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基于区间值毕达哥拉斯Sugeno-Weber Softmax算子的WASPAS多属性决策方法

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摘 要

针对属性值为区间值毕达哥拉斯模糊数且权重信息完全未知的决策问题，本文提出基于Sugeno-Weber Softmax算子的区间值毕达哥拉斯模糊多属性决策方法。首先，定义了基于Sugeno-Weber三角模的区间值毕达哥拉斯模糊数运算法则。其次，为考虑属性间的优先关系，提出四种区间值毕达哥拉斯模糊Sugeno-Weber Softmax平均和几何集成算子并讨论所提算子的幂等性、有界性和单调性等性质。为确定属性的权重信息，提出基于区间值毕达哥拉斯模糊PSI (Preference Selection Index)方法确定属性的客观权重。进一步提出基于Sugeno-Weber Softmax算子的区间值毕达哥拉斯模糊WASPAS (Weighted Aggregated Sum Product Assessment)方法确定备选方案的排序。通过实际案例验证所提方法的有效性及合理性，并由比较分析和参数分析讨论所提方法的鲁棒性和优越性。

关键词

多属性决策，区间值毕达哥拉斯模糊集，Sugeno-Weber Softmax算子，WASPAS方法

WASPAS Multi-Attribute Decision-Making Method Based on Interval Valued Pythagorean Sugeno-Weber Softmax Operator

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Abstract

This paper addresses decision-making problems characterized by attribute values expressed as interval-valued Pythagorean fuzzy numbers (IVPFNs) and completely unknown weight information. We propose a novel interval-valued Pythagorean fuzzy multi-attribute decision-making (MADM) methodology based on the Sugeno-Weber Softmax operator. Firstly, arithmetic operations for IVPFNs are defined utilizing the Sugeno-Weber t-norm and t-conorm. Secondly, to account for priority relationships among attributes, four Pythagorean fuzzy Sugeno-Weber Softmax averaging and geometric aggregation operators are introduced. Fundamental properties of these operators, including idempotency, boundedness, and monotonicity, are rigorously investigated. To resolve the unknown weight issue, an objective weight determination method for attributes is developed using the Pythagorean fuzzy Preference Selection Index approach. Furthermore, an enhanced Pythagorean fuzzy WASPAS (Weighted Aggregated Sum Product Assessment) method, integrated with the proposed Sugeno-Weber Softmax operators, is presented to determine the ranking of alternatives. The effectiveness and rationality of the proposed methodology are validated through an empirical case study. Comparative analysis and parameter sensitivity analysis further demonstrate its superior robustness and performance over existing methods.

Keywords

Multi-Attribute Decision-Making, Interval-Valued Pythagorean Fuzzy Set, Sugeno-Weber Softmax Aggregation Operator, WASPAS Method

1. 引言

现实决策问题的复杂性、模糊性及决策者认知能力的不确定性使得决策者难以用精确的数值表达其偏好信息。为此，诸多学者从不确定性角度出发，通过模糊数、模糊语言和区间数等模型表征实际问题中存在的模糊不确定性。1965 年，Zadeh [1]首次提出模糊集理论，通过隶属度函数刻画元素属于给定目标的模糊性。此后，Atanassov [2]从隶属度、非隶属度和犹豫度的角度提出直觉模糊集理论以更加精确的表征模糊不确定现象。但直觉模糊集在应用时具有极高的限制条件，使得决策者并不能完全表达其不确定性评估信息。为增大决策者表达偏好的范围和自由度，Yager [3] [4]提出毕达哥拉斯模糊集理论，将直觉模糊集的隶属度和非隶属度和小于等于 1 的条件扩充至隶属度和非隶属度的平方和小于等于 1。毕达哥拉斯模糊集的提出使得其在决策分析和信息融合等领域产生广泛的应用前景并取得丰硕的成果[5]-[8]。但决策者在表达隶属度和非隶属度时往往存在不确定性而难以给出准确的隶属度和非隶属度，因此 Peng 和 Yang [9]提出区间值毕达哥拉斯模糊集的概念，通过区间值来表达决策者提供隶属度和非隶属度的深层次不确定性。此后，诸多学者针对区间值毕达哥拉斯模糊集的理论和应用展开了广泛研究[10] [11]。

集成算子是模糊信息融合过程的重要组成部分并被广泛用于模糊信息融合和决策方法构建过程中。集成算子的基础是基于不同的三角模定义不同模糊环境下的基本运算法则，然后基于优先算子、幂算子和考虑属性间相互关系的 Hamy 均值、Heronian 均值等聚合函数提出一系列新的集成算子。Sugeno-Weber [12]作为一种新型的三角模且具有极高的灵活性，目前已被拓展到不同的模糊环境中定义集成算子并构建决策方法[13]-[16]。Softmax 函数是机器学习和深度学习中多分类问题的核心函数，它可以将一个包含

任意实数的向量压缩成一个概率分布[17]。考虑 Softmax 函数的优势，目前已被用于决策分析中构建集成算子以考虑融合数据间的优先关系[17] [18]。在多属性决策方法中，WASPAS 作为一种融合加权和和加权积模型的决策方法，同时兼顾了加权和和加权积模型的优势并获得更佳合理的决策分析结果[19]。WASPAS 方法自提出以来备受学者关注并将其与模糊集理论和不确定性理论结合，提出了不同特征下的扩展 WASPAS 方法，并被广泛应用于不同领域的实际决策中[20]-[22]。据文献所知，目前尚未有关于 Sugeno-Weber Softmax 集成算子，WASPAS 方法在区间值毕达哥拉斯模糊环境中的研究。

基于上述讨论和分析，本文针对属性值为区间值毕达哥拉斯模糊数且权重信息完全未知的决策问题，首先定义区间值毕达哥拉斯模糊数 Sugeno-Weber 运算法则并联合 Softmax 函数提出四种区间值毕达哥拉斯模糊 Sugeno-Weber Softmax 集成算子，同时讨论所提算子的幂等性、有界性和单调性等性质。为确定属性的权重，基于区间值毕达哥拉斯模糊数得分函数，提出改进的 PSI 方法[23]确定属性的客观权重。进一步提出基于区间值毕达哥拉斯模糊 Sugeno-Weber Softmax 算子的 WASPAS 方法确定备选方案的优先级。

2. 预备知识

定义 1 [9]设 Y 为一个非空经典集合，则区间值毕达哥拉斯模糊集 F 表示为：

$$F=\left\{\left\langle y_i,\left(\left[\alpha_F^-(y_i),\alpha_F^+(y_i)\right],\left[\beta_F^-(y_i),\beta_F^+(y_i)\right]\right)\right\rangle \middle| y_i\in Y\right\}, \tag{1}$$

其中 $\alpha_F^-(y_i),\alpha_F^+(y_i)\in[0,1]$， $\beta_F^-(y_i),\beta_F^+(y_i)\in[0,1]$ 且 $0\le\left(\alpha_F^+(y_i)\right)^2+\left(\beta_F^+(y_i)\right)^2\le 1$。$\alpha_F(y_i)=\left[\alpha_F^-(y_i),\alpha_F^+(y_i)\right]$ 和 $\beta_F(y_i)=\left[\beta_F^-(y_i),\beta_F^+(y_i)\right]$ 分别表示元素 $y_i\in Y$ 的区间隶属度和非隶属度。$\pi_F(y_i)=\left[\pi_F^-(y_i),\pi_F^+(y_i)\right]$ 表示元素 $y_i\in Y$ 的区间犹豫度，其中 $\pi_F^-(y_i)=\sqrt{1-\left(\alpha_F^+(y_i)\right)^2-\left(\beta_F^+(y_i)\right)^2}$，$\pi_F^+(y_i)=\sqrt{1-\left(\alpha_F^-(y_i)\right)^2-\left(\beta_F^-(y_i)\right)^2}$。为方便起见，称 $\kappa=\left(\left[\alpha_\kappa^-,\alpha_\kappa^+\right],\left[\beta_\kappa^-,\beta_\kappa^+\right]\right)$ 为一个区间值毕达哥拉斯模糊数(IVPFN)。

定义 2 [9]设 $\kappa_1=\left(\left[\alpha_{\kappa_1}^-,\alpha_{\kappa_1}^+\right],\left[\beta_{\kappa_1}^-,\beta_{\kappa_1}^+\right]\right)$ 为一个 IVPFN。则其得分函数和精确函数分别定义为：

$$SF(\kappa_1)=\frac{1}{2}\left(\frac{1}{2}\left(\left(\alpha_{\kappa_1}^-\right)^2+\left(\alpha_{\kappa_1}^+\right)^2-\left(\beta_{\kappa_1}^-\right)^2-\left(\beta_{\kappa_1}^+\right)^2\right)+1\right), \tag{2}$$

$$AF(\kappa_1)=\frac{1}{2}\left(\left(\alpha_{\kappa_1}^-\right)^2+\left(\alpha_{\kappa_1}^+\right)^2-\left(\beta_{\kappa_1}^-\right)^2-\left(\beta_{\kappa_1}^+\right)^2\right). \tag{3}$$

定义 3 [9]设 $\kappa_1=\left(\left[\alpha_{\kappa_1}^-,\alpha_{\kappa_1}^+\right],\left[\beta_{\kappa_1}^-,\beta_{\kappa_1}^+\right]\right)$ 和 $\kappa_2=\left(\left[\alpha_{\kappa_2}^-,\alpha_{\kappa_2}^+\right],\left[\beta_{\kappa_2}^-,\beta_{\kappa_2}^+\right]\right)$ 为两个 IVPFNs。若 $SF(\kappa_1)>SF(\kappa_2)$，则 $\kappa_1\succ\kappa_2$；若 $SF(\kappa_1)=SF(\kappa_2)$ 且 $AF(\kappa_1)>AF(\kappa_2)$，则 $\kappa_1\succ\kappa_2$；若 $SF(\kappa_1)=SF(\kappa_2)$ 且 $AF(\kappa_1)<AF(\kappa_2)$，则 $\kappa_1\prec\kappa_2$；若 $SF(\kappa_1)=SF(\kappa_2)$ 且 $AF(\kappa_1)=AF(\kappa_2)$，则 $\kappa_1\sim\kappa_2$。

定义 4 [17]逻辑函数 Softmax 定义如下：

$$\psi^{\xi}(j,T_1,T_2,\cdots,T_n)=\psi_j^{\xi}=\exp\left(T_j/\xi\right)\Big/\left(\sum_{j=1}^{n}\exp\left(T_j/\xi\right)\right),\xi>0 \tag{4}$$

其中 ξ 为参数，对任意区间值毕达哥拉斯模糊数，$T_j=\prod_{l=1}^{j-1}SF(\kappa_j),(j=2,3,\cdots,n)$，$T_1=1$ 且 $SF(\kappa_j)$ 是 κ_j 的得分函数。

定义 5 [12]. Sugeno-Weber T-范数 $\left(T_{sw}^{\aleph}\right)_{\aleph\in[0,\infty)}$ 和 S-范数 $\left(S_{sw}^{\aleph}\right)_{\aleph\in[0,\infty)}$ 定义如下：

$$T_{sw}^{\aleph}(a,b)=\begin{cases}T_D(a,b),\text{if }\aleph=-1\\ \max\left(0,\dfrac{a+b-1+\aleph ab}{1+\aleph}\right),\aleph\in(-1,+\infty)\\ T_P(a,b),\text{if }\aleph=+\infty\end{cases},\quad S_{sw}^{\aleph}(a,b)=\begin{cases}S_D(a,b),\text{if }\aleph=-1\\ \min\left(1,a+b-\dfrac{\aleph}{1+\aleph}ab\right),\aleph\in(-1,+\infty)\\ S_P(a,b),\text{if }\aleph=+\infty\end{cases}$$

其中 $T_D(a,b)$ 和 $S_D(a,b)$ 表示 Drastic T-范数和 S-范数，$T_P(a,b)$ 和 $S_P(a,b)$ 分别表示和 Product T-范数和 S-范数。

3. 区间值毕达哥拉斯模糊环境下的 Sugeno-Weber 算子

本节定义区间值毕达哥拉斯模糊 Sugeno-Weber 运算法则。基于 Softmax 函数提出几种区间值毕达哥拉斯模糊 Sugeno-Weber Softmax 集成算子并讨论所提算子的相关性质。Ω 表示区间值毕达哥拉斯模糊数集合，$\varpi_j\left(j=1(1)n\right)$ 是区间值毕达哥拉斯模糊数 κ_j 的权重且满足 $\sum_{j=1}^{n}\varpi_j=1,\varpi_j\in[0,1]$。

3.1. 基于 Sugeno-Weber 模的区间值毕达哥拉斯模糊数运算法则

基于 Sugeno-Weber 模的区间值毕达哥拉斯模糊运算法则定义如下。

定义 6 设 $\kappa_1=\left(\left[\alpha_{\kappa_1}^-,\alpha_{\kappa_1}^+\right],\left[\beta_{\kappa_1}^-,\beta_{\kappa_1}^+\right]\right)$ 和 $\kappa_2=\left(\left[\alpha_{\kappa_2}^-,\alpha_{\kappa_2}^+\right],\left[\beta_{\kappa_2}^-,\beta_{\kappa_2}^+\right]\right)$ 为两个 IVPFNs。则：

1) $$\kappa_1\oplus_{sw}\kappa_2=\left(\left[\begin{array}{l}\sqrt{\left(\alpha_{\kappa_1}^-\right)^2+\left(\alpha_{\kappa_{12}}^-\right)^2-\dfrac{\aleph}{1+\aleph}\left(\alpha_{\kappa_1}^-\right)^2\left(\alpha_{\kappa_2}^-\right)^2},\\ \sqrt{\left(\alpha_{\kappa_1}^+\right)^2+\left(\alpha_{\kappa_{12}}^+\right)^2-\dfrac{\aleph}{1+\aleph}\left(\alpha_{\kappa_1}^+\right)^2\left(\alpha_{\kappa_2}^+\right)^2}\end{array}\right],\left[\begin{array}{l}\sqrt{\dfrac{\left(\beta_{\kappa_1}^-\right)^2+\left(\beta_{\kappa_2}^-\right)^2-1+\aleph\left(\beta_{\kappa_1}^-\right)^2\left(\beta_{\kappa_2}^-\right)^2}{1+\aleph}},\\ \sqrt{\dfrac{\left(\beta_{\kappa_1}^+\right)^2+\left(\beta_{\kappa_2}^+\right)^2-1+\aleph\left(\beta_{\kappa_1}^+\right)^2\left(\beta_{\kappa_2}^+\right)^2}{1+\aleph}}\end{array}\right]\right);$$

2) $$\kappa_1\otimes_{sw}\kappa_2=\left(,\left[\begin{array}{l}\sqrt{\dfrac{\left(\alpha_{\kappa_1}^-\right)^2+\left(\alpha_{\kappa_2}^-\right)^2-1+\aleph\left(\alpha_{\kappa_1}^-\right)^2\left(\alpha_{\kappa_2}^-\right)^2}{1+\aleph}},\\ \sqrt{\dfrac{\left(\alpha_{\kappa_1}^+\right)^2+\left(\alpha_{\kappa_2}^+\right)^2-1+\aleph\left(\alpha_{\kappa_1}^+\right)^2\left(\alpha_{\kappa_2}^+\right)^2}{1+\aleph}}\end{array}\right],\left[\begin{array}{l}\sqrt{\left(\beta_{\kappa_1}^-\right)^2+\left(\beta_{\kappa_{12}}^-\right)^2-\dfrac{\aleph}{1+\aleph}\left(\beta_{\kappa_1}^-\right)^2\left(\beta_{\kappa_2}^-\right)^2},\\ \sqrt{\left(\beta_{\kappa_1}^+\right)^2+\left(\beta_{\kappa_{12}}^+\right)^2-\dfrac{\aleph}{1+\aleph}\left(\beta_{\kappa_1}^+\right)^2\left(\beta_{\kappa_2}^+\right)^2}\end{array}\right]\right);$$

3) $$\lambda\kappa_1=\left(\left[\begin{array}{l}\sqrt{\dfrac{1+\aleph}{\aleph}\left(1-\left(1-\left(\alpha_{\kappa_1}^-\right)^2\left(\dfrac{\aleph}{1+\aleph}\right)\right)^{\lambda}\right)},\\ \sqrt{\dfrac{1+\aleph}{\aleph}\left(1-\left(1-\left(\alpha_{\kappa_1}^+\right)^2\left(\dfrac{\aleph}{1+\aleph}\right)\right)^{\lambda}\right)}\end{array}\right],\left[\begin{array}{l}\sqrt{\dfrac{1}{\aleph}\left((1+\aleph)\left(\dfrac{\aleph\left(\beta_{\kappa_1}^-\right)^2+1}{1+\aleph}\right)^{\lambda}-1\right)},\\ \sqrt{\dfrac{1}{\aleph}\left((1+\aleph)\left(\dfrac{\aleph\left(\beta_{\kappa_1}^+\right)^2+1}{1+\aleph}\right)^{\lambda}-1\right)}\end{array}\right]\right),\lambda>0;$$

4) $$\left(\kappa_1\right)^{\lambda}=\left(\left[\begin{array}{l}\sqrt{\dfrac{1}{\aleph}\left((1+\aleph)\left(\dfrac{\aleph\left(\alpha_{\kappa_1}^-\right)^2+1}{1+\aleph}\right)^{\lambda}-1\right)},\\ \sqrt{\dfrac{1}{\aleph}\left((1+\aleph)\left(\dfrac{\aleph\left(\alpha_{\kappa_1}^+\right)^2+1}{1+\aleph}\right)^{\lambda}-1\right)}\end{array}\right],\left[\begin{array}{l}\sqrt{\dfrac{1+\aleph}{\aleph}\left(1-\left(1-\left(\beta_{\kappa_1}^-\right)^2\left(\dfrac{\aleph}{1+\aleph}\right)\right)^{\lambda}\right)},\\ \sqrt{\dfrac{1+\aleph}{\aleph}\left(1-\left(1-\left(\beta_{\kappa_1}^+\right)^2\left(\dfrac{\aleph}{1+\aleph}\right)\right)^{\lambda}\right)}\end{array}\right]\right),\lambda>0;$$

基于区间值毕达哥拉斯模糊 Sugeno-Weber 运算法则，下面提出区间值毕达哥拉斯模糊环境下的

Sugeno-Weber Softmax 集成算子。

3.2. 区间值毕达哥拉斯模糊环境下的 Sugeno-Weber Softmax 算子

定义 7 设 $\kappa_j=\left(\left[\alpha_{\kappa_j}^-,\alpha_{\kappa_j}^+\right],\left[\beta_{\kappa_j}^-,\beta_{\kappa_j}^+\right]\right)(j=1,2,\cdots,n)$ 为一组 IVPFNs。则区间值毕达哥拉斯模糊 Sugeno-Weber Softmax (IVPFSWSF)算子 IVPFSWSF : $\Omega^n\to\Omega$ 定义如下：

$$\mathrm{IVPFSWSF}\left(\kappa_1,\kappa_2,\cdots,\kappa_n\right)=\underset{j=1}{\overset{n}{\oplus}}{}_{sw}\,\psi_j^{\xi}\kappa_j \tag{5}$$

其中 $\psi_j^{\xi}=\exp\left(T_j/\xi\right)\Big/\sum_{j=1}^{n}\exp\left(T_j/\xi\right)$ 满足 $\sum_{j=1}^{n}\psi_j^{\xi}=1,\psi_j^{\xi}\in[0,1]$， $\xi>0$。 $T_1=1$，$T_j=\prod_{l=1}^{j-1}SF\left(\kappa_j\right),(j=2,3,\cdots,n)$， $SF\left(\kappa_j\right)$ 是 κ_j 的得分函数。

定理 1 设 $\kappa_j=\left(\left[\alpha_{\kappa_j}^-,\alpha_{\kappa_j}^+\right],\left[\beta_{\kappa_j}^-,\beta_{\kappa_j}^+\right]\right)(j=1,2,\cdots,n)$ 为一组 IVPFNs。则利用 IVPFSWSF 算子集成这组区间值毕达哥拉斯模糊数后的结果仍是区间值毕达哥拉斯模糊数且

$$\begin{aligned}&\mathrm{IVPFSWSF}\left(\kappa_1,\kappa_2,\cdots,\kappa_n\right)=\underset{j=1}{\overset{n}{\oplus}}{}_{sw}\,\psi_j^{\xi}\kappa_j\\&=\left(\left[\begin{array}{l}\sqrt{\frac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{n}\left(1-\left(\alpha_{\kappa_j}^-\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\psi_j^{\xi}}\right)},\\\sqrt{\frac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{n}\left(1-\left(\alpha_{\kappa_j}^+\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\psi_j^{\xi}}\right)}\end{array}\right],\left[\begin{array}{l}\sqrt{\frac{1}{\aleph}\left(\left(1+\aleph\right)\prod_{j=1}^{n}\left(\frac{\aleph\left(\beta_{\kappa_j}^-\right)^2+1}{1+\aleph}\right)^{\psi_j^{\xi}}-1\right)},\\\sqrt{\frac{1}{\aleph}\left(\left(1+\aleph\right)\prod_{j=1}^{n}\left(\frac{\aleph\left(\beta_{\kappa_j}^+\right)^2+1}{1+\aleph}\right)^{\psi_j^{\xi}}-1\right)}\end{array}\right]\right)\end{aligned} \tag{6}$$

证明： 下面通过数学归纳法证明公式(6)成立。当 $n=2$ 时，根据定义 7 可得：

$$\begin{aligned}&\mathrm{IVPFSWSFWA}\left(\kappa_1,\kappa_2,\cdots,\kappa_n\right)=\psi_1^{\xi}\kappa_1\oplus_{sw}\psi_2^{\xi}\kappa_2\\&=\left(\left[\begin{array}{l}\sqrt{\frac{1+\aleph}{\aleph}\left(1-\left(1-\left(\alpha_{\kappa_1}^-\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\psi_1^{\xi}}\right)},\\\sqrt{\frac{1+\aleph}{\aleph}\left(1-\left(1-\left(\alpha_{\kappa_1}^+\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\psi_1^{\xi}}\right)}\end{array}\right],\left[\begin{array}{l}\sqrt{\frac{1}{\aleph}\left(\left(1+\aleph\right)\left(\frac{\aleph\left(\beta_{\kappa_1}^-\right)^2+1}{1+\aleph}\right)^{\psi_1^{\xi}}-1\right)},\\\sqrt{\frac{1}{\aleph}\left(\left(1+\aleph\right)\left(\frac{\aleph\left(\beta_{\kappa_1}^+\right)^2+1}{1+\aleph}\right)^{\psi_1^{\xi}}-1\right)}\end{array}\right]\right)\\&\oplus_{sw}\left(\left[\begin{array}{l}\sqrt{\frac{1+\aleph}{\aleph}\left(1-\left(1-\left(\alpha_{\kappa_2}^-\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\psi_2^{\xi}}\right)},\\\sqrt{\frac{1+\aleph}{\aleph}\left(1-\left(1-\left(\alpha_{\kappa_2}^+\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\psi_2^{\xi}}\right)}\end{array}\right],\left[\begin{array}{l}\sqrt{\frac{1}{\aleph}\left(\left(1+\aleph\right)\left(\frac{\aleph\left(\beta_{\kappa_2}^-\right)^2+1}{1+\aleph}\right)^{\psi_2^{\xi}}-1\right)},\\\sqrt{\frac{1}{\aleph}\left(\left(1+\aleph\right)\left(\frac{\aleph\left(\beta_{\kappa_2}^+\right)^2+1}{1+\aleph}\right)^{\psi_2^{\xi}}-1\right)}\end{array}\right]\right)\end{aligned}$$

$$=\left(\left[\sqrt{\frac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{2}\left(1-\left(\alpha_{\kappa_j}^{-}\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\psi_j^{\xi}}\right)},\sqrt{\frac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{2}\left(1-\left(\alpha_{\kappa_j}^{+}\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\psi_j^{\xi}}\right)}\right],\left[\sqrt{\frac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{2}\left(\frac{\aleph\left(\beta_{\kappa_j}^{-}\right)^2+1}{1+\aleph}\right)^{\psi_j^{\xi}}-1\right)},\sqrt{\frac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{2}\left(\frac{\aleph\left(\beta_{\kappa_j}^{+}\right)^2+1}{1+\aleph}\right)^{\psi_j^{\xi}}-1\right)}\right]\right)$$

因此，则当 $n=2$ 时，公式(6)成立。假设当 $n=\ell$，公式(6)成立。

$$\mathrm{IVPFSWSFWA}\left(\kappa_1,\kappa_2,\cdots,\kappa_\ell\right)=\bigoplus_{sw\,j=1}^{\ell}\psi_j^{\xi}\kappa_j$$

$$=\left(\left[\sqrt{\frac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{\ell}\left(1-\left(\alpha_{\kappa_j}^{-}\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\psi_j^{\xi}}\right)},\sqrt{\frac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{\ell}\left(1-\left(\alpha_{\kappa_j}^{+}\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\psi_j^{\xi}}\right)}\right],\left[\sqrt{\frac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{\ell}\left(\frac{\aleph\left(\beta_{\kappa_j}^{-}\right)^2+1}{1+\aleph}\right)^{\psi_j^{\xi}}-1\right)},\sqrt{\frac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{\ell}\left(\frac{\aleph\left(\beta_{\kappa_j}^{+}\right)^2+1}{1+\aleph}\right)^{\psi_j^{\xi}}-1\right)}\right]\right)$$

则当 $n=\ell+1$时，

$$\mathrm{IVPFSWSF}\left(\kappa_1,\kappa_2,\cdots,\kappa_\ell\right)=\mathrm{IVPFSWSF}\left(\kappa_1,\kappa_2,\cdots,\kappa_\ell\right)\oplus_{sw}\psi_{\ell+1}^{\xi}\kappa_{\ell+1}$$

$$=\left(\left[\sqrt{\frac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{\ell}\left(1-\left(\alpha_{\kappa_j}^{-}\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\psi_j^{\xi}}\right)},\sqrt{\frac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{\ell}\left(1-\left(\alpha_{\kappa_j}^{+}\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\psi_j^{\xi}}\right)}\right],\left[\sqrt{\frac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{\ell}\left(\frac{\aleph\left(\beta_{\kappa_j}^{-}\right)^2+1}{1+\aleph}\right)^{\psi_j^{\xi}}-1\right)},\sqrt{\frac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{\ell}\left(\frac{\aleph\left(\beta_{\kappa_j}^{+}\right)^2+1}{1+\aleph}\right)^{\psi_j^{\xi}}-1\right)}\right]\right)$$

$$\oplus_{sw}\left(\left[\sqrt{\frac{1+\aleph}{\aleph}\left(1-\left(1-\left(\alpha_{\kappa_{\ell+1}}^{-}\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\psi_{\ell+1}^{\xi}}\right)},\sqrt{\frac{1+\aleph}{\aleph}\left(1-\left(1-\left(\alpha_{\kappa_{\ell+1}}^{+}\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\psi_{\ell+1}^{\xi}}\right)}\right],\left[\sqrt{\frac{1}{\aleph}\left((1+\aleph)\left(\frac{\aleph\left(\beta_{\kappa_{\ell+1}}^{-}\right)^2+1}{1+\aleph}\right)^{\psi_{\ell+1}^{\xi}}-1\right)},\sqrt{\frac{1}{\aleph}\left((1+\aleph)\left(\frac{\aleph\left(\beta_{\kappa_{\ell+1}}^{+}\right)^2+1}{1+\aleph}\right)^{\psi_{\ell+1}^{\xi}}-1\right)}\right]\right)$$

$$=\left(\left[\begin{array}{l}\sqrt{\frac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{\ell+1}\left(1-\left(\alpha_{\kappa_j}^-\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\psi_j^\xi}\right)},\\ \sqrt{\frac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{\ell+1}\left(1-\left(\alpha_{\kappa_j}^+\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\psi_j^\xi}\right)}\end{array}\right],\left[\begin{array}{l}\sqrt{\frac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{\ell+1}\left(\frac{\aleph\left(\beta_{\kappa_j}^-\right)^2+1}{1+\aleph}\right)^{\psi_j^\xi}-1\right)},\\ \sqrt{\frac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{\ell+1}\left(\frac{\aleph\left(\beta_{\kappa_j}^+\right)^2+1}{1+\aleph}\right)^{\psi_j^\xi}-1\right)}\end{array}\right]\right)$$

因此，当 $n=\ell+1$ 时，公式(6)成立。因此，公式(6)对所有 n 成立。

接下来，我们探讨 IVPFSWSF 算子的性质。

性质 1 (幂等性) 设 $\kappa_j=\left(\left[\alpha_{\kappa_j}^-,\alpha_{\kappa_j}^+\right],\left[\beta_{\kappa_j}^-,\beta_{\kappa_j}^+\right]\right)(j=1,2,\cdots,n)$ 为一组区间值毕达哥拉斯模糊数。若 $\kappa_j=\kappa_0=\left(\left[\alpha_{\kappa_0}^-,\alpha_{\kappa_0}^+\right],\left[\beta_{\kappa_0}^-,\beta_{\kappa_0}^+\right]\right)$，则 $\text{IVPFSWSF}(\kappa_1,\kappa_2,\cdots,\kappa_n)=\kappa_0$。

证明： 因为 $\kappa_j=\kappa_0=\left(\left[\alpha_{\kappa_0}^-,\alpha_{\kappa_0}^+\right],\left[\beta_{\kappa_0}^-,\beta_{\kappa_0}^+\right]\right)$，$\text{IVPFSWSF}(\kappa_1,\kappa_2,\cdots,\kappa_n)=\oplus_{sw\,j=1}^{n}\psi_j^\xi\kappa_j=\kappa_0\sum_{j=1}^{n}\psi_j^\xi=\kappa_0$。

性质 2 (单调性) 设 $\kappa_j=\left(\left[\alpha_{\kappa_j}^-,\alpha_{\kappa_j}^+\right],\left[\beta_{\kappa_j}^-,\beta_{\kappa_j}^+\right]\right)$ 和 $\tilde{\kappa}_j=\left(\left[\tilde{\alpha}_{\tilde{\kappa}_j}^-,\tilde{\alpha}_{\tilde{\kappa}_j}^+\right],\left[\tilde{\beta}_{\tilde{\kappa}_j}^-,\tilde{\beta}_{\tilde{\kappa}_j}^+\right]\right)(j=1(1)n)$ 是 IVPFNs。若 $\alpha_{\kappa_j}^-\ge\tilde{\alpha}_{\tilde{\kappa}_j}^-$，$\alpha_{\kappa_j}^+\ge\tilde{\alpha}_{\tilde{\kappa}_j}^+$，$\beta_{\kappa_j}^-\le\tilde{\beta}_{\tilde{\kappa}_j}^-$，$\beta_{\kappa_j}^+\le\tilde{\beta}_{\tilde{\kappa}_j}^+$，则 $\text{IVPFSWSF}(\kappa_1,\kappa_2,\cdots,\kappa_n)\ge\text{IVPFSWSF}(\tilde{\kappa}_1,\tilde{\kappa}_2,\cdots,\tilde{\kappa}_n)$.

证明： 因为 $\alpha_{\kappa_j}^-\ge\tilde{\alpha}_{\tilde{\kappa}_j}^-$，则 $1-(\aleph/(1+\aleph))\cdot\alpha_{\kappa_j}^-\le1-(\aleph/(1+\aleph))\cdot\tilde{\alpha}_{\tilde{\kappa}_j}^-$。此外，

$1-\prod_{j=1}^{n}\left(1-\left(\frac{\aleph}{1+\aleph}\right)\cdot\alpha_{\kappa_j}^-\right)^{\psi_j^\xi}\ge1-\prod_{j=1}^{n}\left(1-\left(\frac{\aleph}{1+\aleph}\right)\cdot\tilde{\alpha}_{\tilde{\kappa}_j}^-\right)^{\psi_j^\xi}$，则

$$\frac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\aleph}{1+\aleph}\right)\cdot\alpha_{\kappa_j}^-\right)^{\psi_j^\xi}\right)\ge\frac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\aleph}{1+\aleph}\right)\cdot\tilde{\alpha}_{\tilde{\kappa}_j}^-\right)^{\psi_j^\xi}\right),$$

$$\frac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\aleph}{1+\aleph}\right)\cdot\alpha_{\kappa_j}^+\right)^{\psi_j^\xi}\right)\ge\frac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\aleph}{1+\aleph}\right)\cdot\tilde{\alpha}_{\tilde{\kappa}_j}^+\right)^{\psi_j^\xi}\right).$$

此外，因为 $\beta_{\kappa_j}^-\le\tilde{\beta}_{\tilde{\kappa}_j}^-$，则 $\left(\aleph\cdot\beta_{\kappa_j}^-+1\right)/(1+\aleph)\le\left(\aleph\cdot\tilde{\beta}_{\tilde{\kappa}_j}^-+1\right)/(1+\aleph)$，则

$$\frac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{n}\left(\frac{\aleph\cdot\beta_{\kappa_j}^-+1}{1+\aleph}\right)^{\psi_j^\xi}-1\right)\le\frac{1}{\wp}\left((1+\wp)\prod_{j=1}^{n}\left(\frac{\aleph\cdot\tilde{\beta}_{\tilde{\kappa}_j}^-+1}{1+\aleph}\right)^{\psi_j^\xi}-1\right),$$

$$\frac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{n}\left(\frac{\aleph\cdot\beta_{\kappa_j}^++1}{1+\aleph}\right)^{\psi_j^\xi}-1\right)\le\frac{1}{\wp}\left((1+\wp)\prod_{j=1}^{n}\left(\frac{\aleph\cdot\tilde{\beta}_{\tilde{\kappa}_j}^++1}{1+\aleph}\right)^{\psi_j^\xi}-1\right).$$

基于定义 2 可得 $\text{IVPFSWSF}(\kappa_1,\kappa_2,\cdots,\kappa_n)\ge\text{IVPFSWSF}(\tilde{\kappa}_1,\tilde{\kappa}_2,\cdots,\tilde{\kappa}_n)$。

性质 3 (有界性) 设 $\kappa_j=\left(\left[\alpha_{\kappa_j}^-,\alpha_{\kappa_j}^+\right],\left[\beta_{\kappa_j}^-,\beta_{\kappa_j}^+\right]\right)(j=1,2,\cdots,n)$ 为一组区间值毕达哥拉斯模糊数。若

$\kappa^-=\left(\left[\min_{1\le j\le n}\left\{\alpha_{\kappa_j}^-\right\},\min_{1\le j\le n}\left\{\alpha_{\kappa_j}^+\right\}\right],\left[\max_{1\le j\le n}\left\{\beta_{\kappa_j}^-\right\},\max_{1\le j\le n}\left\{\beta_{\kappa_j}^+\right\}\right]\right)$，

$\kappa^+=\left(\left[\max_{1\le j\le n}\left\{\alpha_{\kappa_j}^-\right\},\max_{1\le j\le n}\left\{\alpha_{\kappa_j}^+\right\}\right],\left[\min_{1\le j\le n}\left\{\beta_{\kappa_j}^-\right\},\min_{1\le j\le n}\left\{\beta_{\kappa_j}^+\right\}\right]\right)$。则 $\kappa^-\le\text{IVPFSWSF}(\kappa_1,\kappa_2,\cdots,\kappa_n)\le\kappa^+$。

定义 8 设 $\kappa_j=\left(\left[\alpha_{\kappa_j}^-,\alpha_{\kappa_j}^+\right],\left[\beta_{\kappa_j}^-,\beta_{\kappa_j}^+\right]\right)(j=1,2,\cdots,n)$ 为一组 IVPFNs。则区间值毕达哥拉斯模糊 Sugeno-Weber Softmax 加权平均(IVPFSWSFWA)算子 $\text{IVPFSWSFWA}:\Omega^n\to\Omega$ 定义如下：

$$\text{IVPFSWSFWA}\left(\kappa_1,\kappa_2,\cdots,\kappa_n\right)=\mathop{\oplus_{sw}}_{j=1}^{n}\Phi_j^{\xi}\kappa_j \tag{7}$$

其中 $\Phi_j^{\xi}=\varpi_j\exp\left(T_j/\xi\right)\Big/\sum_{j=1}^{n}\left(\varpi_j\exp\left(T_j/\xi\right)\right)(\xi>0)$ 满足 $\sum_{j=1}^{n}\psi_j^{\xi}=1,\psi_j^{\xi}\in[0,1]$，$\xi$ 为参数且 $\xi>0$。$T_j=\prod_{l=1}^{j-1}SF\left(\kappa_1\right),(j=2,3,\cdots,n)$，$T_1=1$ 且 $SF\left(\kappa_j\right)$ 是 κ_j 的得分函数。

定理 2 设 $\kappa_j=\left(\left[\alpha_{\kappa_j}^-,\alpha_{\kappa_j}^+\right],\left[\beta_{\kappa_j}^-,\beta_{\kappa_j}^+\right]\right)(j=1,2,\cdots,n)$ 为一组 IVPFNs。则利用 IVPFSWSFWA 算子集成这组区间值毕达哥拉斯模糊数后的结果仍是区间值毕达哥拉斯模糊数且

$$\text{IVPFSWSFWA}\left(\kappa_1,\kappa_2,\cdots,\kappa_n\right)=\mathop{\oplus_{sw}}_{j=1}^{n}\Phi_j^{\xi}\kappa_j$$

$$=\left(\begin{bmatrix}\sqrt{\dfrac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{n}\left(1-\left(\alpha_{\kappa_j}^-\right)^2\left(\dfrac{\aleph}{1+\aleph}\right)\right)^{\Phi_j^{\xi}}\right)},\\ \sqrt{\dfrac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{n}\left(1-\left(\alpha_{\kappa_j}^+\right)^2\left(\dfrac{\aleph}{1+\aleph}\right)\right)^{\Phi_j^{\xi}}\right)}\end{bmatrix},\begin{bmatrix}\sqrt{\dfrac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{n}\left(\dfrac{\aleph\left(\beta_{\kappa_j}^-\right)^2+1}{1+\aleph}\right)^{\Phi_j^{\xi}}-1\right)},\\ \sqrt{\dfrac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{n}\left(\dfrac{\aleph\left(\beta_{\kappa_j}^+\right)^2+1}{1+\aleph}\right)^{\Phi_j^{\xi}}-1\right)}\end{bmatrix}\right) \tag{8}$$

证明：与定理 1 相似。

3.3. 区间值毕达哥拉斯模糊环境下的 Sugeno-Weber Softmax 几何算子

定义 9 设 $\kappa_j=\left(\left[\alpha_{\kappa_j}^-,\alpha_{\kappa_j}^+\right],\left[\beta_{\kappa_j}^-,\beta_{\kappa_j}^+\right]\right)(j=1,2,\cdots,n)$ 为一组 IVPFNs。则区间值毕达哥拉斯模糊 Sugeno-Weber Softmax 几何(IVPFSWSFG)算子 $\text{IVPFSWSFG}:\Omega^n\to\Omega$ 定义如下：

$$\text{IVPFSWSFG}\left(\kappa_1,\kappa_2,\cdots,\kappa_n\right)=\mathop{\otimes_{sw}}_{j=1}^{n}\left(\kappa_j\right)^{\psi_j^{\xi}}. \tag{9}$$

定理 3 设 $\kappa_j=\left(\left[\alpha_{\kappa_j}^-,\alpha_{\kappa_j}^+\right],\left[\beta_{\kappa_j}^-,\beta_{\kappa_j}^+\right]\right)(j=1,2,\cdots,n)$ 为一组 IVPFNs。则利用 IVPFSWSFG 算子集成这组区间值毕达哥拉斯模糊数后的结果仍是区间值毕达哥拉斯模糊数且集成结果表示为

$$\text{IVPFSWSFG}\left(\kappa_1,\kappa_2,\cdots,\kappa_n\right)=\mathop{\otimes_{sw}}_{j=1}^{n}\left(\kappa_j\right)^{\psi_j^{\xi}}$$

$$=\left(\begin{bmatrix}\sqrt{\dfrac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{n}\left(\dfrac{\aleph\left(\alpha_{\kappa_j}^-\right)^2+1}{1+\aleph}\right)^{\psi_j^{\xi}}-1\right)},\\ \sqrt{\dfrac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{n}\left(\dfrac{\aleph\left(\alpha_{\kappa_j}^+\right)^2+1}{1+\aleph}\right)^{\psi_j^{\xi}}-1\right)}\end{bmatrix},\begin{bmatrix}\sqrt{\dfrac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{n}\left(1-\left(\beta_{\kappa_j}^-\right)^2\left(\dfrac{\aleph}{1+\aleph}\right)\right)^{\psi_j^{\xi}}\right)},\\ \sqrt{\dfrac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{n}\left(1-\left(\beta_{\kappa_j}^+\right)^2\left(\dfrac{\aleph}{1+\aleph}\right)\right)^{\psi_j^{\xi}}\right)}\end{bmatrix}\right) \tag{10}$$

IVPFSWSFG 算子与 IVPFSWSF 算子都具有幂等性、单调性和有界性性质，此处不再赘述。

定义 10 设 $\kappa_j=\left(\left[\alpha_{\kappa_j}^-,\alpha_{\kappa_j}^+\right],\left[\beta_{\kappa_j}^-,\beta_{\kappa_j}^+\right]\right)(j=1,2,\cdots,n)$ 为一组 IVPFNs。则区间值毕达哥拉斯模糊

Sugeno-Weber Softmax 加权几何(IVPFSWSFWG)算子 $\text{IVPFSWSFWG}:\Omega^n\to\Omega$ 定义如下：

$$\text{IVPFSWSFWG}\left(\kappa_1,\kappa_2,\cdots,\kappa_n\right)=\underset{j=1}{\overset{n}{\otimes_{sw}}}\left(\kappa_j\right)^{\Phi_j^{\xi}}. \tag{11}$$

定理 4 设 $\kappa_j=\left(\left[\alpha_{\kappa_j}^-,\alpha_{\kappa_j}^+\right],\left[\beta_{\kappa_j}^-,\beta_{\kappa_j}^+\right]\right)(j=1,2,\cdots,n)$ 为一组 IVPFNs。则利用 IVPFSWSFWG 算子集成这组区间值毕达哥拉斯模糊数后的结果仍是区间值毕达哥拉斯模糊数且

$$\text{IVPFSWSFWG}\left(\kappa_1,\kappa_2,\cdots,\kappa_n\right)=\underset{j=1}{\overset{n}{\otimes_{sw}}}\left(\kappa_j\right)^{\Phi_j^{\xi}}$$

$$=\left(\left[\sqrt{\frac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{n}\left(\frac{\aleph\left(\alpha_{\kappa_j}^-\right)^2+1}{1+\aleph}\right)^{\Phi_j^{\xi}}-1\right)},\sqrt{\frac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{n}\left(\frac{\aleph\left(\alpha_{\kappa_j}^+\right)^2+1}{1+\aleph}\right)^{\Phi_j^{\xi}}-1\right)}\right],\left[\sqrt{\frac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{n}\left(1-\left(\beta_{\kappa_j}^-\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\Phi_j^{\xi}}\right)},\sqrt{\frac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{n}\left(1-\left(\beta_{\kappa_j}^+\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\Phi_j^{\xi}}\right)}\right]\right) \tag{12}$$

证明：与定理 2 相似。

4. 基于区间值毕达哥拉斯模糊 Sugeno-Weber 算子的 WASPAS 多属性决策方法

本章基于区间值毕达哥拉斯模糊多属性决策问题涉及的概念和符号定义如下。$G=\left\{G_i\,|\,i=1(1)m\right\}$ 为一组备选方案，$C=\left\{C_j\,|\,j=1(1)n\right\}$ 为属性集合且其权重向量为 $\varpi=\left\{\varpi_j\,|\,j=1(1)n\right\}$ 且满足 $\varpi_j\in[0,1],\sum_{j=1}^{n}\varpi_j=1$。专家对备选方案 $G_i\ (i=1,2,\cdots,m)$ 在属性 $C_j\ (j=1,2,\cdots,n)$ 下的评价值用区间值毕达哥拉斯模糊表示并构成决策矩阵 $\tilde{F}=\left(\tilde{\kappa}_{ij}\right)_{m\times n}$，$\tilde{\kappa}_{ij}=\left(\left[\tilde{\alpha}_{\tilde{\kappa}_{ij}}^-,\tilde{\alpha}_{\tilde{\kappa}_{ij}}^+\right],\left[\tilde{\beta}_{\tilde{\kappa}_{ij}}^-,\tilde{\beta}_{\tilde{\kappa}_{ij}}^+\right]\right)\ (i=1,2,\cdots,m;j=1,2,\cdots,n)$。基于上述定义和符号，所提基于区间值毕达哥拉斯模糊 Sugeno-Weber Softmax 算子的 WASPAS 多属性决策方法步骤如下：

步骤 1：确定归一化决策矩阵 $F=\left(\kappa_{ij}\right)_{m\times n}$。

$$\kappa_{ij}=\begin{cases}\tilde{\kappa}_{ij}=\left(\left[\tilde{\alpha}_{\tilde{\kappa}_{ij}}^-,\tilde{\alpha}_{\tilde{\kappa}_{ij}}^+\right],\left[\tilde{\beta}_{\tilde{\kappa}_{ij}}^-,\tilde{\beta}_{\tilde{\kappa}_{ij}}^+\right]\right),j\in C_b\\ \left(\tilde{\kappa}_{ij}\right)^c=\left(\left[\tilde{\beta}_{\tilde{\kappa}_{ij}}^-,\tilde{\beta}_{\tilde{\kappa}_{ij}}^+\right],\left[\tilde{\alpha}_{\tilde{\kappa}_{ij}}^-,\tilde{\alpha}_{\tilde{\kappa}_{ij}}^+\right]\right),j\in C_c\end{cases} \tag{13}$$

步骤 2：确定属性权重。

步骤 2.1：计算每个属性下的归一化均值：

$$\phi_j=\text{IVPFSWSFWA}\left(\kappa_{1j},\kappa_{2j},\cdots,\kappa_{mj}\right)=\underset{i=1}{\overset{m}{\oplus_{sw}}}\left(1/m\right)\kappa_{ij}$$

$$=\left(\left[\sqrt{\frac{1+\aleph}{\aleph}\left(1-\prod_{i=1}^{m}\left(1-\left(\alpha_{\kappa_{ij}}^-\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{1/m}\right)},\sqrt{\frac{1+\aleph}{\aleph}\left(1-\prod_{i=1}^{m}\left(1-\left(\alpha_{\kappa_{ij}}^+\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{1/m}\right)}\right],\left[\sqrt{\frac{1}{\aleph}\left((1+\aleph)\prod_{i=1}^{m}\left(\frac{\aleph\left(\beta_{\kappa_{ij}}^-\right)^2+1}{1+\aleph}\right)^{1/m}-1\right)},\sqrt{\frac{1}{\aleph}\left((1+\aleph)\prod_{i=1}^{m}\left(\frac{\aleph\left(\beta_{\kappa_{ij}}^+\right)^2+1}{1+\aleph}\right)^{1/m}-1\right)}\right]\right) \tag{14}$$

步骤 2.2：计算所有方案的评估值与每个属性下的归一化均值的偏差：

$$\mu_j = \sum_{i=1}^{m} \left(SF\left(\kappa_{ij}\right) - SF\left(\phi_j\right) \right)^2, j = 1, 2, \cdots, n \tag{15}$$

步骤 2.3：计算属性的权重：

$$\varpi_j = \frac{\mu_j}{\sum_{j=1}^{n} \mu_j}, j = 1, 2, \cdots, n. \tag{16}$$

步骤 3：基于 IVPFSWSFWA 算子确定备选方案的加权和测度。

$$\begin{aligned} {}^a\kappa_i &= \left(\left[{}^a\alpha_i^-, {}^a\alpha_i^- \right], \left[{}^a\beta_i^-, {}^a\beta_i^- \right] \right) = \mathrm{IVPFSWSFWA}\left(\kappa_{i1}, \kappa_{i2}, \cdots, \kappa_{in}\right) = \oplus_{sw\, j=1}^{n} \psi_j^{\xi} \kappa_{ij} \\ &= \left(\left[\sqrt{\frac{1+\aleph}{\aleph}\left(1 - \prod_{j=1}^{n}\left(1-\left(\alpha_{\kappa_{ij}}^-\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\psi_j^{\xi}}\right)}, \sqrt{\frac{1+\aleph}{\aleph}\left(1 - \prod_{j=1}^{n}\left(1-\left(\alpha_{\kappa_{ij}}^+\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\psi_j^{\xi}}\right)} \right], \left[\sqrt{\frac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{n}\left(\frac{\aleph\left(\beta_{\kappa_{ij}}^-\right)^2+1}{1+\aleph}\right)^{\psi_j^{\xi}} - 1\right)}, \sqrt{\frac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{n}\left(\frac{\aleph\left(\beta_{\kappa_{ij}}^+\right)^2+1}{1+\aleph}\right)^{\psi_j^{\xi}} - 1\right)} \right] \right) \end{aligned} \tag{17}$$

步骤 4：基于 IVPFSWSFWG 算子确定备选方案的加权积测度。

$$\begin{aligned} {}^b\kappa_i &= \left(\left[{}^b\alpha_i^-, {}^b\alpha_i^+ \right], \left[{}^b\beta_i^-, {}^b\beta_i^+ \right] \right) = \mathrm{IVPFSWSFWG}\left(\kappa_1, \kappa_2, \cdots, \kappa_n\right) = \otimes_{sw\, j=1}^{n} \left(\kappa_{ij}\right)^{\Phi_j^{\xi}} \\ &= \left(\left[\sqrt{\frac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{n}\left(\frac{\aleph\left(\alpha_{\kappa_{ij}}^-\right)^2+1}{1+\aleph}\right)^{\Phi_j^{\xi}} - 1\right)}, \sqrt{\frac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{n}\left(\frac{\aleph\left(\alpha_{\kappa_{ij}}^+\right)^2+1}{1+\aleph}\right)^{\Phi_j^{\xi}} - 1\right)} \right], \left[\sqrt{\frac{1+\aleph}{\aleph}\left(1 - \prod_{j=1}^{n}\left(1-\left(\beta_{\kappa_{ij}}^-\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\Phi_j^{\xi}}\right)}, \sqrt{\frac{1+\aleph}{\aleph}\left(1 - \prod_{j=1}^{n}\left(1-\left(\beta_{\kappa_{ij}}^+\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\Phi_j^{\xi}}\right)} \right] \right) \end{aligned} \tag{18}$$

步骤 5：计算每个备选方案的综合测度得分 H_i：

$$H_i = \vartheta \cdot SF\left({}^a\kappa_i\right) + (1-\vartheta) \cdot SF\left({}^b\kappa_i\right), \tag{19}$$

其中 $SF\left({}^a\kappa_i\right)$ 和 $SF\left({}^b\kappa_i\right)$ 表示加权和测度和加权积测度的得分函数，$\vartheta \in [0,1]$ 为策略系数。

步骤 6：根据每个备选方案的综合测度得分 H_i 对备选方案进行降序排列确定其优先级及最优方案。

5. 实例分析

本章通过案例分析和讨论分析所提基于区间值毕达哥拉斯模糊 Sugeno-Weber 算子的 WASPAS 多属性决策方法的有效性和鲁棒性。

5.1. 决策实施过程

为验证所提决策方法的有效性，本节通过逆向物流供应商选择案例进行讨论。某新能源企业为了从

备选的五个逆向物流供应商中选择最优的供应商，基于逆向物流成本(C_1)、技术能力(C_2)、服务质量(C_3)信息共享能力(C_4)和运输能力(C_5)五个属性和决策者的认知确定备选供应商的优先级，考虑决策者的认知不确定性和权重完全未知的特征，通过所提区间值毕达哥拉斯模糊 Sugeno-Weber 算子的 WASPAS 多属性决策方法确定最优逆向物流供应商。决策者基于其知识背景和认知能力，以毕达哥拉斯模糊数形式给出五个逆向物流供应商在属性下的评价信息并构成决策矩阵如表 1 所示。

Table 1. Pythagorean fuzzy decision matrix
表 1. 毕达哥拉斯模糊决策矩阵

	C_1	C_2	C_3	C_4	C_5
G_1	([0.3, 0.4], [0.7, 0.8])	([0.4, 0.5], [0.6, 0.7])	([0.9, 0.95], [0.1, 0.15])	([0.5, 0.65], [0.5, 0.60])	([0.8, 0.9], [0.2, 0.35])
G_2	([0.2, 0.3], [0.8, 0.85])	([0.8, 0.9], [0.2, 0.35])	([0.4, 0.5], [0.6, 0.7])	([0.5, 0.65], [0.5, 0.60])	([0.5, 0.65], [0.5, 0.60])
G_3	([0.4, 0.5], [0.6, 0.7])	([0.5, 0.65], [0.5, 0.60])	([0.5, 0.65], [0.5, 0.60])	([0.4, 0.5], [0.6, 0.7])	([0.8, 0.9], [0.2, 0.35])
G_4	([0.3, 0.4], [0.7, 0.8])	([0.65, 0.8], [0.4, 0.50])	([0.4, 0.5], [0.6, 0.7])	([0.8, 0.9], [0.2, 0.35])	([0.5, 0.65], [0.5, 0.60])
G_5	([0.4, 0.5], [0.6, 0.7])	([0.5, 0.65], [0.5, 0.60])	([0.8, 0.9], [0.2, 0.35])	([0.4, 0.5], [0.6, 0.7])	([0.4, 0.5], [0.6, 0.7])

步骤 1：通过公式(13)确定归一化决策矩阵如表 2 所示。

Table 2. Normalized pythagorean fuzzy decision matrix
表 2. 归一化的毕达哥拉斯模糊决策矩阵

	C_1	C_2	C_3	C_4	C_5
G_1	([0.7, 0.8], [0.3, 0.4])	([0.4, 0.5], [0.6, 0.7])	([0.9, 0.95], [0.1, 0.15])	([0.5, 0.65], [0.5, 0.60])	([0.8, 0.9], [0.2, 0.35])
G_2	([0.8, 0.85], [0.2, 0.3])	([0.8, 0.9], [0.2, 0.35])	([0.4, 0.5], [0.6, 0.7])	([0.5, 0.65], [0.5, 0.60])	([0.5, 0.65], [0.5, 0.60])
G_3	([0.6, 0.7] , [0.4, 0.5])	([0.5, 0.65], [0.5, 0.60])	([0.5, 0.65], [0.5, 0.60])	([0.4, 0.5], [0.6, 0.7])	([0.8, 0.9], [0.2, 0.35])
G_4	([0.7, 0.8], [0.3, 0.4])	([0.65, 0.8], [0.4, 0.50])	([0.4, 0.5], [0.6, 0.7])	([0.8, 0.9], [0.2, 0.35])	([0.5, 0.65], [0.5, 0.60])
G_5	([0.6, 0.7], [0.4, 0.5])	([0.5, 0.65], [0.5, 0.60])	([0.8, 0.9], [0.2, 0.35])	([0.4, 0.5], [0.6, 0.7])	([0.4, 0.5], [0.6, 0.7])

步骤 2：确定属性权重。通过公式(14)~(16)计算属性的权重为

$$\varpi_1 = 0.0424，\ \varpi_2 = 0.1641，\ \varpi_3 = 0.3729，\ \varpi_4 = 0.1867，\ \varpi_5 = 0.2339。$$

步骤 3：通过公式(17)确定备选方案的加权和测度($\aleph = 2, \xi = 2$)：

$$^{a}\kappa_1 = \left(\left[0.7514, 0.8322\right], \left[0.3767, 0.4197\right]\right)，\ ^{a}\kappa_2 = \left(\left[0.5714, 0.6836\right], \left[0.5141, 0.5785\right]\right)，$$

$$^{a}\kappa_3 = \left(\left[0.5788, 0.7038\right], \left[0.4793, 0.5582\right]\right)，\ ^{a}\kappa_4 = \left(\left[0.5841, 0.7057\right], \left[0.4980, 0.5661\right]\right)，$$

$$^{a}\kappa_5 = \left(\left[0.6220, 0.7314\right], \left[0.4608, 0.5421\right]\right)。$$

步骤 4：通过公式(18)确定备选方案的加权积测度($\aleph=2,\xi=2$)：

$$^{b}\kappa_1=\left(\left[0.7006,0.7882\right],\left[0.3698,0.4659\right]\right),\quad ^{b}\kappa_2=\left(\left[0.5376,0.6491\right],\left[0.3218,0.6011\right]\right),$$

$$^{b}\kappa_3=\left(\left[0.5534,0.6788\right],\left[0.3674,0.5769\right]\right),\quad ^{b}\kappa_4=\left(\left[0.5558,0.6721\right],\left[0.3111,0.5887\right]\right),$$

$$^{b}\kappa_5=\left(\left[0.5825,0.6876\right],\left[0.4525,0.5722\right]\right)。$$

步骤 5：通过公式(19)计算每个备选方案的综合测度得分 H_i （$\vartheta=0.5$）：

$$H_1=0.7122,\quad H_2=0.5550,\quad H_3=0.5735,\quad H_4=0.5735,\quad H_5=0.5870。$$

步骤 6：根据每个备选方案的综合测度得分 H_i 对备选方案进行降序排列得到备选逆向物流供应商的排序为 $G_1\succ G_5\succ G_3\sim G_4\succ G_2$。

5.2. 灵敏度分析

本节将对所提方法中涉及的参数进行讨论，分析参数的变化对供应商排序的影响，进而讨论所提方法的稳定性。首先对区间值毕达哥拉斯模糊 Sugeno-Weber Softmax 集成算子中的参数进行讨论，当参数 $\aleph\in\left[2,9\right]$ 时，基于不同参数 $\aleph$ 值的供应商的综合测度和排序结果如表 3 所示，从结果可以发现不同参数 $\aleph$ 值下的供应商的综合测度随着参数值的增大而增大，但是供应商的排序未发生变化，故所提决策方法不受参数 $\aleph$ 的影响，具有较强的稳定性。其次，对 WASPAS 方法中加权和和加权积测度的平衡系数进行讨论，当参数 $\vartheta\in\left[0.1,1\right]$ 时，基于不同参数 ϑ 值的决策结果如图 1 所示。从图 1 可知，当参数 ϑ 不断变化时，供应商的排序有细微的变化，不同参数下的供应商排序为 $G_1\succ G_5\succ G_2\succ G_3\succ G_4$，$G_1\succ G_5\succ G_3\succ G_4\succ G_2$，$G_1\succ G_5\succ G_4\succ G_3\succ G_2$ 和 $G_1\succ G_5\succ G_4\succ G_2\succ G_3$，但是最优选择均是 G_1，决策者可根据其偏好选定不同的参数以确定稳定的供应商排序。

Table 3. Comprehensive measurement and ranking results of suppliers based on different parameter $\aleph$ values
表 3. 基于不同参数 $\aleph$ 值的供应商的综合测度和排序结果

$\aleph$	G_1	G_2	G_3	G_4	G_5	排序
2	0.7122	0.5550	0.5735	0.5735	0.5870	$G_1\succ G_5\succ G_3\sim G_4\succ G_2$
3	0.7142	0.5555	0.5742	0.5738	0.5883	$G_1\succ G_5\succ G_3\succ G_4\succ G_2$
4	0.7160	0.5559	0.5747	0.5739	0.5895	$G_1\succ G_5\succ G_3\succ G_4\succ G_2$
5	0.7175	0.5563	0.5752	0.5741	0.5905	$G_1\succ G_5\succ G_3\succ G_4\succ G_2$
6	0.7188	0.5565	0.5755	0.5742	0.5914	$G_1\succ G_5\succ G_3\succ G_4\succ G_2$
7	0.7200	0.5567	0.5759	0.5743	0.5922	$G_1\succ G_5\succ G_3\succ G_4\succ G_2$
8	0.7210	0.5569	0.5762	0.5743	0.5928	$G_1\succ G_5\succ G_3\succ G_4\succ G_2$
9	0.7219	0.5571	0.5764	0.5744	0.5935	$G_1\succ G_5\succ G_3\succ G_4\succ G_2$

5.3. 比较分析

为验证本文所提方法的有效性，本节将所提方法与基于区间值毕达哥拉斯模糊加权平均(IVPFWA)算子的决策方法和基于区间值毕达哥拉斯模糊加权几何(IVPFWG)算子决策方法进行比较，比较结果如表 4 所示。通过表 4 可以发现，所提方法确定的供应商排序与基于 IVPFWA 算子[9]、IVPFWA 算子[9]、IVPF-WSM 方法和 IVPF-WSM 方法所得到的供应商排序是一致的，说明所提方法是有效的，最优选择均是 G_1。但是所提

决策方法在确定供应商排序时考虑了属性之间的优先级和权重未知的情形，因而具有更加广泛的适用性。

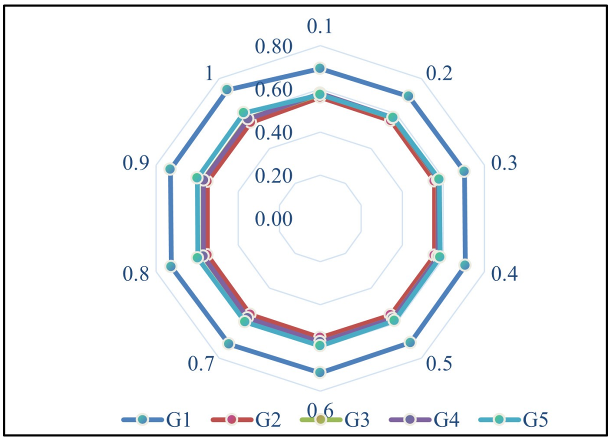

Figure 1. Comprehensive evaluation of alternatives based on different parameter ϑ values
图 1. 基于不同参数 ϑ 值的备选方案的综合测度

Table 4. Decision results based on different methods
表 4. 基于不同方法的决策结果

	G_1	G_2	G_3	G_4	G_5	排序
IVPFWA 算子[9]	0.8089	0.5744	0.6077	0.6014	0.6423	$G_1 \succ G_5 \succ G_3 \succ G_4 \succ G_2$
IVPFWG 算子[9]	0.6811	0.5061	0.5449	0.5294	0.5416	$G_1 \succ G_3 \succ G_5 \succ G_4 \succ G_2$
IVPF-WSM 方法	0.7348	0.5487	0.5723	0.5677	0.6039	$G_1 \succ G_5 \succ G_3 \succ G_4 \succ G_2$
IVPF-WPM 方法	0.6896	0.5613	0.5748	0.5793	0.5700	$G_1 \succ G_5 \succ G_3 \succ G_4 \succ G_2$
所提方法	0.7122	0.5550	0.5735	0.5735	0.5870	$G_1 \succ G_5 \succ G_3 \sim G_4 \succ G_2$

6. 结论

针对属性值为区间值毕达哥拉斯模糊数且属性权重完全未知的复杂决策问题，本研究提出了一种基于 Sugeno-Weber Softmax 算子的 WASPAS 方法。主要结论如下：1) 基于 Sugeno-Weber 模定义了区间值毕达哥拉斯模糊数的运算法则，扩展了模糊集理论在不确定性建模中的适用性。2) 提出四种区间值毕达哥拉斯模糊 Sugeno-Weber Softmax 加权平均和几何集成算子并通过数学证明验证了算子的幂等性、有界性和单调性等性质。3) 设计基于区间值毕达哥拉斯模糊 PSI 方法的属性客观权重确定模型，克服了传统主观赋权的局限性。4) 融合所提算子与权重模型，建立改进的区间值毕达哥拉斯模糊 WASPAS 方法实现了备选方案的排序。最后，通过实际案例验证了方法的有效性与合理性，比较分析表明该方法在区分度与稳定性上优于现有方法，参数分析进一步证实了模型具有强鲁棒性。进一步研究将探索 Sugeno-Weber 模与其他模糊算子，如，幂算子、优先算子、Choquet 算子的混合形式以进一步提升信息融合的灵活性。

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