

具有积分边界条件的分数阶微分方程边值问题正解的存在性

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摘 要

分数阶微积分是对函数进行非整数阶积分和微分的定量分析, 它是整数阶微积分的推广, 且在实际应用过程中一直彰显着其独特优势和不可替代性。分数阶微分方程已应用于生物传染病、金融市场、控制系统、异常扩散等多个领域。目前, 分数阶微分方程边值问题解的存在唯一性是研究的重点课题之一。本文将借助泛函分析等相关工具, 对具有积分边界条件的非线性边值问题进行深入探讨。首先讨论对应线性边值问题的Green函数解, 接着分析Green函数的性质, 然后利用锥上的不动点定理得到边值问题正解的存在性结果, 最后举例说明结果的正确性。

关键词

分数微分方程, 积分边界条件, 正解, 存在性, 不动点定理

The Existence of Positive Solutions for Boundary Value Problem of Fractional Differential Equation with Integral Boundary Conditions

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Abstract

Fractional calculus is a quantitative analysis of non-integer order integration and differentiation of

functions. It is an extension of integer order calculus and has always demonstrated its unique advantages and irreplaceability in practical applications. Fractional order differential equations have been applied in various fields such as biological infectious diseases, financial markets, control systems, and anomalous diffusion. At present, the existence and uniqueness of solutions to boundary value problems of fractional differential equations is one of the key research topics. This article will use functional analysis and other related tools to explore in depth nonlinear boundary value problems with integral boundary conditions. Firstly, we will discuss the Green function solution for the corresponding linear boundary value problem. Then, we will analyze the properties of the Green function and use the fixed point theorem on cones to obtain the existence of positive solutions for the boundary value problem. Finally, we will give an example to demonstrate the correctness of the results.

Keywords

Fractional Differential Equation, Integral Boundary Condition, Positive Solution, Existence, Fixed Point Theorem

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1. 引言

微积分理论自产生至今,其地位举足轻重。整数阶微积分作为描述经典物理及其相关学科理论的数学工具已被人们普遍接受,但随着科学的发展和复杂工程应用需求的增加,整数阶微积分不再是一个优秀的工具[1]-[5]。此时,分数阶微积分强势闯入人们的视野,弥补了整数阶微积分的局限性。20世纪以来,分数阶微积分理论在许多领域得到了广泛的应用和发展,如物理学、血流问题、种群动力学、系统控制、复杂粘弹性材料的经济学和力学,彰显着其独特的优势和不可替代性[6]-[8]。此外,它的非局域特性使得分数阶微分方程的求解比整数阶微分方程更复杂[9]。

边值问题是具有边界条件的微分方程的定解问题,极其重要且被广泛应用于其他领域[10] [11]。因此,近几十年来,许多学者研究了分数阶微分方程边值问题解的存在性。例如,在文献[12]中,Liu和Zhuang得到了如下具有积分边界条件的分数阶微分方程边值问题

$$\begin{cases} {}^c D^\alpha u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = \int_0^1 u(t) dt, u(1) = \int_0^1 t u(t) dt \end{cases}$$

解的存在性,其中, $0 < \alpha < 1$, $f: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ 是连续函数。文章结果基于Banach压缩原理, Leray-Schauder非线性抉择, Boyed and Wong不动点定理以及Krasnoselskii不动点定理。

在文献[13]中, Chandran等人给出了一个基于给定的受控b-Branciari度量空间的不动点方法,并且运用此方法研究了下述分数阶微分方程边值问题

$$\begin{cases} {}^c D^\alpha v(t) + h(t, v(t)) = 0, & 0 < t < 1, \\ v''(0) = v'''(0) = 0, \\ v'(0) = v(1) = \beta \int_0^1 v(s) ds \end{cases}$$

解的存在唯一性, 其中, $0 < \alpha \leq 1$, $1 < \beta < 2$, h 在 $v=0$ 时可能奇异。

此外, 分数阶微分方程边值问题的许多相关研究成果可见于文献[14]-[18]。

迄今为止, 关于分数阶微分方程边值问题解的存在性与唯一性的研究已成果颇丰, 但在实际应用中正解更有意义。因此, 本章考虑下述具有积分边界条件的分数阶微分方程边值问题

$$\begin{cases} {}^C D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u''(0) = u'''(0) = 0, \\ u'(0) = -\int_0^1 h_1(s) u(s) ds, \\ u(1) = \int_0^1 h_2(s) u(s) ds \end{cases} \quad (1)$$

正解的存在性, 其中, $3 < \alpha < 4$, $f: [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ 是连续函数, $h_i (i=1, 2): [0, 1] \rightarrow [0, \infty)$ 是连续函数。比较于上述文献, 本文的研究目标具有更一般的边界条件。

一个函数 $u: [0, 1] \rightarrow [0, \infty)$ 被称作边值问题(1)的正解, 如果 u 满足方程和边界条件且有 $u(t) > 0$, $t \in [0, 1]$ 。

2. 预备知识

首先给出与分数阶微分方程相关的一些基本定义和引理。本节中, $[\gamma]$ 表示 γ 的整数部分。

2.1. 定义 1 [19]

$[0, 1]$ 上的 $\gamma > 0$ 阶 Riemann-Liouville 分数阶积分 $I_{0+}^{\gamma} u$ 分别定义为

$$(I_{0+}^{\gamma} u)(t) := \frac{1}{\Gamma(\gamma)} \int_0^t \frac{u(s) ds}{(t-s)^{1-\gamma}}$$

其中,

$$\Gamma(\gamma) = \int_0^{+\infty} s^{\gamma-1} e^{-s} ds.$$

2.2. 定义 2 [19]

$[0, 1]$ 上的 $\gamma > 0$ 阶 Riemann-Liouville 分数阶导数 $D_{0+}^{\gamma} u$ 分别定义为

$$\begin{aligned} (D_{0+}^{\gamma} u)(t) &:= \left(\frac{d}{dt} \right)^m (I_{0+}^{m-\gamma} u)(t) \\ &= \frac{1}{\Gamma(m-\gamma)} \left(\frac{d}{dt} \right)^m \int_0^t \frac{u(s) ds}{(t-s)^{\gamma-m+1}} \end{aligned}$$

其中, $m = [\gamma] + 1$ 。

2.3. 定义 3 [19]

令 $D_{0+}^{\gamma} [u(s)](t) \equiv (D_{0+}^{\gamma} u)(t)$ 为 γ 阶的 Riemann-Liouville 分数阶导数, 那么 $[0, 1]$ 上的 γ 阶 Caputo 分数阶导数 ${}^C D_{0+}^{\gamma} u$ 定义为

$$({}^C D_{0+}^{\gamma} u)(t) := \left(D_{0+}^{\gamma} \left[u(s) - \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} s^k \right] \right)(t)$$

其中,

$$n = \begin{cases} [\gamma] + 1, & \gamma \notin N^+, \\ \gamma, & \gamma \in N^+. \end{cases} \quad (2)$$

2.4. 引理 1 [19]

假设 n 如(2)式所示且 $u \in C^n[0,1]$, 则

$$(I_{0+}^\gamma {}^C D_{0+}^\gamma u)(t) = u(t) + C_0 + C_1 t + C_2 t^2 + \cdots + C_{n-1} t^{n-1},$$

其中, $C_i \in \mathbb{R}$, $i = 0, 1, \dots, n-1$ 。

2.5. 引理 2 [20]

设 $\gamma > 0$, $\rho > 0$, $u \in C(0,1) \cap L(0,1)$, 则有 $D_{0+}^\gamma I_{0+}^\rho u(t) = I_{0+}^{\rho-\gamma} u(t)$ 成立。

2.6. 引理 3 [20]

假设 n 由(2)式所示, 则如下两个关系成立:

- (1) 对 $k \in \{0, 1, 2, \dots, n-1\}$, ${}^C D_{0+}^\gamma t^k = 0$;
- (2) 如果 $\nu > n$, 则 ${}^C D_{0+}^\gamma t^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\nu-\gamma)} t^{\nu-\gamma-1}$ 。

2.7. 定理 1 [21] [22]

设 X 是 Banach 空间, P 是 X 中的锥, Ω_1 和 Ω_2 是 X 中的有界开子集, 使得 $\theta \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$ 。

若 $T: P \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow P$ 是全连续算子且满足下列条件之一:

- (1) $\|Tx\| \leq \|x\|$, $\forall x \in P \cap \partial\Omega_1$ 且 $\|Tx\| \geq \|x\|$, $\forall x \in P \cap \partial\Omega_2$;
- (2) $\|Tx\| \geq \|x\|$, $\forall x \in P \cap \partial\Omega_1$ 且 $\|Tx\| \leq \|x\|$, $\forall x \in P \cap \partial\Omega_2$,

那么 T 在 $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$ 中有一个不动点。

2.8. 定理 2 [23]

假设 X 是一个 Banach 空间且 E 是 X 的一个闭凸集, Ω 是 E 的相对开子集且 $\theta \in \Omega$, $T: \bar{\Omega} \rightarrow E$ 是一个连续的紧算子, 则有

- (1) T 在 $\bar{\Omega}$ 上有一个不动点, 或者
- (2) 存在 $u \in \partial\Omega$ 和 $\eta \in (0,1)$ 使得 $u = \eta Tu$ 。

3. 主要结果

为方便起见, 在本章中总是记

$$P_i = \int_0^1 h_i(s) ds, \quad Q_i = \int_0^1 (1-s) h_i(s) ds, \quad W_i = \int_0^1 (1-s)^{\alpha-1} h_i(s) ds, \quad i = 1, 2,$$

并且令 $X = C[0,1]$ 为定义在 $[0,1]$ 上的 Banach 空间且具有范数 $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$ 。

3.1. 引理 4

令 $(1-Q_1)(1-P_2) \neq P_1 Q_2$, 则对任意给定的 $y \in X$, 边值问题

$$\begin{cases} {}^C D_{0+}^\alpha u(t) + y(t) = 0, 0 < t < 1, \\ u''(0) = u'''(0) = 0, \\ u'(0) = -\int_0^1 h_1(s)u(s)ds, \\ u(1) = \int_0^1 h_2(s)u(s)ds \end{cases} \quad (3)$$

有唯一解

$$u(t) = \int_0^1 K(t,s)y(s)ds, t \in [0,1],$$

其中

$$K(t,s) = G(t,s) + \sum_{i=1}^2 \phi_i(t) \int_0^1 G(\tau,s)h_i(\tau)d\tau, (t,s) \in [0,1] \times [0,1] \quad (4)$$

且

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ (1-s)^{\alpha-1}, & 0 \leq t \leq s \leq 1, \end{cases} \quad (5)$$

$$\phi_1(t) = \frac{(1-P_2)(1-t) + Q_2}{(1-Q_1)(1-P_2) - P_1Q_2}, t \in [0,1],$$

$$\phi_2(t) = \frac{P_1(1-t) + (1-Q_1)}{(1-Q_1)(1-P_2) - P_1Q_2}, t \in [0,1],$$

证明 由引理 1 可知边值问题(3)中的方程等价于积分方程

$$\begin{aligned} u(t) &= -I_{0+}^\alpha y(t) - C_0 - C_1 t - C_2 t^2 - C_3 t^3 \\ &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds + Y_0 + Y_1 t + Y_2 t^2 + Y_3 t^3, \end{aligned}$$

其中常数 $Y_i \in \mathbb{R}, Y_i = -C_i, i = 0, 1, 2, 3$ 。另外, 由引理 2 和数学分析的相关知识可得

$$u'(t) = -\frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} y(s)ds + Y_1 + 2Y_2 t + 3Y_3 t^2,$$

$$u''(t) = -\frac{1}{\Gamma(\alpha-2)} \int_0^t (t-s)^{\alpha-3} y(s)ds + 2Y_2 + 6Y_3 t,$$

$$u'''(t) = -\frac{1}{\Gamma(\alpha-3)} \int_0^t (t-s)^{\alpha-4} y(s)ds + 6Y_3.$$

由边值条件 $u''(0) = u'''(0) = 0$ 可得 $Y_2 = Y_3 = 0$, 则

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds + Y_0 + Y_1 t, \quad (6)$$

又由边值条件 $u'(0) = -\int_0^1 h_1(s)u(s)ds, u(1) = \int_0^1 h_2(s)u(s)ds$ 可得

$$u'(0) = \Upsilon_1 = -\int_0^1 h_1(s)u(s)ds \quad (7)$$

$$u(1) = -\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)ds + \Upsilon_0 + \Upsilon_1 = \int_0^1 h_2(s)u(s)ds$$

因此,

$$\Upsilon_0 = \int_0^1 h_1(s)u(s)ds + \int_0^1 h_2(s)u(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)ds. \quad (8)$$

故结合式子(6), (7)和(8)可得

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds + \int_0^1 h_1(s)u(s)ds \\ &\quad + \int_0^1 h_2(s)u(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)ds - t \int_0^1 h_1(s)u(s)ds \\ &= \int_0^1 G(t,s)y(s)ds + (1-t) \int_0^1 h_1(s)u(s)ds + \int_0^1 h_2(s)u(s)ds, t \in [0,1]. \end{aligned} \quad (9)$$

接下来, 对方程(9)进行积分可得

$$\begin{aligned} \int_0^1 h_1(s)u(s)ds &= \int_0^1 h_1(s) \int_0^1 G(s,\tau)y(\tau)d\tau ds + \int_0^1 (1-s)h_1(s) \int_0^1 h_1(\tau)u(\tau)d\tau ds \\ &\quad + \int_0^1 h_1(s) \int_0^1 h_2(\tau)u(\tau)d\tau ds, \\ \int_0^1 h_2(s)u(s)ds &= \int_0^1 h_2(s) \int_0^1 G(s,\tau)y(\tau)d\tau ds + \int_0^1 (1-s)h_2(s) \int_0^1 h_1(\tau)u(\tau)d\tau ds \\ &\quad + \int_0^1 h_2(s) \int_0^1 h_2(\tau)u(\tau)d\tau ds. \end{aligned}$$

接着对上述两式进行整理可得

$$(1-Q_1) \int_0^1 h_1(s)u(s)ds - P_1 \int_0^1 h_2(s)u(s)ds = \int_0^1 h_1(s) \int_0^1 G(s,\tau)y(\tau)d\tau ds \quad (10)$$

和

$$-Q_2 \int_0^1 h_1(s)u(s)ds + (1-P_2) \int_0^1 h_2(s)u(s)ds = \int_0^1 h_2(s) \int_0^1 G(s,\tau)y(\tau)d\tau ds. \quad (11)$$

结合(10)式和(11)式可得

$$\begin{aligned} \int_0^1 h_1(s)u(s)ds &= \frac{(1-P_2) \int_0^1 h_1(s) \int_0^1 G(s,\tau)y(\tau)d\tau ds + P_1 \int_0^1 h_2(s) \int_0^1 G(s,\tau)y(\tau)d\tau ds}{(1-Q_1)(1-P_2) - P_1 Q_2}, \\ \int_0^1 h_2(s)u(s)ds &= \frac{Q_2 \int_0^1 h_1(s) \int_0^1 G(s,\tau)y(\tau)d\tau ds + (1-Q_1) \int_0^1 h_2(s) \int_0^1 G(s,\tau)y(\tau)d\tau ds}{(1-Q_1)(1-P_2) - P_1 Q_2}. \end{aligned}$$

综上所述可得

$$\begin{aligned}
u(t) &= \int_0^1 G(t,s)y(s)ds + (1-t)\int_0^1 h_1(s)u(s)ds + \int_0^1 h_2(s)u(s)ds \\
&= \int_0^1 G(t,s)y(s)ds + \sum_{i=1}^2 \phi_i(t)\int_0^1 h_i(s)\int_0^1 G(\tau,s)y(\tau)d\tau ds \\
&= \int_0^1 G(t,s)y(s)ds + \sum_{i=1}^2 \phi_i(t)\int_0^1 h_i(\tau)\int_0^1 G(\tau,s)y(s)dsd\tau \\
&= \int_0^1 G(t,s)y(s)ds + \sum_{i=1}^2 \phi_i(t)\int_0^1 y(s)\int_0^1 G(\tau,s)h_i(\tau)d\tau ds \\
&= \int_0^1 \left(G(t,s) + \sum_{i=1}^2 \phi_i(t)\int_0^1 G(\tau,s)h_i(\tau)d\tau \right) y(s)ds \\
&= \int_0^1 K(t,s)y(s)ds, t \in [0,1].
\end{aligned}$$

3.2. 引理 5

$G(t,s)$ 满足如下性质:

- (1) $G(t,s)$ 在 $[0,1] \times [0,1]$ 上连续;
- (2) $\frac{(1-t)^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} \leq G(t,s) \leq \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}, (t,s) \in [0,1] \times [0,1]$ 。

证明 (1) 显然成立;

(2) 既然右端的不等式是显然的, 故只需要证明左端的不等式成立即可。当 $0 \leq s \leq t \leq 1$ 时,

$$\begin{aligned}
G(t,s) &= \frac{(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} \geq \frac{(1-s)^{\alpha-1} - (t-ts)^{\alpha-1}}{\Gamma(\alpha)} \\
&= \frac{(1-t^{\alpha-1})(1-s)^{\alpha-1}}{\Gamma(\alpha)} \geq \frac{(1-t)^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)},
\end{aligned}$$

$$\text{即 } G(t,s) \geq \frac{(1-t)^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}, (t,s) \in [0,1] \times [0,1].$$

在本章的后续行文中, 总是认定下面的假设成立:

(H₁) $Q_1 < 1, P_2 < 1, (1-Q_1)(1-P_2) > P_1 Q_2$ 。

3.3. 引理 6

$K(t,s)$ 满足如下性质:

- (1) $K(t,s)$ 在 $[0,1] \times [0,1]$ 上连续;
- (2) $\frac{q(t)(1-s)^{\alpha-1}}{\Gamma(\alpha)} \leq K(t,s) \leq \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}, (t,s) \in [0,1] \times [0,1]$,

其中

$$q(t) = (1-t)^{\alpha-1} + m, t \in [0,1], m = \sum_{i=1}^2 W_i \min_{t \in [0,1]} \phi_i(t), M = 1 + \sum_{i=1}^2 P_i \max_{t \in [0,1]} \phi_i(t).$$

证明 (1) 显然成立;

(2) 一方面, 由引理 5 的第二条性质和式子(4)可得

$$\begin{aligned}
K(t,s) &= G(t,s) + \sum_{i=1}^2 \phi_i(t) \int_0^1 G(\tau,s) h_i(\tau) d\tau \\
&\leq \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \sum_{i=1}^2 \phi_i(t) \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h_i(\tau) d\tau \\
&= \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \left(1 + \sum_{i=1}^2 P_i \phi_i(t) \right) \\
&\leq \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \left(1 + \sum_{i=1}^2 P_i \max_{t \in [0,1]} \phi_i(t) \right) \\
&= \frac{M(1-s)^{\alpha-1}}{\Gamma(\alpha)}, (t,s) \in [0,1] \times [0,1].
\end{aligned}$$

同理,

$$\begin{aligned}
K(t,s) &= G(t,s) + \sum_{i=1}^2 \phi_i(t) \int_0^1 G(\tau,s) h_i(\tau) d\tau \\
&\geq \frac{(1-t)^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \sum_{i=1}^2 \phi_i(t) \int_0^1 \frac{(1-\tau)^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} h_i(\tau) d\tau \\
&= \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \left((1-t)^{\alpha-1} + \sum_{i=1}^2 W_i \phi_i(t) \right) \\
&\geq \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \left((1-t)^{\alpha-1} + \sum_{i=1}^2 W_i \min_{t \in [0,1]} \phi_i(t) \right) \\
&= \frac{q(t)(1-s)^{\alpha-1}}{\Gamma(\alpha)}, (t,s) \in [0,1] \times [0,1].
\end{aligned}$$

现在, 假设

$$P = \left\{ u \in X \mid u(t) \geq \frac{q(t)}{M} \|u\|, t \in [0,1] \right\},$$

显然, P 是 X 中的一个锥。定义 P 上的算子 T 为

$$Tu(t) = \int_0^1 K(t,s) f(s, u(s)) ds, t \in [0,1]. \quad (12)$$

由引理 4 可知, 算子 T 的不动点恰好是边值问题(1)的解。

3.4. 引理 7

$T: P \rightarrow P$ 是全连续的。

证明 首先, 证明 $T: P \rightarrow P$ 。由 $K(t,s)$ 的性质和函数 f 的定义可得对于任意 $u \in P$, 算子 T 是非负连续的。又对于任意 $u \in P$, 由引理 6 可得

$$0 \leq Tu(t) = \int_0^1 K(t,s) f(s, u(s)) ds \leq \frac{M}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, u(s)) ds, t \in [0,1],$$

则

$$\|Tu\| \leq \frac{M}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, u(s)) ds.$$

结合引理 6 中的第二条性质可以得出

$$\begin{aligned} Tu(t) &\geq \frac{q(t)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, u(s)) ds \\ &\geq \frac{q(t)}{M} \|Tu\|, \quad t \in [0, 1], \end{aligned}$$

因而 $Tu \in P$, 即 $T: P \rightarrow P$ 成立。

算子 T 的一致有界性和等度连续性易证。因此, 由 Arzela-Ascoli 定理, 算子 $T: P \rightarrow P$ 是全连续的。

3.5. 定理 3

假设存在三个连续函数 $a: [0, 1] \rightarrow [0, \infty)$, $g_1: [0, \infty) \rightarrow (0, \infty)$ 和 $g_2: [0, \infty) \rightarrow [0, \infty)$ 且 g_1 是不增的, $\frac{g_2}{g_1}$ 是不减的, 使得

$$f(t, x) = a(t)(g_1(x) + g_2(x)), \quad (t, x) \in [0, 1] \times [0, \infty),$$

并且假设存在 $r_2 > r_1 > 0$ 使得

$$a(t)g_1\left(\frac{r_1}{M}q(t)\right) \leq \frac{\Gamma(\alpha+1)}{M\left(1+\frac{g_2(r_1)}{g_1(r_1)}\right)}r_1, \quad t \in [0, 1] \quad (13)$$

和

$$a(t)\left[1+\frac{g_2\left(\frac{r_2}{M}q(t)\right)}{g_1\left(\frac{r_2}{M}q(t)\right)}\right] \geq \frac{\Gamma(\alpha+1)}{(1+m)g_1(r_2)}r_2, \quad t \in [0, 1], \quad (14)$$

则边值问题(1)存在一个正解 u^* 且满足 $r_1 \leq \|u^*\| \leq r_2$ 。

证明 令 $\Omega_i = \{u \in X \mid \|u\| < r_i\}, i = 1, 2$ 。

一方面, 如果 $u \in P \cap \partial\Omega_1$, 则 $\|u\| = r_1$ 且对于 $t \in [0, 1]$, 有 $r_1 \geq u(t) \geq \frac{q(t)}{M}\|u\| = \frac{r_1}{M}q(t)$ 。由式子(12)和(13)可以得出

$$\begin{aligned} Tu(t) &= \int_0^1 K(t, s) f(s, u(s)) ds \\ &= \int_0^1 K(t, s) a(s) [g_1(u(s)) + g_2(u(s))] ds \\ &= \int_0^1 K(t, s) a(s) g_1(u(s)) \left[1 + \frac{g_2(u(s))}{g_1(u(s))}\right] ds \\ &\leq \frac{M}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} a(s) g_1\left(\frac{r_1}{M}q(s)\right) \left[1 + \frac{g_2(r_1)}{g_1(r_1)}\right] ds \\ &\leq \frac{\Gamma(\alpha+1)}{M\left[1+\frac{g_2(r_1)}{g_1(r_1)}\right]} \frac{M}{\Gamma(\alpha)} \left[1 + \frac{g_2(r_1)}{g_1(r_1)}\right] r_1 \int_0^1 (1-s)^{\alpha-1} ds \\ &= r_1, \quad t \in [0, 1], \end{aligned}$$

因此,

$$\|Tu\| \leq \|u\|, \quad \forall u \in P \cap \partial\Omega_1. \quad (15)$$

另一方面, 如果 $u \in P \cap \partial\Omega_2$, 则 $\|u\| = r_2$ 且对于 $t \in [0, 1]$, 有 $r_2 \geq u(t) \geq \frac{q(t)}{M} \|u\| = \frac{r_2}{M} q(t)$ 。由式子(12)和(14)可以得出

$$\begin{aligned} Tu(0) &= \int_0^1 K(0, s) f(s, u(s)) ds \\ &= \int_0^1 K(0, s) a(s) [g_1(u(s)) + g_2(u(s))] ds \\ &\geq \frac{q(0)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} a(s) g_1(u(s)) \left[1 + \frac{g_2(u(s))}{g_1(u(s))} \right] ds \\ &\geq \frac{q(0) g_1(r_2)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} a(s) \left[1 + \frac{g_2\left(\frac{r_2}{M} q(s)\right)}{g_1\left(\frac{r_2}{M} q(s)\right)} \right] ds \\ &\geq \frac{\Gamma(\alpha+1)}{(1+m) g_1(r_2)} \frac{(1+m) g_1(r_2)}{\Gamma(\alpha)} r_2 \int_0^1 (1-s)^{\alpha-1} ds \\ &= r_2, \end{aligned}$$

因而,

$$\|Tu\| = \max_{0 \leq t \leq 1} |Tu(t)| \geq |Tu(0)| \geq \|u\|, \quad \forall u \in P \cap \partial\Omega_2. \quad (16)$$

综上所述, 结合不等式(15)和(16), 由定理 1 可以推断出算子 T 有一个不动点 $u^* \in P \cap (\overline{\Omega_2} \setminus \Omega_1)$ 。显然, 这个不动点是边值问题(1)的一个正解且满足 $r_1 \leq \|u^*\| \leq r_2$ 。

3.6. 定理 4

假设存在两个连续函数 $b: [0, 1] \rightarrow [0, \infty)$ 和 $\varphi: [0, \infty) \rightarrow [0, \infty)$ 使得

$$f(t, x) \leq b(t) \varphi(x), \quad (t, x) \in [0, 1] \times [0, \infty)$$

且 $f(t, 0) \neq 0, t \in (0, 1)$ 以及存在 $\tilde{r} > 0$ 使得

$$\int_0^1 (1-t)^{\alpha-1} b(t) \varphi^*(t) dt < \frac{\Gamma(\alpha)}{M} \tilde{r},$$

其中, $\varphi^*(t) = \max_{\frac{\tilde{r}}{M} q(t) \leq x \leq \tilde{r}} \varphi(x), t \in [0, 1]$, 则边值问题(1)存在一个正解。

证明 令 $\Omega = \{u \in X \mid \|u\| < \tilde{r}\} \cap P$ 。由引理 7 可知算子 $T: P \rightarrow P$ 是全连续的, 因而 $T: \bar{\Omega} \rightarrow P$ 是连续且紧的。

接下来, 证明定理 2 中的(1)成立, 只需证明定理中的(2)不成立。

反设定理 2 中的(2)成立, 则存在 $u \in \partial\Omega$ 和 $\eta \in (0, 1)$ 使得 $u = \eta Tu$ 。对于 $u \in \partial\Omega$, 有

$\frac{\tilde{r}}{M} q(t) \leq u(t) \leq \tilde{r}, t \in [0, 1]$ 。又根据 φ^* 的定义可以得到 $\varphi(u(t)) \leq \varphi^*(t), t \in [0, 1]$ 。现在, 依照引理 6 和式子(12)可推断

$$\begin{aligned}
u(t) &= (\eta Tu)(t) \leq Tu(t) = \int_0^1 K(t,s) f(s, u(s)) ds \\
&\leq \frac{M}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} b(s) \varphi(u(s)) ds \\
&\leq \frac{M}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} b(s) \varphi^*(s) ds \\
&< \frac{M}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{M} \tilde{r} = \tilde{r}, t \in [0,1],
\end{aligned}$$

因此, $\|u\| < \tilde{r}$ 与 $u \in \partial\Omega$ 矛盾。

综上所述, 由定理 2 可以推出 T 有一个不动点 $u^* \in \bar{\Omega}$, 即为边值问题(1)的一个非负解。又因为 $f(t, 0) \neq 0, t \in (0, 1)$, 则零函数不是边值问题(1)的解, 故 u^* 是边值问题(1)的一个正解。

3.7. 注

为确保文章的正确性, 下面对结论即定理 3 和定理 4 进行验证, 故只讨论实例分析中边值问题正解的存在性。

4. 实例分析

4.1. 实例 1

考虑下述边值问题

$$\begin{cases} {}^c D_{0+}^{\frac{7}{2}} u(t) + \cos \frac{\pi}{3} t \left(\frac{1}{u(t)+1} + u(t) \right) = 0, & 0 < t < 1, \\ u''(0) = u'''(0) = 0, \\ u'(0) = -\int_0^1 s u(s) ds, \\ u(1) = \int_0^1 (1-s) u(s) ds. \end{cases} \quad (17)$$

由于 $\alpha = \frac{7}{2}, h_1(s) = s, h_2(s) = 1-s, s \in [0, 1]$, 经过直接的计算可以得到

$$P_1 = P_2 = \frac{1}{2}, Q_1 = \frac{1}{6}, Q_2 = \frac{1}{3}, W_1 = \frac{4}{63}, W_2 = \frac{2}{9},$$

$$\phi_1(t) = -2t + \frac{10}{3}, \phi_2(t) = -2t + \frac{16}{3}, t \in [0, 1], m = \frac{52}{63}, M = \frac{16}{3}, q(t) = (1-t)^{\frac{5}{2}} + \frac{52}{63}, t \in [0, 1],$$

显然, 假设 (H_1) 成立。又因为 $f(t, x) = \cos \frac{\pi}{3} t \left(\frac{1}{x+1} + x \right), (t, x) \in [0, 1] \times [0, \infty)$, 则 $a(t) = \cos \frac{\pi}{3} t$,

$g_1(x) = \frac{1}{x+1}, g_2(x) = x$ 。现在取 $r_1 = \frac{1}{50}, r_2 = 8.5 > r_1$, 经过计算可得

$$\frac{\Gamma(\alpha+1)}{M \left(1 + \frac{g_2(r_1)}{g_1(r_1)} \right)} r_1 = \frac{9}{2} \times \frac{105\sqrt{\pi}}{16} \times \frac{0.02}{1.0204} \approx 1.0259,$$

$$\frac{\Gamma(\alpha+1)}{(1+m)g_1(r_2)} r_2 = \frac{63}{115} \times \frac{105\sqrt{\pi}}{16} \times 80.75 \approx 514.5523.$$

因此,

$$\begin{aligned}
 a(t)g_1\left(\frac{r_1}{M}q(t)\right) &= \frac{\cos\frac{\pi}{3}t}{\frac{9r_2}{2}q(t)+1} \leq \frac{1}{\frac{13}{175}+1} \approx 0.9309 < \frac{\Gamma(\alpha+1)}{M\left(1+\frac{g_2(r_1)}{g_1(r_1)}\right)}r_1 \approx 1.0259, \\
 a(t)\left[1+\frac{g_2\left(\frac{r_2}{M}q(t)\right)}{g_1\left(\frac{r_2}{M}q(t)\right)}\right] &= \cos\frac{\pi}{3}t\left[1+\left(\frac{9r_2}{2}q(t)\right)^2+\frac{9r_2}{2}q(t)\right] \\
 &\geq \cos\frac{\pi}{3}t\left(1+\frac{676}{49}r_2^2+\frac{26}{7}r_2\right) \\
 &\geq \frac{1}{2}\left(1+\frac{676}{49}\times 8.5^2+\frac{26}{7}\times 8.5\right) \approx 514.6632 \\
 &> \frac{\Gamma(\alpha+1)}{(1+m)g_1(r_2)}r_2 \approx 514.5523,
 \end{aligned}$$

这表明定理 3 中的全部条件均满足。

因此, 由定理 3 可知边值问题(17)存在一个正解 u^* 且满足 $\frac{1}{50} \leq u^* \leq 8.5$ 。

4.2. 实例 2

考虑下述边值问题

$$\begin{cases} {}^C D_{0+}^{3.6}u(t) + t + \frac{t}{2}(u(t))^2 = 0, 0 < t < 1, \\ u''(0) = u'''(0) = 0, \\ u'(0) = -\frac{1}{3}\int_0^1 s^2 u(s) ds, \\ u(1) = \frac{2}{3}\int_0^1 s^2 u(s) ds. \end{cases} \quad (18)$$

由于 $\alpha = 3.6, h_1(s) = \frac{1}{3}s^2, h_2(s) = \frac{2}{3}s^2, s \in [0, 1]$, 经过计算可得

$$\begin{aligned}
 P_1 &= \frac{1}{9}, P_2 = \frac{2}{9}, Q_1 = \frac{1}{36}, Q_2 = \frac{1}{18}, W_1 \approx 0.0072, W_2 \approx 0.0144, \\
 \phi_1(t) &= -1.3334t + 1.1111, \phi_2(t) = -0.1481t + 1.4444, t \in [0, 1], \\
 m &\approx 0.0192, M \approx 1.4445, q(t) = (1-t)^{2.6} + 0.0192, t \in [0, 1].
 \end{aligned}$$

显然, 假设 (H_1) 成立。又因为 $f(t, x) = t\left(1 + \frac{x^2}{2}\right), (t, x) \in [0, 1] \times [0, \infty)$, 则 $f(t, 0) = t \neq 0, t \in (0, 1)$ 且

$$f(t, x) = t\left(1 + \frac{x^2}{2}\right) \leq \frac{1+t^2}{2}\left(1 + \frac{x^2}{2}\right), (t, x) \in [0, 1] \times [0, \infty), \text{ 故 } b(t) = \frac{1+t^2}{2}, t \in [0, 1], \varphi(x) = 1 + \frac{x^2}{2}, x \in [0, \infty)。$$

选择 $\tilde{r} = \frac{1}{10}$, 则 $\varphi^*(t) = \max_{\frac{\tilde{r}}{M}q(t) \leq x \leq \tilde{r}} \varphi(x) = \max_{0.0692q(t) \leq x \leq \frac{1}{10}} \left(1 + \frac{x^2}{2}\right) = \frac{201}{200}, t \in [0, 1]$ 。因而, 有

$$\int_0^1 (1-t)^{\alpha-1} b(t) \varphi^*(t) dt = \frac{201}{200} \int_0^1 (1-t)^{2.6} \left(\frac{1+t^2}{2} \right) dt = \frac{201}{400} \int_0^1 (1-t)^{2.6} (1+t^2) dt \approx 0.1504,$$

$$\frac{\Gamma(\alpha)}{M} \tilde{r} = \frac{\Gamma(3.6)}{1.4445} \times \frac{1}{10} \approx 0.2574,$$

显然, 定理 4 全部条件均满足。

因此, 由定理 4 可知边值问题(18)存在一个正解。

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