases}D_{0+}^{\alpha}u(t)+h(t)=0, t\in[0,1],\\ u^{(i)}(0)=0, i=0,1,\cdots,n-2,\\ D_{0+}^{\beta}u(1)=\sum_{i=1}^{\infty}\beta_i\int_0^{\eta_i}u(s)\,\mathrm{d}s+\sum_{i=1}^{\infty}\gamma_i u(\xi_i)\end{cases}\tag{2}$$

有唯一解

$$u(t)=\int_0^1 G(t,s)h(s)\,\mathrm{d}s,$$

其中

$$G(t,s)=\frac{1}{m(0)\Gamma(\alpha)}\begin{cases}m(s)t^{\alpha-1}(1-s)^{\alpha-\beta-1}-m(0)(t-s)^{\alpha-1}, & 0\le s\le t\le 1,\\ m(s)t^{\alpha-1}(1-s)^{\alpha-\beta-1}, & 0\le t\le s\le 1.\end{cases}$$

$$m(s)=\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}-\frac{1}{\alpha}\sum_{s\le\eta_i}\beta_i\left(\frac{\eta_i-s}{1-s}\right)^{\alpha}(1-s)^{\beta+1}-\sum_{s\le\xi_i}\gamma_i\left(\frac{\xi_i-s}{1-s}\right)^{\alpha-1}(1-s)^{\beta},$$

所以

$$D_{0+}^{p}u(t)=\int_0^1 H(t,s)h(s)\mathrm{d}s$$

其中

$$H(t,s)=\frac{1}{m(0)\Gamma(\alpha-\beta)}\begin{cases} m(s)t^{\alpha-p-1}(1-s)^{\alpha-\beta-1}-m(0)(t-s)^{\alpha-p-1}, & 0\le s\le t\le 1,\\ m(s)t^{\alpha-p-1}(1-s)^{\alpha-\beta-1}, & 0\le t\le s\le 1. \end{cases}$$

证明 令 $C_i=-C_i'$，通过 2.3 引理 1 和边值问题(2)得到

$$\begin{aligned} u(t)&=-I_{0+}^{\alpha}h(t)-C_1't^{\alpha-1}-C_2't^{\alpha-2}-\cdots-C_n't^{\alpha-n}\\ &=-I_{0+}^{\alpha}h(t)+C_1t^{\alpha-1}+C_2t^{\alpha-2}+\cdots+C_nt^{\alpha-n}. \end{aligned}$$

由边值条件 $u^{(i)}(0)=0,i=0,1,\cdots,n-2$，可推出 $C_2=C_3=\cdots=C_n=0$。那么

$$u(t)=-I_{0+}^{\alpha}h(t)+C_1t^{\alpha-1}.$$

考虑到边值条件

$$D_{0+}^{\beta}u(1)=\sum_{i=1}^{\infty}\beta_i\int_0^{\eta_i}u(s)\mathrm{d}s+\sum_{i=1}^{\infty}\gamma_i u(\xi_i),$$

可以得出

$$\begin{aligned} D_{0+}^{\beta}u(1)&=\sum_{i=1}^{\infty}\beta_i\int_0^{\eta_i}u(s)\mathrm{d}s+\sum_{i=1}^{\infty}\gamma_i u(\xi_i)\\ &=-\frac{1}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-\beta-1}h(s)\mathrm{d}s+C_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\\ &=\sum_{i=1}^{\infty}\beta_i\left[-I^{\alpha+1}h(\eta_i)+\frac{C_1\eta_i^{\alpha}}{\alpha}\right]+\sum_{i=1}^{\infty}\gamma_i\left[-I_{0+}^{\alpha}h(\xi_i)+C_1\xi_i^{\alpha-1}\right], \end{aligned}$$

经过推导可以得出

$$\begin{aligned} &C_1\left[\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}-\sum_{i=1}^{\infty}\frac{\beta_i\eta_i^{\alpha}}{\alpha}-\sum_{i=1}^{\infty}\gamma_i\xi_i^{\alpha-1}\right]\\ &=\frac{1}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-\beta-1}h(s)\mathrm{d}s-\sum_{i=1}^{\infty}\frac{\beta_i}{\Gamma(\alpha+1)}\int_0^{\eta_i}(\eta_i-s)^{\alpha}h(s)\mathrm{d}s\\ &-\sum_{i=1}^{\infty}\frac{\gamma_i}{\Gamma(\alpha)}\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}h(s)\mathrm{d}s, \end{aligned}$$

所以

$$C_1=\frac{1}{m(0)}\left[\int_0^1\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}h(s)\mathrm{d}s-\sum_{i=1}^{\infty}\beta_i\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha}}{\Gamma(\alpha+1)}h(s)\mathrm{d}s-\sum_{i=1}^{\infty}\gamma_i\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s\right],$$

所以，边值问题(2)的解为

$$
\begin{aligned}
u(t) &= -I_{0+}^{\alpha} h(t) + C_1 t^{\alpha-1} \\
&= -\int_0^t \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} h(s) \mathrm{d}s + \frac{1}{\Gamma(\alpha) m(0)} \Bigg[\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} h(s) \mathrm{d}s \\
&\quad - \sum_{i=1}^{\infty} \beta_i \int_0^{\eta_i} \frac{(\eta_i - s)^{\alpha}}{\Gamma(\alpha+1)} h(s) \mathrm{d}s - \sum_{i=1}^{\infty} \gamma_i \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} h(s) \mathrm{d}s \Bigg] \cdot t^{\alpha-1} \\
&= -\int_0^t \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} h(s) \mathrm{d}s + \frac{1}{\Gamma(\alpha) m(0)} \int_0^1 (1-s)^{\alpha-\beta-1} \Bigg[\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \\
&\quad - \frac{1}{\alpha} \sum_{s \le \eta_i} \beta_i (\eta_i - s)^{\alpha} (1-s)^{\beta+1-\alpha} - \sum_{s \le \xi_i} \gamma_i (\xi_i - s)^{\alpha-1} (1-s)^{\beta+1-\alpha} \Bigg] \cdot h(s) \mathrm{d}s \cdot t^{\alpha-1} \\
&= -\int_0^t \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} h(s) \mathrm{d}s + \int_0^1 \frac{(1-s)^{\alpha-\beta-1} m(s) h(s)}{\Gamma(\alpha) m(0)} \mathrm{d}s \cdot t^{\alpha-1} \\
&= -\int_0^t \frac{m(0)(t-s)^{\alpha-1}}{\Gamma(\alpha) m(0)} h(s) \mathrm{d}s + \int_0^t \frac{(1-s)^{\alpha-\beta-1} m(s) t^{\alpha-1}}{\Gamma(\alpha) m(0)} h(s) \mathrm{d}s + \int_t^1 \frac{(1-s)^{\alpha-\beta-1} m(s) t^{\alpha-1}}{\Gamma(\alpha) m(0)} h(s) \mathrm{d}s \\
&= \int_0^t \frac{m(s)(1-s)^{\alpha-\beta-1} t^{\alpha-1} - m(0)(t-s)^{\alpha-1}}{\Gamma(\alpha) m(0)} h(s) \mathrm{d}s + \int_t^1 \frac{m(s)(1-s)^{\alpha-\beta-1} t^{\alpha-1}}{\Gamma(\alpha) m(0)} h(s) \mathrm{d}s \\
&= \int_0^1 G(t,s) h(s) \mathrm{d}s.
\end{aligned}
$$

更多地，经过推导有

$$
\begin{aligned}
D_{0+}^{p} u(t) &= -D_{0+}^{p} I_{0+}^{\alpha} h(t) + C_1 D_{0+}^{p} t^{\alpha-1} \\
&= -\frac{1}{\Gamma(\alpha-p)} \int_0^t (t-s)^{\alpha-p-1} h(s) \mathrm{d}s \\
&\quad + \frac{t^{\alpha-p-1}}{m(0)\Gamma(\alpha-p)} \int_0^1 (1-s)^{\alpha-\beta-1} m(s) h(s) \mathrm{d}s \\
&= \int_0^t \frac{m(s) t^{\alpha-p-1} (1-s)^{\alpha-\beta-1} - m(0)(t-s)^{\alpha-p-1}}{m(0)\Gamma(\alpha-p)} h(s) \mathrm{d}s \\
&\quad + \int_t^1 \frac{m(s) t^{\alpha-p-1} (1-s)^{\alpha-\beta-1}}{m(0)\Gamma(\alpha-p)} h(s) \mathrm{d}s \\
&= \int_0^1 H(t,s) h(s) \mathrm{d}s
\end{aligned}
$$

证明结束。

2.8. 引理 6

如果 $m(0) > 0$，那么函数 $m(s)$ 在 $s \in [0,1]$ 上是不减的并且 $m(s) > 0$。

证明 通过公式

$$
m(s) = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} - \frac{1}{\alpha} \sum_{s \le \eta_i} \beta_i \left(\frac{\eta_i - s}{1-s} \right)^{\alpha} (1-s)^{\beta+1} - \sum_{s \le \xi_i} \gamma_i \left(\frac{\xi_i - s}{1-s} \right)^{\alpha-1} (1-s)^{\beta}, s \in [0,1]
$$

可以得出

$$m(0)=\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}-\frac{1}{\alpha}\sum_{i=1}^{\infty}\beta_i\eta_i^{\alpha}-\sum_{i=1}^{\infty}\gamma_i\xi_i^{\alpha-1}>0$$

并且

$$\begin{aligned}m'(s)&=\sum_{s\le\eta_i}\beta_i(\eta_i-s)^{\alpha-1}(1-s)^{\beta+1-\alpha}+\frac{\beta+1-\alpha}{\alpha}\sum_{s\le\eta_i}\beta_i(\eta_i-s)^{\alpha}(1-s)^{\beta-\alpha}\\&\quad+(\alpha-1)\sum_{s\le\xi_i}\gamma_i(\xi_i-s)^{\alpha-2}(1-s)^{\beta+1-\alpha}+(\beta+1-\alpha)\sum_{s\le\xi_i}\gamma_i(\xi_i-s)^{\alpha-1}(1-s)^{\beta-\alpha}\\&=\sum_{s\le\eta_i}\beta_i(\eta_i-s)^{\alpha-1}(1-s)^{\beta-\alpha}\left[(1-\eta_i)+\frac{\beta+1}{\alpha}(\eta_i-s)\right]\\&\quad+\sum_{s\le\xi_i}\gamma_i(\xi_i-s)^{\alpha-2}(1-s)^{\beta-\alpha}\left[(\alpha-1)(1-\xi_i)+\beta(\xi_i-s)\right]\ge0,s\in[0,1],\end{aligned}$$

所以 $m(s)$ 在 $[0,1]$ 上是不减的并且 $m(s)>0$。证明结束。

2.9. 引理 7

函数 $G(t,s)$ 满足下列条件：

(1) $G(t,s)\ge0,\frac{\partial G(t,s)}{\partial t}\ge0,0\le t,s\le1$；

(2) $G(t,s)\ge t^{\alpha-1}G(1,s),0\le t,s\le1$；

(3) $\max_{t\in[0,1]}G(t,s)=G(1,s),0\le s\le1;$；

(4) $H(t,s)\ge0,\frac{\partial H(t,s)}{\partial t}\ge0,0\le t,s\le1$；

(5) $H(t,s)\ge t^{\alpha-p-1}H(1,s),0\le t,s\le1$；

(6) $\max_{t\in[0,1]}H(t,s)=H(1,s),0\le s\le1$。

证明 (1) 当 $0\le s\le t\le1$ 时，通过 2.7 引理 5 可以得到

$$\begin{aligned}G(t,s)&=\frac{1}{m(0)\Gamma(\alpha)}\left[m(s)t^{\alpha-1}(1-s)^{\alpha-\beta-1}-m(0)(t-s)^{\alpha-1}\right]\\&=\frac{t^{\alpha-1}}{m(0)\Gamma(\alpha)}\left[m(s)(1-s)^{\alpha-\beta-1}-m(0)\left(1-\frac{s}{t}\right)^{\alpha-1}\right]\\&\ge\frac{m(s)t^{\alpha-1}}{m(0)\Gamma(\alpha)}\left[(1-s)^{\alpha-\beta-1}-\left(1-\frac{s}{t}\right)^{\alpha-1}\right]\\&\ge\frac{m(s)t^{\alpha-1}}{m(0)\Gamma(\alpha)}\left[(1-s)^{\alpha-1}-\left(1-\frac{s}{t}\right)^{\alpha-1}\right]\\&=\frac{m(s)}{m(0)\Gamma(\alpha)}\left[(t-ts)^{\alpha-1}-(t-s)^{\alpha-1}\right]\ge0.\end{aligned}$$

同样地，当 $0\le t\le s\le1$ 时可以得到

$$G(t,s)=\frac{1}{m(0)\Gamma(\alpha)}\left[m(s)t^{\alpha-1}(1-s)^{\alpha-\beta-1}\right]\ge0.$$

对 Green 函数 $G(t,s)$ 关于 t 求偏导可以得到

$$\frac{\partial G(t,s)}{\partial t}=\frac{1}{m(0)\Gamma(\alpha)}\begin{cases}(\alpha-1)m(s)t^{\alpha-2}(1-s)^{\alpha-\beta-1}-(\alpha-1)m(0)(t-s)^{\alpha-2}, & 0\le s\le t\le 1,\\(\alpha-1)m(s)t^{\alpha-2}(1-s)^{\alpha-\beta-1}, & 0\le t\le s\le 1.\end{cases}$$

显然 $\frac{\partial G(t,s)}{\partial t}$ 在 $[0,1]\times[0,1]$ 上是连续的并且当 $0\le t\le s\le 1$ 时 $\frac{\partial G(t,s)}{\partial t}\ge 0$。接下来只需考虑当 $0\le s\le t\le 1$ 时 $\frac{\partial G(t,s)}{\partial t}$ 的性质。当 $0\le s\le t\le 1$ 时，

$$\begin{aligned}\frac{\partial G(t,s)}{\partial t}&=\frac{1}{m(0)\Gamma(\alpha)}\left[(\alpha-1)m(s)t^{\alpha-2}(1-s)^{\alpha-\beta-1}-(\alpha-1)m(0)(t-s)^{\alpha-2}\right]\\&\ge\frac{(\alpha-1)m(0)t^{\alpha-2}}{m(0)\Gamma(\alpha)}\left[(1-s)^{\alpha-\beta-1}-\left(1-\frac{s}{t}\right)^{\alpha-2}\right]\\&\ge\frac{(\alpha-1)t^{\alpha-2}}{\Gamma(\alpha)}\left[(1-s)^{\alpha-2}-\left(1-\frac{s}{t}\right)^{\alpha-2}\right]\\&=\frac{(\alpha-1)}{\Gamma(\alpha)}\left[(t-ts)^{\alpha-2}-(t-s)^{\alpha-2}\right]\ge 0.\end{aligned}$$

(2) 当 $0\le s\le t\le 1$ 时可以得到

$$\begin{aligned}G(t,s)&=\frac{1}{m(0)\Gamma(\alpha)}\left[m(s)t^{\alpha-1}(1-s)^{\alpha-\beta-1}-m(0)(t-s)^{\alpha-1}\right]\\&=\frac{t^{\alpha-1}}{m(0)\Gamma(\alpha)}\left[m(s)(1-s)^{\alpha-\beta-1}-m(0)\left(1-\frac{s}{t}\right)^{\alpha-1}\right]\\&\ge\frac{t^{\alpha-1}}{m(0)\Gamma(\alpha)}\left[m(s)(1-s)^{\alpha-\beta-1}-m(0)(1-s)^{\alpha-1}\right]\\&=t^{\alpha-1}G(1,s).\end{aligned}$$

当 $0\le t\le s\le 1$ 时可以得出

$$G(t,s)=\frac{1}{m(0)\Gamma(\alpha)}t^{\alpha-1}m(s)(1-s)^{\alpha-\beta-1}\ge t^{\alpha-1}G(1,s).$$

(3) 由于 $\frac{\partial G(t,s)}{\partial t}\ge 0$，所以可以得出 $G(t,s)$ 在 $0\le t\le 1$ 上是不减的。所以

$$\max_{t\in[0,1]}G(t,s)=G(1,s)=\frac{1}{m(0)\Gamma(\alpha)}\left[m(s)(1-s)^{\alpha-\beta-1}-m(0)(1-s)^{\alpha-1}\right],0\le s\le 1.$$

(4) 当 $0\le s\le t\le 1$ 时有

$$\begin{aligned}H(t,s)&=\frac{m(s)t^{\alpha-p-1}(1-s)^{\alpha-\beta-1}-m(0)(t-s)^{\alpha-p-1}}{m(0)\Gamma(\alpha-\beta)}\\&\ge\frac{m(s)t^{\alpha-p-1}}{m(0)\Gamma(\alpha-p)}\left[(1-s)^{\alpha-\beta-1}-\left(1-\frac{s}{t}\right)^{\alpha-p-1}\right]\\&\ge\frac{m(s)t^{\alpha-p-1}}{m(0)\Gamma(\alpha-p)}\left[(1-s)^{\alpha-\beta-1}-(1-s)^{\alpha-p-1}\right]\ge 0.\end{aligned}$$

并且
$$\frac{\partial H(t,s)}{\partial t}=\frac{(\alpha-p-1)m(s)t^{\alpha-p-2}(1-s)^{\alpha-\beta-1}-(\alpha-p-1)m(0)(t-s)^{\alpha-p-2}}{m(0)\Gamma(\alpha-p)}$$
$$\geq\frac{(\alpha-p-1)m(0)t^{\alpha-p-2}\left[(1-s)^{\alpha-\beta-1}-(1-s)^{\alpha-p-2}\right]}{m(0)\Gamma(\alpha-p)}\geq 0.$$

当 $0\leq t\leq s\leq 1$ 时，$H(t,s)\geq 0$ 并且 $\frac{\partial H(t,s)}{\partial t}\geq 0$ 是显然的。

(5) 当 $0\leq t\leq s\leq 1$ 时有
$$H(t,s)=\frac{m(s)t^{\alpha-p-1}(1-s)^{\alpha-\beta-1}}{m(0)\Gamma(\alpha-p)}\geq t^{\alpha-p-1}H(1,s).$$

当 $0\leq s\leq t\leq 1$ 时有
$$H(t,s)=\frac{1}{m(0)\Gamma(\alpha-p)}\left[m(s)t^{\alpha-p-1}(1-s)^{\alpha-\beta-1}-m(0)(t-s)^{\alpha-p-1}\right]$$
$$=\frac{t^{\alpha-p-1}}{m(0)\Gamma(\alpha-p)}\left[m(s)(1-s)^{\alpha-\beta-1}-m(0)\left(1-\frac{s}{t}\right)^{\alpha-p-1}\right]$$
$$\geq\frac{t^{\alpha-p-1}}{m(0)\Gamma(\alpha-p)}\left[m(s)(1-s)^{\alpha-\beta-1}-m(0)(1-s)^{\alpha-p-1}\right]$$
$$=t^{\alpha-p-1}H(1,s).$$

(6) 通过(4)的性质可以得到
$$\max_{t\in[0,1]}H(t,s)=H(1,s)=\frac{1}{m(0)\Gamma(\alpha-p)}\left[m(s)(1-s)^{\alpha-\beta-1}-m(0)(1-s)^{\alpha-p-1}\right],0\leq s\leq 1.$$

3. 主要结果

在本节中，得到了关于问题(1)正解的存在性的一些主要结果。

在后续部分，定义空间 $E=\left\{u\,\middle|\,u\in C[0,1],D_{0+}^{p}u\in C[0,1]\right\}$，则 E 是一个巴拿赫空间[22]。其上的范数为
$$\|u\|=\|u\|_1+\|u\|_2=\max_{t\in[0,1]}|u(t)|+\max_{t\in[0,1]}\left|D_{0+}^{p}u(t)\right|.$$

在该空间上定义锥为
$$P=\left\{u(t)\in E\,\middle|\,u(t)\geq 0,u(t)+D_{0+}^{p}u(t)\geq\left(\frac{1}{4}\right)^{\alpha-1}\|u\|,t\in\left[\frac{1}{4},1\right]\right\}.$$

之后，定义算子 $T:P\to P$ 为
$$(Tu)(t)=\int_0^1 G(t,s)f\left(s,u(s),D_{0+}^{p}u(s)\right)\mathrm{d}s,t\in\left[\frac{1}{4},1\right].$$

根据 2.7 引理 5 可以得到
$$D_{0+}^{p}(Tu)(t)=\int_0^1 H(t,s)f\left(s,u(s),D_{0+}^{p}u(s)\right)\mathrm{d}s,t\in\left[\frac{1}{4},1\right].$$

在接下来的证明过程中，定义

$$\|Tu\| = \|Tu\|_1 + \|Tu\|_2 = \max_{t\in\left[\frac{1}{4},1\right]} \left|(Tu)(t)\right| + \max_{t\in\left[\frac{1}{4},1\right]} \left|D_{0+}^{p}(Tu)(t)\right|.$$

那么，讨论边值问题(1)正解的存在性就转变为讨论算子 T 的不动点问题。由于使用不动点定理可以得到算子 T 的不动点的存在性，算子 T 的不动点就是边值问题(1)的正解，故边值问题(1)正解的存在性就可以得出，也即本文讨论的问题。本节主要应用了 Guo-Krasnoselskii 不动点定理和 Twin 不动点定理得到了边值问题(1)正解的存在性。

3.1. 定理 1

$T: P \to P$ 是完全连续算子。

证明 对于 $\forall u \in P$ 都有

$$\begin{aligned}(Tu)(t) + D_{0+}^{p}(Tu)(t) &= \int_0^1 G(t,s) f\left(s,u(s),D_{0+}^{p}u(s)\right)\mathrm{d}s + \int_0^1 H(t,s) f\left(s,u(s),D_{0+}^{p}u(s)\right)\mathrm{d}s \\ &\ge t^{\alpha-1}\int_0^1 G(1,s) f\left(s,u(s),D_{0+}^{p}u(s)\right)\mathrm{d}s + t^{\alpha-p-1}\int_0^1 H(1,s) f\left(s,u(s),D_{0+}^{p}u(s)\right)\mathrm{d}s \\ &\ge t^{\alpha-1}\left[\max_{t\in\left[\frac{1}{4},1\right]}\left|(Tu)(t)\right| + \max_{t\in\left[\frac{1}{4},1\right]}\left|D_{0+}^{p}(Tu)(t)\right|\right] = t^{\alpha-1}\|Tu\|,\end{aligned}$$

当 $t\in\left[\frac{1}{4},1\right]$ 时可以得到

$$(Tu)(t) + D_{0+}^{p}(Tu)(t) \ge \left(\frac{1}{4}\right)^{\alpha-1}\|Tu\|,$$

所以 $Tu \in P$ 并且 T 是一个 $P \to P$ 的算子。

接下来，证明 T 在 P 是有界的。

令

$$k_1 = \int_0^1 G(1,s)\mathrm{d}s,\ k_2 = \int_0^1 H(1,s)\mathrm{d}s.$$

若 Z 是 P 上的任意有界集，那么 $\exists L \ge 0,$ 对于 $\forall u \in Z$，使得 $\|u\| \le L$，由于 f 是连续函数，所以对于 $\forall u \in Z$，$\left(t,u(t),D_{0+}^{p}u(t)\right) \in [0,1]\times[0,L]\times[0,L]$，$\exists M > 0$，使得 $f\left(t,u(t),D_{0+}^{p}u(t)\right) \le M$。所以可以推出

$$\|Tu\|_1 = \max_{t\in\left[\frac{1}{4},1\right]}\left|(Tu)(t)\right| = \int_0^1 G(1,s) f\left(s,u(s),D_{0+}^{p}u(s)\right)\mathrm{d}s \le k_1 M,$$

$$\|Tu\|_2 = \max_{t\in\left[\frac{1}{4},1\right]}\left|D_{0+}^{p}(Tu)(t)\right| = \int_0^1 H(1,s) f\left(s,u(s),D_{0+}^{p}u(s)\right)\mathrm{d}s \le k_2 M.$$

那么

$$\begin{aligned}\|Tu\| &= \|Tu\|_1 + \|Tu\|_2 = \max_{t\in\left[\frac{1}{4},1\right]}\left|(Tu)(t)\right| + \max_{t\in\left[\frac{1}{4},1\right]}\left|D_{0+}^{p}(Tu)(t)\right| \\ &= \int_0^1 G(1,s) f\left(s,u(s),D_{0+}^{p}u(s)\right)\mathrm{d}s + \int_0^1 H(1,s) f\left(s,u(s),D_{0+}^{p}u(s)\right)\mathrm{d}s \\ &\le M(k_1 + k_2),\end{aligned}$$

所以 $T(Z)$ 是一致有界的。

接下来，证明 T 是等度连续的。由于 $G(t,s)$ 和 $H(t,s)$ 在 $[0,1]\times[0,1]$ 上是连续的，令 $t_1,t_2,s\in\left[\frac{1}{4},1\right]$，对

于 $\forall\varepsilon>0$，$\exists\delta>0$，当 $|t_2-t_1|<\delta$ 时有

$$\left|G(t_2,s)-G(t_1,s)\right|<\frac{\varepsilon}{2M+1},\ \left|H(t_2,s)-H(t_1,s)\right|<\frac{\varepsilon}{2M+1}.$$

那么

$$\begin{aligned}\left\|(Tu)(t_2)-(Tu)(t_1)\right\|&=\left\|(Tu)(t_2)-(Tu)(t_1)\right\|_1+\left\|(Tu)(t_2)-(Tu)(t_1)\right\|_2\\&\le\int_0^1\left|G(t_2,s)-G(t_1,s)\right|\left|f\left(s,u(s),D_{0+}^{p}u(s)\right)\right|\mathrm{d}s\\&\quad+\int_0^1\left|H(t_2,s)-H(t_1,s)\right|\left|f\left(s,u(s),D_{0+}^{p}u(s)\right)\right|\mathrm{d}s\\&<\frac{\varepsilon}{2M+1}M+\frac{\varepsilon}{2M+1}M<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

由此可以得到算子 T 满足 Arzela-Ascoli 定理的所有条件，根据 Arzela-Ascoli 定理可以得到算子 T 是相对紧的，这就说明算子 T 是完全连续的，证明结束。

在本章剩余部分，记

$$f_\infty=\lim_{(u,v)\to+\infty}\inf_{t\in[0,1]}\frac{f(t,u,v)}{u+v},\ f^0=\lim_{(u,v)\to0}\sup_{t\in[0,1]}\frac{f(t,u,v)}{u+v}$$

$$L_1=\frac{1}{\int_0^1G(1,s)\mathrm{d}s+\int_0^1H(1,s)\mathrm{d}s},\ L_2=\frac{1}{\int_0^1G\left(\frac{1}{4},s\right)\mathrm{d}s+\int_0^1H\left(\frac{1}{4},s\right)\mathrm{d}s}$$

3.2. 定理 2

对于 $f\in C\left([0,1]\times[0,+\infty)\times[0,+\infty),[0,+\infty)\right)$，若 $f^0=0$， $f_\infty=+\infty$，那么边值问题(1)有一个正解。

证明 由于 $f_\infty=+\infty$，所以可以得到 $\exists r_2>0$，使得

$$f(t,u,v)\ge\delta_1(u+v)\ge\delta_1\left(\frac{1}{4}\right)^{\alpha-1}\|u\|$$

对于 $(t,u,v)\in[0,1]\times[r_2,+\infty)\times[r_2,+\infty)$，其中 $\delta_1\ge4^{\alpha-1}\left[\int_0^1G(1,s)\mathrm{d}s+\int_0^1H(1,s)\mathrm{d}s\right]^{-1}$。

同样地，由于 $f^0=0$，那么 $\exists r_2>r_1>0$ 使得

$$f(t,u,v)\le\delta_2(u+v)\le\delta_2\|u\|$$

对于 $(t,u,v)\in[0,1]\times[0,r_1]\times[0,r_1]$，其中 $\delta_2\le\left[\int_0^1G(1,s)\mathrm{d}s+\int_0^1H(1,s)\mathrm{d}s\right]^{-1}$。

令 $\Omega_1=\{u\in P:\|u\|<r_1\}$， $\Omega_2=\{u\in P:\|u\|<r_2\}$。

当 $u\in P\cap\partial\Omega_2$ 时，可以得到

$$\begin{aligned}\|Tu\|&=\max_{t\in\left[\frac{1}{4},1\right]}\left|(Tu)(t)\right|+\max_{t\in\left[\frac{1}{4},1\right]}\left|D_{0+}^{p}(Tu)(t)\right|\\&=\int_0^1G(1,s)f\left(s,u(s),D_{0+}^{p}u(s)\right)\mathrm{d}s+\int_0^1H(1,s)f\left(s,u(s),D_{0+}^{p}u(s)\right)\mathrm{d}s\\&\ge\delta_1\left(\frac{1}{4}\right)^{\alpha-1}\|u\|\left[\int_0^1G(1,s)\mathrm{d}s+\int_0^1H(1,s)\mathrm{d}s\right]\ge\|u\|.\end{aligned}$$

当 $u\in P\cap\partial\Omega_1$ 时，可以推出

$$\begin{aligned}\|Tu\| &= \max_{t\in\left[\frac{1}{4},1\right]}\left|(Tu)(t)\right| + \max_{t\in\left[\frac{1}{4},1\right]}\left|D_{0+}^{p}(Tu)(t)\right| \\ &= \int_0^1 G(1,s)\, f\left(s,u(s),D_{0+}^{p}u(s)\right)\mathrm{d}s + \int_0^1 H(1,s)\, f\left(s,u(s),D_{0+}^{p}u(s)\right)\mathrm{d}s \\ &\le \delta_2\|u\|\left[\int_0^1 G(1,s)\mathrm{d}s + \int_0^1 H(1,s)\mathrm{d}s\right] \le \|u\|.\end{aligned}$$

根据 Guo-Krasnoselskii 不动点定理，边值问题(1)在 $P\cap\left(\bar{\Omega}_2\setminus\Omega_1\right)$ 上有一个正解，证明完成。

3.3. 定理 3

对于 $f\in C\left([0,1]\times[0,+\infty)\times[0,+\infty),[0,+\infty)\right)$，如果存在三个常数 a_1,a_2,a_3 满足 $0<a_1<a_2<a_3$，并且下列条件也成立：

(1) $f(t,u,v)>a_1L_1$，对于 $(t,u,v)\in[0,1]\times[0,a_1]\times[0,a_1]$；

(2) $f(t,u,v)>a_3L_2$，对于 $(t,u,v)\in\left[\frac{1}{4},1\right]\times[0,Na_3]\times[0,Na_3]$；

(3) $f(t,u,v)<a_2L_1$，对于 $(t,u,v)\in[0,1]\times[0,Na_2]\times[0,Na_2]$；

那么，边值问题(1)至少有两个正解 u_1,u_2 并且满足 $a_1<\alpha(u_1)$，$\theta(u_1)<a_2<\theta(u_2)$，$\gamma(u_2)<a_3$。

证明 在证明的过程中，

$$\bar{P}(\gamma,a_1)=\left\{u\in P:\gamma(u)\le a_1\right\}, P(\gamma,a_1)=\left\{u\in P:\gamma(u)<a_1\right\}.$$

令

$$\gamma(u)=\min_{t\in\left[\frac{1}{4},1\right]}\left|u(t)\right|+\min_{t\in\left[\frac{1}{4},1\right]}\left|D_{0+}^{p}u(t)\right|,$$

$$\theta(u)=\alpha(u)=\max_{t\in[0,1]}\left|u(t)\right|+\max_{t\in[0,1]}\left|D_{0+}^{p}u(t)\right|.$$

显然 $\gamma(u)$ 和 $\alpha(u)$ 是非负递增的连续函数并且 $\theta(u)$ 是非负的连续函数，且 $\gamma(u)\le\theta(u)=\alpha(u)$，$\theta(0)=0$。

另外，可以推出

$$\begin{aligned}\gamma(u) &= \min_{t\in\left[\frac{1}{4},1\right]}\left|u(t)\right|+\min_{t\in\left[\frac{1}{4},1\right]}\left|D_{0+}^{p}u(t)\right| \\ &= \int_0^1 G\left(\frac{1}{4},s\right) f\left(s,u(s),D_{0+}^{p}u(s)\right)\mathrm{d}s+\int_0^1 H\left(\frac{1}{4},s\right) f\left(s,u(s),D_{0+}^{p}u(s)\right)\mathrm{d}s \\ &\ge \left(\frac{1}{4}\right)^{\alpha-1}\left[\int_0^1 G(1,s)\, f\left(s,u(s),D_{0+}^{p}u(s)\right)\mathrm{d}s+\int_0^1 H(1,s)\, f\left(s,u(s),D_{0+}^{p}u(s)\right)\mathrm{d}s\right] \\ &= \left(\frac{1}{4}\right)^{\alpha-1}\|u\|,\end{aligned}$$

所以对于所有 $u\in\bar{P}(\gamma,a_3)$，$\|u\|\le 4^{\alpha-1}\gamma(u)=N\gamma(u)$。上述证明表明该定理满足 2.6 引理 4 的条件(1)。

对于 $\forall\lambda\in[0,1]$，$u\in\partial P(\theta,a_2)$ 可以得到

$$\begin{aligned}\theta(\lambda u) &= \max_{t\in[0,1]}\left|\lambda u(t)\right|+\max_{t\in[0,1]}\left|D_{0+}^{p}\lambda u(t)\right| \\ &= \lambda\max_{t\in[0,1]}\left|u(t)\right|+\lambda\max_{t\in[0,1]}\left|D_{0+}^{p}u(t)\right|=\lambda\theta(u),\end{aligned}$$

这表明该定理满足 2.6 引理 4 的条件(2)。

运用和 3.1 定理 1 同样的证明方法可以得到 $T:\bar{P}(\gamma,a_3)\to P$ 是一个完全连续算子，这表明该定理满

足 2.6 引理 4 的条件(3)。

对于 $\forall u\in\partial P(\alpha,a_1)$，可以得到 $\alpha(u)=a_1$， $0\le u(t)\le a_1$， $0\le v=D_{0+}^{p}u(t)\le a_1$，并且通过该定理的条件(1)可以得到

$$\begin{aligned}\alpha(Tu)&=\max_{t\in[0,1]}|Tu|+\max_{t\in[0,1]}\left|D_{0+}^{p}Tu\right|\\&=\int_0^1 G(1,s)f\left(s,u(s),D_{0+}^{p}u(s)\right)ds+\int_0^1 H(1,s)f\left(s,u(s),D_{0+}^{p}u(s)\right)ds\\&>a_1L_1\left[\int_0^1 G(1,s)ds+\int_0^1 H(1,s)ds\right]=a_1,\end{aligned}$$

这表明 $\alpha(Tu)>a_1$ 对于 $\forall u\in\partial P(\alpha,a_1)$。令 $u=\frac{a_1}{3}$，那么 $\alpha(u)=\alpha\left(\frac{a_1}{3}\right)=\frac{a_1}{3}<a_1$。这表面 $P(\alpha,a_1)\ne\phi$。对于 $\forall u\in\partial P(\gamma,a_3)$，可以得到 $\gamma(u)=a_3$， $0\le u(t)\le\|u\|\le N\gamma(u)\le Na_3$， $0\le v=D_{0+}^{p}u(t)\le Na_3$，通过该定理的条件(2)可以得到

$$\begin{aligned}\gamma(Tu)&=\min_{t\in\left[\frac{1}{4},1\right]}|Tu|+\min_{t\in\left[\frac{1}{4},1\right]}\left|D_{0+}^{p}Tu\right|\\&=\int_0^1 G\left(\frac{1}{4},s\right)f\left(s,u(s),D_{0+}^{p}u(s)\right)ds+\int_0^1 H\left(\frac{1}{4},s\right)f\left(s,u(s),D_{0+}^{p}u(s)\right)ds\\&>a_3L_2\left[\int_0^1 G\left(\frac{1}{4},s\right)ds+\int_0^1 H\left(\frac{1}{4},s\right)ds\right]=a_3,\end{aligned}$$

所以得出 $\gamma(Tu)>a_3$ 对于 $\forall u\in\partial P(\gamma,a_3)$。

同样地，对于 $\forall u\in\partial P(\theta,a_2)$，可以得到 $\theta(u)=a_2$， $0\le u=u(t)\le\|u\|\le Na_2$， $0\le v=D_{0+}^{p}u(t)\le Na_2$，并且通过该定理的条件(3)可以得出

$$\begin{aligned}\theta(Tu)&=\max_{t\in[0,1]}|Tu|+\max_{t\in[0,1]}\left|D_{0+}^{p}Tu\right|\\&=\int_0^1 G(1,s)f\left(s,u(s),D_{0+}^{p}u(s)\right)ds+\int_0^1 H(1,s)f\left(s,u(s),D_{0+}^{p}u(s)\right)ds\\&<a_2L_1\left[\int_0^1 G(1,s)ds+\int_0^1 H(1,s)ds\right]=a_2,\end{aligned}$$

这表明 $\theta(Tu)<a_2$ 对于 $\forall u\in\partial P(\theta,a_2)$，这表明该定理满足 2.6 引理 4 的条件(4)。

综上，2.6 引理 4 的所有条件都满足。所以边值问题(1)至少有两个正解 u_1,u_2 并且满足 $a_1<\alpha(u_1)$，$\theta(u_1)<a_2<\theta(u_2)$， $\gamma(u_2)<a_3$。

证明完成。

4. 实例分析

在这一部分，给出该实例来说明得到定理的实用性。

考虑下面的边值问题

$$\begin{cases}D_{0+}^{\frac{9}{2}}u(t)+f\left(t,u(t),D_{0+}^{\frac{3}{2}}u(t)\right)=0,t\in[0,1],\\u(0)=u^{(1)}(0)=\cdots=u^{(n-2)}(0)=0,\\D_{0+}^{\frac{5}{2}}u(1)=\sum_{i=1}^{\infty}\frac{1}{2^i}\int_0^{1-\frac{1}{i+1}}u(s)ds+\sum_{i=1}^{\infty}\frac{1}{3^i}u\left(1-\frac{1}{i+2}\right),\end{cases}\tag{3}$$

其中

$$\alpha=\frac{9}{2},1<\beta=\frac{5}{2}\le\alpha-1=\frac{7}{2},0<p=\frac{3}{2}=\beta-1=\frac{3}{2},$$
$$\eta_i=1-\frac{1}{i+1},0<\eta_1<\eta_2<\cdots<\eta_i<\cdots<1,$$
$$\xi_i=1-\frac{1}{i+2},0<\xi_1<\xi_2<\cdots<\xi_i<\cdots<1,$$
$$\beta_i=\frac{1}{2^i}>0,\gamma_i=\frac{1}{3^i}>0,$$

并且

$$\begin{aligned}m(0)&=\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}-\frac{1}{\alpha}\sum_{i=1}^{\infty}\beta_i\eta_i^{\alpha}-\sum_{i=1}^{\infty}\gamma_i\xi_i^{\alpha-1}\\&=\frac{\Gamma\left(\frac{9}{2}\right)}{\Gamma(2)}-\frac{2}{9}\sum_{i=1}^{\infty}\frac{1}{2^i}\left(1-\frac{1}{i+1}\right)^{\frac{9}{2}}-\sum_{i=1}^{\infty}\frac{1}{3^i}\left(1-\frac{1}{i+2}\right)^{\frac{7}{2}}\\&\ge\frac{105\sqrt{\pi}}{16}-\frac{2}{9}\left(\frac{1}{2}\times1^{\frac{9}{2}}+\frac{1}{2^2}\times1^{\frac{9}{2}}+\cdots+\frac{1}{2^n}\times1^{\frac{9}{2}}+\cdots\right)\\&\quad-\left(\frac{1}{3}\times1^{\frac{7}{2}}+\frac{1}{3^2}\times1^{\frac{7}{2}}+\cdots+\frac{1}{3^n}\times1^{\frac{7}{2}}+\cdots\right)\\&\approx10.91>0,\end{aligned}$$

令

$$f\left(t,u(t),D_{0+}^{\frac{3}{2}}u(t)\right)=\frac{\left(u^{\frac{3}{2}}+u\right)^2}{1+\sin t},$$

对于 $\left(t,u(t),D_{0+}^{\frac{3}{2}}u(t)\right)\in[0,1]\times[0,+\infty)\times[0,+\infty).$

容易得出 $f^0=\lim_{\left(u,u^{\frac{3}{2}}\right)\to0}\sup_{t\in[0,1]}\frac{u^{\frac{3}{2}}+u}{1+\sin t}=0$，$f_\infty=\lim_{\left(u,u^{\frac{3}{2}}\right)\to+\infty}\inf_{t\in[0,1]}\frac{u^{\frac{3}{2}}+u}{1+\sin t}=+\infty$。所以根据 3.2 定理 2 得出边值问题(3)有一个正解。

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