Advances in Applied Mathematics 应用数学进展, **2026, 15(7), 23-33**
Published Online July 2026 in Hans. https://www.hanspub.org/journal/aam
https://doi.org/10.12677/aam.2026.157298

一类四次系统的中心焦点问题

马晟焯

河南科技大学莫动理工学院，河南 洛阳

收稿日期：2026年6月7日；录用日期：2026年7月1日；发布日期：2026年7月8日

摘 要

聚焦一类平面四次微分系统在奇点处的类型判定问题，所涉系统仅保留二次项与单一四次幂项。借助极坐标变换导出径向方程，通过幂级数展开依次计算Lyapunov常数，确定前三阶为零的四类参数分布。在此基础上，给出奇点成为四阶细焦点的四组参数关系，同时获得奇点为中心的四个充分条件，并由此构造首次积分函数完成充分性证明。上述结果系统总结了该类模型低阶情形下的中心与高阶焦点特征，为后续极限环分支分析奠定理论基础。

关键词

四次微分系统，中心与焦点判别，Lyapunov常数，高阶细焦点

The Central Focal Issue of a Class IV System

Shengzhuo Ma

Modo Institute of Technology, Henan University of Science and Technology, Luoyang Henan

Received: June 7, 2026; accepted: July 1, 2026; published: July 8, 2026

Abstract

This study focuses on the type determination of a class of planar fourth-order differential systems at singularities, where the systems retain only quadratic terms and a single fourth-power term. Using polar coordinate transformations, radial equations are derived, and Lyapunov constants are calculated sequentially via power series expansions to identify four parameter distributions with zero Lyapunov coefficients up to the third order. Building upon this, four sets of parameter relationships are established for singularities to become fourth-order fine focal points, along with four sufficient conditions centered around these singularities. The sufficiency is demonstrated through the construction of first-integral functions. These results systematically summarize the central and higher-order focal characteristics of such models in low-order cases, laying a theoretical foundation for subsequent analysis of limit cycle bifurcation.

文章引用: 马晟焯. 一类四次系统的中心焦点问题[J]. 应用数学进展, 2026, 15(7): 23-33.
DOI: 10.12677/aam.2026.157298

Keywords

Fourth-Order Differential System, Center and Focus Discrimination, Lyapunov Constant, High-Order Fine Focus

1. 引言

平面多项式微分系统中，初等奇点的中心焦点判别是常微分方程定性理论的经典问题之一。该判别不仅决定奇点附近轨线是螺旋趋近还是闭轨环绕，更直接关联极限环的分支数量与生成条件。自 Andronov 等系统建立平面动力系统的分岔理论以来，相关研究取得了长足进展[1]。

在低次多项式系统方面，二次及三次系统的中心焦点理论已趋成熟。对于四次及更高次系统，由于非线性项数量随次数迅速增加，焦点量的计算复杂度显著提高，相关研究仍面临较大困难。近年来，Li 等针对一类 Z_2-对称五次多项式微分系统，通过计算前几阶 Lyapunov 常数，确定了系统中两个奇点成为中心的充要条件，并分析了细中心的阶数和局部临界周期分岔[2]。赵大虎与卢景苹针对一类四次多项式微分系统，计算了原点的焦点量，得到了原点成为中心的条件，并研究了原点邻域内小振幅极限环的分支问题[3]。张历卓与赵梅春通过计算奇点量，给出了一类四次系统奇点成为中心或焦点的判定条件[4]。陈兴武与张伟年则研究了一类时间可逆四次微分系统的细中心问题，计算了周期常数，分析了细中心的阶数与临界周期分岔[5]。

在高次系统方面，陆飞飞等利用 Poincaré 映射与反射函数方法，研究了十次高次多项式微分系统的中心焦点判定问题，推导了焦点量的计算方法[6]。此外，张齐通过 Poincaré 变换将无穷远点转化为有限奇点，研究了一类多项式微分系统无穷远点的中心焦点判定问题[7]；李时敏研究了一类拟四次解析微分系统的中心与等时中心问题，通过将拟解析系统转化为复系统，计算了奇点量和周期常数[8]；张志文与刘兴国则突破了多项式系统的限制，研究了一类非多项式微分系统的中心焦点判定问题，借助 Taylor 展开和焦点量递推公式给出了判定方法[9]。

本文采用极坐标变换与幂级数递推算法，计算了该系统前若干阶焦点量。计算结果显示，所有偶数阶焦点量恒为零，非零量仅出现在奇数阶。首先令三阶焦点量为零，归纳出四种基本参数情形。在此四分支上，继续施加五阶与七阶焦点量为零且九阶焦点量非零的约束，通过逐次代入与代数化简，最终获得四组独立的参数化条件，使得原点成为四阶弱焦点。这些条件在正文定理二中分别以(f1)至(f4)给出。与此同时，针对原点的中心情形，本文构造了相应的首次积分函数，验证了定理一中(c1)至(c4)四组中心条件的充分性。

2. 主要结果

本文研究了以下平面光滑四次系统的中心焦点判定问题，

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{40}x^4, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = x + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{40}x^4. \end{cases} \tag{1}$$

其中 $a_{20}, a_{11}, a_{02}, a_{40}, b_{20}, b_{11}, b_{02}, b_{40} \in R$，显然(0, 0)点是系统(1)的奇点，通过极坐标变换计算出焦点量，可

以得到相对应的中心条件和焦点条件。

记 V_k 为系统(1)的第 k 阶焦点量，其中 $k \ge 1$ 且为整数。在计算过程中，第一个非零的 n 阶焦点量 V_n 总是一个奇数，即 $n = 2q + 1$，其中 $q > 0$，相应的弱焦点被称为 q 阶。如果对于所有的 $q \ge 1$ 都有 $V_q = 0$，则称(0, 0)点为系统(1)的中心。为了证明主要的研究结果，我们从以下两个定理来进行佐证

定理 1：如果满足以下的条件之一，则原点为系统(1)的中心，

(c1) $b_{11} = 0, a_{20} = 0, a_{02} = 0, a_{40} = 0$，

(c2) $b_{11} = 0, a_{20} = 0, a_{11} = -2b_{02}, a_{40} = 0$，

(c3) $b_{11} = 0, a_{11} = 2b_{20}, a_{02} = 0, a_{20} = 0, a_{40} = 0$，

(c4) $b_{11} = 0, a_{11} = 2b_{20}, a_{11} = -2b_{02}, a_{40} = 0, a_{20} = 0$。

定理 2：如果满足以下条件，则原点为系统(1)的 4 阶弱焦点，

(f1) $b_{11} = 0, a_{20} = 0, a_{02} = 0, 20b_{20} + 22b_{02} + 5a_{11} = 0$，

$$b_{40} = \frac{621}{350}b_{02}^3 - \frac{31}{7000}b_{02}^2 a_{11} - \frac{4}{35}b_{02}a_{11}^2 + \frac{1}{32}a_{11}^3,$$

$$\begin{aligned} & a_{40}\left(-201772993b_{02}^5 + 623097600b_{02}^4 a_{11} + 1222908000b_{02}^3 a_{11}^2\right. \\ & \left. + 1061736000b_{02}^2 a_{11}^3 + 453780000b_{02}a_{11}^4 + 76545000a_{11}^5\right) \ne 0 \end{aligned};$$

(f2) $b_{11} = 0, a_{20} = 0, a_{11} = -2b_{02}, 3b_{02} + 5b_{20} = 0$，

$$b_{40} = \frac{25}{21}a_{02}^2 b_{20} + \frac{460}{189}b_{20}^3,$$

$$a_{40}b_{20}\left(28125a_{02}^4 + \frac{2157500}{9}a_{02}^2 b_{20}^2 + \frac{1525500}{81}b_{20}^4 + 31875a_{02}a_{40}\right) \ne 0;$$

(f3) $b_{11} = 0, a_{11} = 2b_{20}, a_{02} = 0, a_{20} \ne 0, a_{40} \ne 0$，

$$a_{40} = -\frac{a_{20}}{11b_{02} + 15b_{20}}\left(3a_{20}^2 b_{02} - 3b_{02}^3 + 9b_{02}b_{20}^2 + 6b_{20}^3 + 3b_{40}\right),$$

$$b_{40} = \frac{A}{B},$$

$$C \ne 0;$$

(f4) $b_{11} = 0, a_{11} = 2b_{20}, a_{11} = -2b_{02}, a_{20} \ne 0, a_{40} \ne 0$，

$$b_{40} = \frac{1}{6a_{20}}\left(5a_{02}^2 a_{11}a_{20} + 8a_{02}a_{11}a_{20}^2 + 3a_{11}a_{20}^3 - 4a_{11}a_{40}\right),$$

$$a_{40} = \frac{-D \pm \sqrt{D^2 - 4EF}}{336a_{11}},$$

$$G \ne 0;$$

其中

$$\begin{aligned} A = -\Big[& \left(11b_{02} + 15b_{20}\right)\left(240a_{20}^3 b_{02}^2 b_{20} + 180a_{20}^3 b_{02}b_{20}^2 - 240a_{20}b_{02}^4 b_{20} - 180a_{20}b_{02}^3 b_{20}^2\right. \\ & \left. + 720a_{20}b_{02}^2 b_{30}^3 + 1020a_{20}b_{02}b_{40}^4 + 360a_{20}b_{50}^5\right) + a_{20}\left(108b_{02} - 42b_{02}^3 - 610b_{02}^2 b_{20}\right. \\ & \left. - 1434b_{02}b_{20}^2 - 954b_{02}^3\right) \times \left(3a_{20}^2 b_{02} - 3b_{02}^3 + 9b_{02}b_{20}^2 + 6b_{02}^3\right), \end{aligned}$$

$$B = 63a_{20}\left(3a_{20}^2 b_{02} - 3b_{02}^3 + 9b_{02}b_{20}^2 + 6b_{20}^3\right) + 3a_{20}\left(108b_{02} - 42b_{02}^3 - 610b_{02}^2 b_{20} - 1434b_{02}b_{20}^2 - 954b_{20}^3\right),$$

$$
\begin{aligned}
C = {} & 464400a_{20}^5b_{02}^2b_{20} + 348300a_{20}^5b_{02}b_{20}^2 + 810852a_{20}^3b_{02}^4b_{20} + 916920a_{20}^3b_{02}^3b_{20}^2 + 3772368a_{20}^3b_{02}^2b_{20}^3 \\
& + 3494340a_{20}^3b_{02}^4b_{20}^4 + 696600a_{20}^3b_{20}^5 - 1275252a_{20}b_{02}^6b_{20} - 1265220a_{20}b_{02}^5b_{20}^2 + 1446588a_{20}b_{20}^4b_{20}^3 \\
& + 4825524a_{20}b_{20}^3b_{20}^4 + 9667944a_{20}b_{20}^2b_{20}^5 + 9320256a_{20}b_{02}b_{02}^6 + 3041280a_{20}b_{70}^2 - 87075a_{20}^4a_{40}b_{02} \\
& - 802440a_{20}^2a_{40}b_{02}^3 + 425079a_{20}^2a_{40}b_{02}^2b_{20} + 1492155a_{20}^2a_{40}b_{02}b_{20}^2 + 2089800a_{20}^2a_{40}b_{20}^3 \\
& + 324555a_{40}b_{02}^5 + 3188251a_{40}b_{02}^4b_{20} + 8727543a_{40}b_{03}^2b_{20}^2 + 13980417a_{40}b_{02}^2b_{20}^3 + 15279066a_{40}b_{02}b_{20}^4 \\
& + 7460640a_{40}b_{03}^5 + 609525b_{20}a_{20}a_{40}^2 \\
& \neq 0,
\end{aligned}
$$

$$
D = a_{20}\left(90a_{02}^2a_{11} + 300a_{02}a_{11}a_{20} + 44a_{11}^3 + 216a_{11}a_{20}^2\right) - 42\left(5a_{02}^2a_{11}a_{20} + 8a_{02}a_{11}a_{20}^2 + 3a_{11}a_{20}^3\right),
$$

$$
E = 168a_{11},
$$

$$
F = a_{20}\left(180a_{20}a_{11}a_{02}^4 + 450a_{02}^3a_{11}a_{20}^2 + 35a_{02}^2a_{11}^3a_{20} + 360a_{02}^2a_{11}a_{20}^3 + 65a_{02}a_{11}^3a_{20}^2 + 90a_{02}a_{11}a_{20}^4 + 30a_{11}^3a_{20}^3\right),
$$

$$
\begin{aligned}
G = {} & 556920a_{02}^6a_{20} + 1715040a_{02}^5a_{20}^2 + 302400a_{02}^4a_{11}^2a_{20} + 1569492a_{02}^4a_{20}^3 + 755328a_{02}^3a_{11}^2a_{20}^2 \\
& + 43362a_{02}^3a_{20}^4 + 46060a_{02}^2a_{11}^4a_{20} + 544539a_{02}^2a_{11}^2a_{20}^3 - 542160a_{02}^2a_{20}^5 + 85540a_{02}^4a_{11}^2a_{20}^2 \\
& + 33561a_{02}a_{11}^2a_{20}^4 - 174150a_{02}a_{20}^6 + 39480a_{11}^4a_{20}^3 - 58050a_{11}^2a_{20}^5 + 181440a_{02}^4a_{40} \\
& + 1151856a_{02}^3a_{20}a_{40} + 317520a_{02}^2a_{11}^2a_{40} + 1671336a_{02}^2a_{20}^2a_{40} + 609408a_{02}a_{11}^2a_{20}a_{40} \\
& + 549180a_{02}a_{20}^3a_{40} + 13552a_{11}^4a_{40} + 215748a_{11}^2a_{20}^2a_{40} - 174150a_{40}^2a_{40}.
\end{aligned}
$$

3. 讨论

定理 1 显示，原点成为中心时所有四次项系数必须为零，且二次项系数满足若干线性等式。这一结果揭示：系统的中心性完全由低次项内部的对称结构决定，高次项在此情形下不产生贡献。相反，定理 2 给出的四阶弱焦点条件中，四次项系数均非零，且各参数需服从高阶多项式约束，表明高阶细焦点的实现依赖于低次项与高次项之间的非线性耦合与精密平衡。

一个贯穿所有中心与四阶弱焦点条件的关键特征是：某个特定的交叉乘积项系数始终为零。该系数一旦非零，三阶焦点量即无法同时为零，原点至多为一阶弱焦点。由此可以断定，该交叉项系数对系统的中心焦点类型具有决定性影响，是调控奇点局部行为的一个核心参数。

从极限环分支的视角分析，四阶弱焦点的存在意味着在充分小的扰动下，原点邻域内最多可产生四个极限环。定理 2 提供的四组参数化条件，分别对应不同的二次与四次项搭配关系，为后续极限环分支的实例构造提供了多样的参数区域。

相比二次与三次系统，本文所研究的系统尽管仅保留单项四次成分，其焦点量的代数结构仍显著更为复杂。但正是由于高次项的高度稀疏性——即仅含单一方向的四次项——使得计算至九阶焦点量即可完成四阶弱焦点的判定。这体现了“稀疏高次项”结构在代数化简方面的固有优势，为更高次系统的分析提供了可行思路。

最后，定理 1 中各中心条件的充分性已通过构造首次积分严格验证，定理 2 中四阶弱焦点条件也已借助焦点量逐阶计算确认。后续研究可进一步探讨在所得四组条件下弱焦点附近实际可分支出极限环的具体数目，以及是否存在五阶或更高阶细焦点的可能性。

4. 焦点量的计算

我们先对系统(1)用极坐标进行换元，令 $x = r\cos\theta$， $y = r\sin\theta$ 代入系统并消去 dt，可得

$$
\frac{\mathrm{d}r}{\mathrm{d}\theta} = \sum_{i=2}^{\infty} R_i(\theta) r^i \text{ ，其中}
$$

$$R_2(\theta)=(a_{20}-a_{02}-b_{11})\cos^3\theta+(a_{11}+b_{20}-b_{02})\sin\theta\cos^2\theta+(a_{20}+b_{11})\cos\theta+b_{02}\sin\theta.$$

$$\begin{aligned}R_3(\theta)=&-2(a_{11}+b_{20}-b_{02})(a_{20}-a_{02}-b_{11})\cos^6\theta\\&+(a_{20}+a_{11}-a_{02}+b_{20}-b_{11}-b_{02})(a_{20}-a_{02}-b_{11}+a_{11}+b_{20}-b_{02})\sin\theta\cos^5\theta\\&+\left[(-4a_{20}-3b_{11}+5b_{02})a_{02}+(-3a_{20}+4b_{11})b_{02}+(2a_{20}-3b_{02})a_{20}+b_{20}(a_{11}-2b_{20})\right]\cos^4\theta\\&+2\sin\theta\left[-a_{20}^2+\left(a_{20}-\frac{3b_{02}}{2}\right)a_{02}+b_{02}^2\right]\cos^3\theta\\&+\left[\left(-\frac{3a_{20}}{2}-b_{11}\right)b_{02}+\frac{a_{20}^2}{2}+\frac{a_{20}b_{11}}{2}+\frac{b_{11}(a_{11}-b_{02})}{2}\right]\cos^2\theta\\&+\left[(2a_{20}+b_{11}-4b_{02})a_{02}+(a_{11}-2b_{20})b_{02}+a_{20}b_{20}\right]\sin\theta\cos\theta+a_{02}b_{02}.\end{aligned}$$

设系统 $\dfrac{\mathrm{d}r}{\mathrm{d}\theta}=\sum_{i=2}^{\infty}R_i(\theta)r^i$ 满足初值 $r(0)=c$，其解展开为：

$$r(\theta)=\sum_{k=1}^{\infty}r_k(\theta)c^k,\quad r_1(\theta)\equiv 1.$$

$\tilde{f}(\theta)=\int_0^{\theta}f(\varphi)\mathrm{d}\varphi$，则 $r_k(\theta)$ 可递推得到引理 1。

引理 1 当 $k=2,3,4$ 时，$r_k(\theta)$ 可以表示如下：

$$r_2(\theta)=\widetilde{R_2},$$

$$r_3(\theta)=\left(\widetilde{R_2}\right)^2+\widetilde{R_3},$$

$$r_4(\theta)=\left(\widetilde{R_2}\right)^3+2\widetilde{R_2}\widetilde{R_3}+\widetilde{R_2\widetilde{R_3}}+\widetilde{R_4},$$

由第二章的定理 1 可得焦点量 $V_k\ (2\le k\le 4)$ 的表达式如下：

$$V_2=0$$

$$V_3=-\frac{\pi}{4}\left(2a_{20}b_{20}-a_{20}a_{11}+b_{20}b_{11}-a_{11}a_{02}-2a_{02}b_{02}+b_{11}b_{02}\right)$$

$$V_4=0$$

$$\begin{aligned}V_5=\frac{\pi}{24}\big(&5a_{11}a_{02}^3+10a_{02}^3b_{02}+8a_{11}a_{02}^2a_{20}+4a_{11}a_{02}^2b_{11}+6a_{02}^2a_{20}b_{02}+3a_{02}^2b_{02}b_{11}+a_{11}^3a_{02}-a_{02}a_{11}^2b_{02}\\&-4a_{11}^2b_{20}a_{02}+3a_{11}a_{20}^2a_{02}+7a_{11}a_{02}a_{20}b_{11}-6a_{02}a_{11}b_{02}^2-11a_{02}a_{11}b_{02}b_{20}-a_{11}a_{02}b_{11}^2-5a_{11}a_{02}b_{20}^2\\&-10a_{02}a_{20}^2b_{02}+8a_{02}a_{20}b_{02}b_{11}-6a_{02}b_{02}^2b_{20}-a_{02}b_{02}b_{11}^2-10a_{02}b_{02}b_{20}^2+a_{11}^3a_{20}-2a_{11}^2a_{20}b_{02}\\&-5a_{11}^2b_{20}a_{20}+3a_{11}a_{20}^2b_{11}-5a_{11}b_{02}^2a_{20}-5a_{11}a_{20}b_{02}b_{20}-a_{11}a_{20}b_{11}^2-6a_{20}^3b_{02}+5a_{20}^2b_{02}b_{11}\\&+6b_{02}^3a_{20}+10a_{20}b_{02}^2b_{20}-a_{20}b_{02}b_{11}^2-5a_{40}a_{11}-6a_{20}b_{40}-22a_{40}b_{02}-20a_{40}b_{20}-3b_{11}b_{40}\big)\end{aligned}$$

5. 定理证明

定理 1 和定理 2 证明如下：

由表达式

$$\begin{aligned}V_3&=-\frac{\pi}{4}\left(2a_{20}b_{20}-a_{20}a_{11}+b_{20}b_{11}-a_{11}a_{02}-2a_{02}b_{02}+b_{11}b_{02}\right)\\&=\frac{a_{20}(a_{11}-2b_{20})+a_{02}(a_{11}+2b_{02})-b_{11}(b_{20}+b_{02})}{4}\cdot\pi\end{aligned}$$

$$\begin{aligned}V_5=\frac{\pi}{24}\big(&5a_{11}a_{02}^3+10a_{02}^3b_{02}+8a_{11}a_{02}^2a_{20}+4a_{11}a_{02}^2b_{11}+6a_{02}^2a_{20}b_{02}+3a_{02}^2b_{02}b_{11}+a_{11}^3a_{02}-a_{02}a_{11}^2b_{02}\\&-4a_{11}^2b_{20}a_{02}+3a_{11}a_{20}^2a_{02}+7a_{11}a_{02}a_{20}b_{11}-6a_{02}a_{11}b_{02}^2-11a_{02}a_{11}b_{02}b_{20}-a_{11}a_{02}b_{11}^2-5a_{11}a_{02}b_{20}^2\\&-10a_{02}a_{20}^2b_{02}+8a_{02}a_{20}b_{02}b_{11}-6a_{02}b_{02}^2b_{20}-a_{02}b_{02}b_{11}^2-10a_{02}b_{02}b_{20}^2+a_{11}^3a_{20}-2a_{11}^2a_{20}b_{02}\\&-5a_{11}^2b_{20}a_{20}+3a_{11}a_{20}^2b_{11}-5a_{11}b_{02}^2a_{20}-5a_{11}a_{20}b_{02}b_{20}-a_{11}a_{20}b_{11}^2-6a_{20}^3b_{02}+5a_{20}^2b_{02}b_{11}\\&+6b_{02}^3a_{20}+10a_{20}b_{02}^2b_{20}-a_{20}b_{02}b_{11}^2-5a_{40}a_{11}-6a_{20}b_{40}-22a_{40}b_{02}-20a_{40}b_{20}-3b_{11}b_{40}\big)\end{aligned}$$

当 $b_{11}\neq 0$ 时，令 $V_3=0$，计算出 $b_{20}=\dfrac{a_{20}a_{11}+a_{11}a_{02}+2a_{02}b_{02}-b_{11}b_{02}}{2a_{20}+b_{11}}$，把 b_{20} 代入到 V_5 得

$$V_5=\frac{\pi}{24}\cdot\frac{H}{I^2}$$

其中 $I=2a_{20}+b_{11}$

$$\begin{aligned}H=\ &5a_{11}a_{02}^3I^2+10a_{02}^3b_{02}I^2+8a_{11}a_{02}^2a_{20}I^2+4a_{11}a_{02}^2b_{11}I^2+6a_{02}^2a_{20}b_{02}I^2+3a_{02}^2b_{02}b_{11}I^2\\&+a_{11}^3a_{02}I^2-a_{02}a_{11}^2b_{02}I^2-4a_{11}^2a_{02}MI+3a_{11}a_{20}^2a_{02}I^2+7a_{11}a_{02}a_{20}b_{11}I^2-6a_{02}a_{11}b_{02}^2I^2\\&-11a_{02}a_{11}b_{02}MI-a_{11}a_{02}b_{11}^2I^2-5a_{11}a_{02}M^2-10a_{02}a_{20}^2b_{02}I^2+8a_{02}a_{20}b_{02}b_{11}I^2-6a_{02}b_{02}^2MI\\&-a_{02}b_{02}b_{11}^2I^2-10a_{02}b_{02}M^2+a_{11}^3a_{20}I^2-2a_{11}^2a_{20}b_{02}I^2-5a_{11}^2a_{20}MI+3a_{11}a_{20}^2b_{11}I^2\\&-5a_{11}b_{02}^2a_{20}I^2-5a_{11}a_{20}b_{02}MI-a_{11}a_{20}b_{11}^2I^2-6a_{20}^3b_{02}I^2+5a_{20}^2b_{02}b_{11}I^2+6b_{02}^3a_{20}I^2\\&+10a_{20}b_{02}^2MI-a_{20}b_{02}b_{11}^2I^2-5a_{40}a_{11}I^2-6a_{20}b_{40}I^2-22a_{40}b_{02}I^2-20a_{40}MI-3b_{11}b_{40}I^2\end{aligned}$$

$$M=a_{20}a_{11}+a_{11}a_{02}+2a_{02}b_{02}-b_{11}b_{02}$$

此时 V_5 表达式过于繁琐，为了简化计算令 $V_3=0$，只讨论以下四种情况：

$$b_{11}:=0\ a_{20}:=0\ a_{02}:=0\,;$$

$$b_{11}:=0\ a_{20}:=0\ a_{11}:=-2b_{02}\,;$$

$$b_{11}:=0\ a_{11}:=2b_{20}\ \ a_{02}:=0\,;$$

$$b_{11}:=0\ a_{11}:=2b_{20}\ \ a_{11}:=-2b_{02}\,;$$

(1) 当 $b_{11}:=0\ a_{20}:=0\ a_{02}:=0$ 时，$V_2:=0$

$$V_3=0$$

$$V_4=0$$

$$V_5=-\frac{a_{40}\pi\left(20b_{20}+22b_{02}+5a_{11}\right)}{24}$$

$$V_6:=0$$

$$V_7=-\frac{\pi a_{40}\left(4099b_{02}^3+9860b_{02}^2b_{20}+7375b_{02}b_{20}^2+1750b_{20}^3+875b_{40}\right)}{2000}$$

$$V_8:=0$$

$$V_9=-\frac{\pi a_{40}\left(5003307b_{02}^5+21383460b_{02}^4b_{20}+35534500b_{02}^3b_{20}^2+28805875b_{02}^2b_{20}^3+11416875b_{02}b_{20}^4+1771875b_{20}^4\right)}{52500}$$

令 $V_5=0$，根据 V_5 因式有两种情况，即 $a_{40}=0$ 或 $20b_{20}+22b_{02}+5a_{11}=0$。

若 $a_{40}=0$ 可得中心条件(c1)。

若 $20b_{20}+22b_{02}+5a_{11}=0$，得 $b_{20}=-\dfrac{11}{10}b_{02}-\dfrac{1}{4}a_{11}$，代入焦点量 V_7 和 V_9 得，

$$V_7 = -\frac{\pi a_{40}}{2000}\left[4099b_{02}^3 + 9860b_{02}^2\left(-\frac{11}{10}b_{02} - \frac{1}{4}a_{11}\right)\right.$$
$$\left. + 7375b_{02}\left(-\frac{11}{10}b_{02} - \frac{1}{4}a_{11}\right)^2 + 1750\left(-\frac{11}{10}b_{02} - \frac{1}{4}a_{11}\right)^3 + 875b_{40}\right]$$

展开并化简后得

$$V_7 = -\frac{\pi a_{40}}{32000}\left(1729b_{02}^3 - 5340b_{02}^2 a_{11} - 8700b_{02}a_{11}^2 - 2625a_{11}^3 + 14000b_{40}\right)$$

$$V_9 = -\frac{\pi a_{40}}{52500}\left[5003307b_{02}^5 + 21383460b_{02}^4\left(-\frac{11}{10}b_{02} - \frac{1}{4}a_{11}\right)\right.$$
$$+ 35534500b_{02}^3\left(-\frac{11}{10}b_{02} - \frac{1}{4}a_{11}\right)^2 + 28805875b_{02}^2\left(-\frac{11}{10}b_{02} - \frac{1}{4}a_{11}\right)^3$$
$$\left. + 11416875b_{02}\left(-\frac{11}{10}b_{02} - \frac{1}{4}a_{11}\right)^4 + 1771875\left(-\frac{11}{10}b_{02} - \frac{1}{4}a_{11}\right)^5\right]$$

展开并化简后得

$$V_9 = -\frac{\pi a_{40}}{215040000}\left(-201772993b_{02}^5 + 623097600b_{02}^4 a_{11} + 1222908000b_{02}^3 a_{11}^2\right.$$
$$\left. + 1061736000b_{02}^2 a_{11}^3 + 453780000b_{02}a_{11}^4 + 76545000a_{11}^5\right)$$

再令 $V_7 = 0$，得到 $1729b_{02}^3 - 5340b_{02}^2 a_{11} - 8700b_{02}a_{11}^2 - 2625a_{11}^3 + 14000b_{40} = 0$，解得

$$b_{40} = \frac{621}{350}b_{02}^3 - \frac{31}{7000}b_{02}^2 a_{11} - \frac{4}{35}b_{02}a_{11}^2 + \frac{1}{32}a_{11}^3$$

满足 $b_{40} = \frac{621}{350}b_{02}^3 - \frac{31}{7000}b_{02}^2 a_{11} - \frac{4}{35}b_{02}a_{11}^2 + \frac{1}{32}a_{11}^3$ 时，有

$$V_9 = -\frac{\pi a_{40}}{215040000}\left(-201772993b_{02}^5 + 623097600b_{02}^4 a_{11} + 1222908000b_{02}^3 a_{11}^2 + 1061736000b_{02}^2 a_{11}^3 + 453780000b_{02}a_{11}^4 + 76545000a_{11}^5\right)$$

，得到四阶焦点条件(f1)。

(2) $b_{11} := 0\ a_{20} := 0\ a_{11} := -2b_{20}$ 时，

$$V_2 = 0$$
$$V_3 = 0$$
$$V_4 = 0$$
$$V_5 = -\frac{a_{40}\pi\left(3b_{02} + 5b_{20}\right)}{6}$$
$$V_6 = 0$$
$$V_7 = -\frac{a_{40}\pi\left(125ba_{02}^2 b_{02} + 92b_0 b_{02}^2 + 175b_{40}\right)}{400}$$
$$V_8 := 0$$
$$V_9 = -\frac{a_{40}b_{02}\pi\left(28125a_{02}^4 + 107875a_{02}^2 b_{02}^2 + 12204b_{02}^4 + 31875a_{02}a_{40}\right)}{75000}$$

令 $V_5 = 0$，根据 V_5 因式有两种情况，即 $a_{40} = 0$ 或 $3b_{02} + 5b_{20} = 0$。

若 $a_{40} = 0$ 可得中心条件(c2)。

若 $3b_{02}+5b_{20}=0$，得 $b_{02}=-\frac{5}{3}b_{20}$，代入焦点量 V_7 和 V_9 得

$$V_7=-\frac{a_{40}\pi}{400}\left(125a_{02}^2\left(-\frac{5}{3}b_{20}\right)+92\left(-\frac{5}{3}b_{20}\right)^3+175b_{40}\right)$$

$$V_9=-\frac{a_{40}\pi}{75000}\left(-\frac{5}{3}b_{20}\right)\left(28125a_{02}^4+107875a_{02}^2\left(-\frac{5}{3}b_{20}\right)^2+12204\left(-\frac{5}{3}b_{20}\right)^4+31875a_{02}a_{40}\right)$$

展开并化简后得

$$V_7=-\frac{a_{40}\pi}{400}\left(-\frac{625}{3}a_{02}^2b_{20}-\frac{11500}{27}b_{20}^3+175b_{40}\right)$$

$$V_9=\frac{5a_{40}b_{20}\pi}{225000}\left(28125a_{02}^4+\frac{2157500}{9}a_{02}^2b_{20}^2+\frac{1525500}{81}b_{20}^4+31875a_{02}a_{40}\right)$$

再令 $V_7=0$，得到 $-\frac{625}{3}a_{02}^2b_{20}-\frac{11500}{27}b_{20}^3+175b_{40}=0$，解得

$$b_{40}=\frac{25}{21}a_{02}^2b_{20}+\frac{460}{189}b_{20}^3$$

代入 V_9 可得

$a_{40}b_{20}\pi\left(28125a_{02}^4+\frac{2157500}{9}a_{02}^2b_{20}^2+\frac{1525500}{81}b_{20}^4+31875a_{02}a_{40}\right)\neq 0$。得到四阶焦点条件($f_2$)。

(3) $b_{11}:=0$ $a_{11}:=2b_{20}$ $a_{02}:=0$ 时

$$V_2=0$$

$$V_3=0$$

$$V_4=0$$

$$V_5:=-\frac{\pi\left(3a_{20}^3b_{02}-3a_{20}b_{02}^3+9a_{20}b_{02}b_{20}^2+6a_{20}b_{20}^3+3a_{20}b_{40}+11a_{40}b_{02}+15a_{40}b_{20}\right)}{12}$$

$$V_6=0$$

$$V_7=\frac{1}{144}\left(\pi\left(240a_{20}^3b_{20}^2b_{20}+180a_{20}^3b_{02}b_{20}^2-240a_{20}b_{02}{}^4b_{20}-180a_{20}b_{02}^3b_{20}^2+720a_{20}b_{02}{}^2b_{20}^3+1020a_{20}b_{20}^4\right.\right.$$
$$\left.\left.+360a_{20}b_{20}^5-108a_{20}^2a_{40}b_{02}+42a_{40}b_{02}^3+610a_{20}b_{02}^2b_{02}+1434a_{20}b_{02}b_{20}^2+954a_{20}b_{20}^3-63a_{20}b_{40}\right)\right)$$

$$V_8=0$$

$$V_9=\frac{1}{665280}\left(\pi\left(464400a_{20}^5b_{02}^2b_{20}+348300a_{20}^5b_{02}b_{20}^2+810852a_{20}^3b_{02}^4b_{20}+916920a_{20}^3b_{02}^3b_{20}^2\right)\right.$$
$$+3772368a_{20}^3b_{02}^2b_{20}^3+3494340a_{20}^3b_{02}b_{20}^4+696600a_{20}^3b_{20}^5-1275252a_{20}b_{02}^6b_{20}$$
$$-1265220a_{20}b_{02}^5b_{20}^2+1446588a_{20}b_{02}^4b_{20}^3+4825524a_{20}b_{02}^4b_{20}^4+9667944a_{20}b_{02}^2b_{20}^5$$
$$+9320256a_{20}b_{02}b_{20}^6+3041280a_{20}b_{20}^7-87075a_{20}^4a_{40}b_{02}-802440a_{20}^2a_{40}b_{02}^3$$
$$+425079a_{20}^2a_{40}b_{02}^2b_{20}+1492155a_{20}^2a_{40}b_{02}b_{20}^2+2089800a_{20}^2a_{40}b_{20}^3+324555a_{40}b_{02}^5$$
$$+3188251a_{40}b_{02}^4b_{20}+8727543a_{40}b_{02}^3b_{20}^2+13980417a_{40}b_{02}^2b_{20}^3+15279066a_{40}b_{02}b_{20}^4$$
$$\left.+7460640a_{40}b_{20}^5+609525b_{20}a_{20}a_{20}^2\right))$$

令 $V_5=0$，根据 V_5 因式有 2 种情况，即

当 $a_{20}=0$，此时 $V_5=-\frac{\pi}{12}\left(11a_{40}b_{02}+15a_{40}b_{20}\right)=0$，即 $a_{40}\left(11b_{02}+15b_{20}\right)=0$。

若 $a_{40}=0$，可得中心条件(c3)。

若 $b_{02}+15b_{20}=0$，则 $b_{02}=-\frac{15}{11}b_{20}$，此时 $V_7=\frac{\pi}{144}\left(610a_{40}b_{02}^2b_{20}+1434a_{40}b_{02}b_{20}^2+954a_{40}b_{20}^3\right)$。

代入 V_7 得

$$V_7=\frac{\pi a_{40}b_{20}}{144}\left[610\left(-\frac{15}{11}b_{20}\right)^2+1434\left(-\frac{15}{11}b_{20}\right)b_{20}+954b_{20}^2\right]=\frac{\pi a_{40}b_{20}^3}{144}\cdot\frac{16074}{121}=\frac{893\pi a_{40}b_{20}^3}{968}。$$

令 $V_7=0$，因为 $a_{40}\neq 0$，故必须 $b_{20}=0$。但此时 $b_{02}=0$，系统退化为中心，矛盾，此时我们只能得到三阶焦点条件。

当 $a_{20}\neq 0$ 此时 $V_5=0$ 的条件为：

$$a_{20}\left(3a_{20}^2b_{02}-3b_{02}^3+9b_{02}b_{20}^2+6b_{20}^3+3b_{40}\right)+a_{40}\left(11b_{02}+15b_{20}\right)=0。$$

若 $a_{40}=0$，此时条件变为 $3a_{20}^2b_{02}-3b_{02}^3+9b_{02}b_{20}^2+6b_{20}^3+3b_{40}=0$，

$a_{20}^2=b_{02}^2-3b_{20}^2-\frac{2b_{20}^3+b_{40}}{b_{02}}\ \left(b_{02}\neq 0\right)$，此时 $a_{40}=0$ 系统退化为二次系统，所有高阶焦点量 $V_7=V_9=0$，原点为中心(c4)。

若 $a_{40}\neq 0$，此时 $V_5=0$ 的条件为：$a_{40}=-\frac{a_{20}}{11b_{02}+15b_{20}}\left(3a_{20}^2b_{02}-3b_{02}^3+9b_{02}b_{20}^2+6b_{20}^3+3b_{40}\right)$ 代入 V_7 并令 $V_7=0$ 得到：

$$\begin{aligned}&\left(11b_{02}+15b_{20}\right)\times\left(240a_{20}^3b_{02}^2b_{20}+180a_{20}^3b_{02}b_{20}^2-240a_{20}b_{02}^4b_{20}-180a_{20}b_{02}^3b_{20}^2\right.\\&\left.+720a_{20}b_{02}^2b_{20}^3+1020a_{20}b_{02}b_{20}^4+360a_{20}b_{20}^5\right)+a_{20}\left(108b_{02}-42b_{02}^3-610b_{02}^2b_{20}\right.\\&\left.-1434b_{02}b_{20}^2-954b_{20}^3+63b_{40}\right)\times\left(3a_{20}^2b_{02}-3b_{02}^3+9b_{02}b_{20}^2+6b_{20}^3+3b_{40}\right)=0\end{aligned}$$

化简得到 $b_{40}=\frac{N}{D}$

$$\begin{aligned}N=-&\left[\left(11b_{02}+15b_{20}\right)\left(240a_{20}^3b_{02}^2b_{20}+180a_{20}^3b_{02}b_{20}^2-240a_{20}b_{02}^4b_{20}-180a_{20}b_{02}^3b_{20}^2\right.\right.\\&\left.+720a_{20}b_{02}^2b_{30}^3+1020a_{20}b_{02}b_{40}^4+360a_{20}b_{50}^5\right)+a_{20}\left(108b_{02}-42b_{02}^3-610b_{02}^2b_{20}\right.\\&\left.\left.-1434b_{02}b_{20}^2-954b_{02}^3\right)\times\left(3a_{20}^2b_{02}-3b_{02}^3+9b_{02}b_{20}^2+6b_{20}^3\right)\right]\end{aligned}$$

$$D=63a_{20}\left(3a_{20}^2b_{02}-3b_{02}^3+9b_{02}b_{20}^2+6b_{20}^3\right)+3a_{20}\left(108b_{02}-42b_{02}^3-610b_{02}^2b_{20}-1434b_{02}b_{20}^2-954b_{20}^3\right)$$

将 b_{40} 重新代入到 V_9 得 $V_9\neq 0$，此时我们得到了四阶焦点条件(f3)。

(4) $b_{11}:=0\ a_{11}:=2b_{20};a_{11}:=-2b_{02}$ 时

$$V_2=0$$
$$V_3=0$$
$$V_4=0$$
$$V_5=\frac{\pi\left(5a_{02}^2a_{11}a_{20}+8a_{02}a_{11}a_{20}^2+3a_{11}a_{20}^2-4a_{11}a_{40}-6a_{20}b_{40}\right)}{24}$$
$$V_6=0$$

$$\begin{aligned}V_7=\frac{1}{576}&\left(\pi\left(180a_{20}a_{11}a_{02}^4+450a_{02}^3a_{11}a_{20}^2+35a_{02}^2a_{11}{}^3a_{20}+360a_{02}^2a_{11}a_{20}^3+65a_{02}a_{11}^3a_{20}^2+90a_{02}a_{11}a_{20}^4\right.\right.\\&\left.\left.+30a_{11}^3a_{20}^3+90a_{02}^2a_{11}11+300a_{02}a_{11}a_{20}a_{40}+44a_{11}^3a_{40}+216a_{11}a_{20}^2a_{40}-252a_{40}b_{40}\right)\right)\end{aligned}$$

$$V_8 = 0$$

$$V_9 = \frac{1}{967680}\Big(\pi a_{11}\big(556920a_{02}^6 a_{20} + 1715040a_{02}^5 a_{20}^2 + 302400a_{02}^4 a_{11}^2 a_{20} + 1569492a_{02}^4 a_{20}^3 + 755328a_{02}^3 a_{11}^2 a_{20}^2 + 43362a_{02}^3 a_{20}^4 + 46060a_{02}^2 a_{11}^4 a_{20} + 544539a_{02}^2 a_{11}^2 a_{20}^3 - 542160a_{02}^2 a_{20}^5 + 85540a_{02} a_{11}^4 a_{20}^2 + 33561a_{02} a_{11}^2 a_{20}^4 - 174150a_{02} a_{20}^6 + 39480a_{11}^4 a_{20}^3 - 58050a_{11}^4 a_{20}^5 + 181440a_{02}^4 a_{40} + 1151856a_{02}^4 a_{20} a_{40} + 317520a_{02}^2 a_{11}^4 a_{40} + 1671336a_{02}^2 a_{20}^2 a_{40} + 609408a_{02} a_{11}^2 a_{20} a_{40} + 549180a_{02} a_{20}^3 a_{40} + 13552a_{11}^4 a_{40} + 215748a_{11}^2 a_{20}^2 a_{40} - 174150a_{20}^4 a_{40}\big)\Big)$$

令 $V_5 = 0$ 可得 $V_5 = \frac{\pi}{24}\left(5a_{02}^2 a_{11} a_{20} + 8a_{02} a_{11} a_{20}^2 + 3a_{11} a_{20}^3 - 4a_{11} a_{40} - 6a_{20} b_{40}\right) = 0$。

当 $a_{40} = 0$ 时，若 $a_{20} = 0$，则 $V_5 = V_7 = V_9 = 0$，可得中心条件(c4)。

对 V_5 整理得 $b_{40} = \frac{1}{6a_{20}}\left(5a_{02}^2 a_{11} a_{20} + 8a_{02} a_{11} a_{20}^2 + 3a_{11} a_{20}^3 - 4a_{11} a_{40}\right)$ $(a_{20} \neq 0)$。

代入到 V_7 得

$$V_7 = \frac{\pi}{576}\left(180a_{20} a_{11} a_{02}^4 + 450a_{02}^3 a_{11} a_{20}^2 + 35a_{02}^2 a_{11}^3 a_{20} + 360a_{02}^2 a_{11} a_{20}^3\right) + 65a_{02} a_{11}^3 a_{20}^2 + 90a_{02} a_{11} a_{20}^4 + 30a_{11}^3 a_{20}^3 + 90a_{02}^2 a_{11} a_{40} + 300a_{02} a_{11} a_{20} a_{40} + 44a_{11}^3 a_{40} + 216a_{11} a_{20}^2 a_{40} - 42\frac{a_{40}}{a_{20}}\left(5a_{02}^2 a_{11} a_{20} + 8a_{02} a_{11} a_{20}^2 + 3a_{11} a_{20}^3 - 4a_{11} a_{40}\right)$$

令 $V_7 = 0$

$$a_{40} = \frac{-B \pm \sqrt{B^2 - 4AC}}{336a_{11}}，其中$$

$$A = 168a_{11}$$

$$B = a_{20}\left(90a_{02}^2 a_{11} + 300a_{02} a_{11} a_{20} + 44a_{11}^3 + 216a_{11} a_{20}^2\right) - 42\left(5a_{02}^2 a_{11} a_{20} + 8a_{02} a_{11} a_{20}^2 + 3a_{11} a_{20}^3\right)$$

$$C = a_{20}\left(180a_{20} a_{11} a_{02}^4 + 450a_{02}^3 a_{11} a_{20}^2 + 35a_{02}^2 a_{11}^3 a_{20} + 360a_{02}^2 a_{11} a_{20}^3 + 65a_{02} a_{11}^3 a_{20}^2 + 90a_{02} a_{11} a_{20}^4 + 30a_{11}^3 a_{20}^3\right)。$$

把 a_{40} 代入 V_9 得到 $V_9 \neq 0$，此时我们得到了四阶焦点条件(f4)。

接下来我们证明条件(c1)的充分性，当满足条件(c1)时，系统(1)具有首次积分

$$H(x, y) = \frac{1}{2}y^2 - \frac{1}{3}a_{20}x^3 - \frac{1}{2}a_{11}x^2 y - a_{02}xy^2 - \frac{1}{5}a_{40}x^5 + \left(\frac{1}{3}b_{20}x^3 + \frac{1}{2}b_{11}x^2 y + b_{02}xy^2 + \frac{1}{5}b_{40}x^5\right)y + \frac{1}{4}a_{20}b_{11}x^4 + \frac{1}{3}a_{20}b_{02}x^3 y + \frac{1}{6}a_{11}b_{20}x^4 + \frac{1}{4}a_{11}b_{11}x^3 y + \frac{1}{3}a_{11}b_{02}x^2 y^2 + \frac{1}{2}a_{02}b_{20}x^3 y + \frac{1}{2}a_{02}b_{11}x^2 y^2 + a_{02}b_{02}xy^3 + \frac{1}{5}a_{20}b_{40}x^5 + \frac{1}{4}a_{11}b_{40}x^4 y + \frac{1}{3}a_{02}b_{40}x^3 y^2 + \frac{1}{6}a_{40}b_{20}x^6 + \frac{1}{5}a_{40}b_{11}x^5 y + \frac{1}{4}a_{40}b_{02}x^4 y^2 + \frac{1}{7}a_{40}b_{40}x^7 + \left(\frac{1}{12}a_{20}^2 b_{11}x^5 + \frac{1}{10}a_{20}a_{11}b_{20}x^5 + \frac{1}{8}a_{20}a_{11}b_{11}x^4 y + \frac{1}{6}a_{20}a_{02}b_{20}x^4 y + \cdots\right) + C$$

其中 C 为任意常数，同理可证明中心条件(c2)，(c3)，(c4)的充分性。

6. 结语

本文给出了四次系统的四组中心条件和四组四阶弱焦点条件，为极限环分支研究提供了理论依据；更高阶细焦点及极限环个数有待进一步探讨。

致　谢

感谢河南科技大学数学与统计学院樊智辉老师的悉心指导。

参考文献

[1] Andronov, A.A., Leontovich, E.A., Gordon, I.I. and Maier, A.G. (1973) Theory of Bifurcations of Dynamic Systems on a Plane. Wiley.

[2] Li, H., Li, F. and Yu, P. (2020) Bi-Center Problem in a Class of Z_2-Equivariant Quintic Vector Fields. *Journal of Nonlinear Modeling and Analysis*, **2**, 57-78.

[3] 赵大虎，卢景苹. 一类四次多项式系统原点的中心条件与极限环分支[J]. 黑龙江大学自然科学学报, 2012(6): 767-770, 775.

[4] 张历卓，赵梅春. 一类四次系统奇点的中心-焦点判定[J]. 贵州大学学报(自然科学版), 2001(1): 6-8.

[5] 陈兴武，张伟年. 一类可逆四次微分系统的细中心(英文) [J]. 四川大学学报(自然科学版), 2004, 41(2): 251-255.

[6] 陆飞飞，谢丽娜，周正新. 一类十次多项式微分系统的 *Poincaré* 中心焦点问题[J]. 扬州大学学报(自然科学版), 2020, 23(5): 9-14.

[7] 张齐. 一类多项式微分系统无穷远点的中心-焦点判定[J]. 经济数学, 2007(1): 98-103.

[8] 李时敏. 一类拟四次系统的中心与等时中心(II) [J]. 桂林电子科技大学学报, 2007, 27(5): 405-408.

[9] 张志文，刘兴国. 一类非多项式微分系统中心焦点的判定[J]. 湖南工业大学学报, 2009, 23(5): 10-13.