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时延分数阶复值惯性神经网络的有限时间控制

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摘 要

本文研究带有时延的分数阶复值惯性神经网络的有限时间控制问题。首先使用变量代换法将高阶复值系统转化为四个低阶实值系统，然后根据新提出的有限时间稳定性引理，构造李亚普洛夫函数，使得驱动和响应系统可以在设计的非线性控制器下达到同步且得到其沉降时间。最后，给出一个数值仿真去检验得到的理论结果的正确性。

关键词

有限时间，惯性，复值，时延，分数阶神经网络

Finite-Time Control of Fractional-Order Complex-Valued Inertial Neural Networks with Time Delays

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Abstract

This paper studies the finite-time control problem of time-delayed fractional-order complex-valued inertial neural networks. Firstly, the higher-order complex-valued system is converted into four lower-order real-valued systems using the variable substitution method. Then, based on the newly proposed finite-time stability lemma, a Lyapunov function is constructed and a nonlinear controller is designed to guarantee that the response system can be synchronized to the drive system in

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finite time and that the settling time is derived simultaneously. Finally, a numerical example is given to check the correctness of the theoretical results.

Keywords

Finite-Time, Inertial, Complex-Valued, Time Delays, Fractional-Order Neural Networks

1. 引言

在过去的几十年中，由于神经网络具有并行计算、离散存储、自学习以及自适应的良好特性，被广泛使用到信息、金融、和医学等领域[1]。惯性神经网络作为一类重要的神经网络模型，具有很强的生物学背景以及强大的存储能力，可以解决更加复杂的实际问题，受到了很多学者的关注[2]。在神经网络的具体应用中，神经元之间的信号传输不可避免地会产生时延，这可能会导致神经网络不稳定或产生振荡，因此研究时延惯性神经网络是有意义的。

复值变量在许多领域中被广泛使用，例如图像重建、模式识别等[3]。与实值变量相比，复值变量具有更快的计算能力和更强的处理能力，可以解决一些实值变量不能解决的问题，如：XOR 问题、电磁波成像和对称性检测[4]。

分数阶微积分可以看作整数阶的推广，具有记忆、遗传等性质，广泛应用于非牛顿流体研究和新材料的力学特性分析。目前，将分数阶与神经网络结合已经成为一个热点话题，关于分数阶神经网络的动力学行为已经取得了许多有趣的结果[5]。相较整数阶系统，分数阶系统描述更加精确。

同步作为一种重要的动力学行为，表示系统的运动状态逐渐趋于相同，在安全通信、信息科学、图像传输都有重要应用[6]。为了实现同步，许多学者提出了连续控制[7]、不连续的控制方法[8]去实现同步，以上的结果是渐进的，即同步时间是趋于无穷的，无法得出具体的收敛时间，限制了其在实际中的应用。有限时间控制因具有更快的收敛速度以及可以求得明确的沉降时间[9]，在图像加密等实际应用中被广泛使用。

基于上述讨论，本文将研究时延分数阶复值惯性神经网络的有限时间控制。尽管目前已有关于时延分数阶复值神经网络的相关研究，但关于惯性神经网络的研究相对较少。本文提出了一个较为宽松的可有效判断分数阶系统达到有限时间稳定的引理，基于此得到了驱动和响应系统达到同步需要的时间。

符号说明：$\mathbb{Z}^+,\mathbb{C},\mathbb{R}$ 分别代表正整数、复数和实数，j 代表虚数单位，$j^2=-1$，$Re(\cdot)$ 表示数的实部，$Im(\cdot)$ 表示数的虚部，$\mathbb{S}=\{1,2,3,4\}$。

2. 预备知识与模型介绍

2.1. 预备知识

定义 1 [5]：函数 f 的 α 阶分数阶积分的定义为

$$ {}_{t_0}I_t^{\alpha} f(t)=\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau , $$

其中 $t\ge t_0$，$\alpha>0$，$\Gamma(\alpha)=\int_0^{+\infty}\tau^{\alpha-1}\mathrm{e}^{-\tau}\mathrm{d}\tau$。

定义 2 [5]：函数 f 的 α 阶 Caputo 分数阶导数定义为

$$ {}_{t_0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\mathrm{d}\tau , $$

其中 $n-1<\alpha<n$， $n\in\mathbb{Z}^{+}$。

特别地，当 $t_0=0$， $0<\alpha<1$ 时，有 ${}_{0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{0}^{t} (t-\tau)^{-\alpha} f'(\tau)\mathrm{d}\tau$。

2.2. 模型介绍

本文研究时延分数阶复值惯性神经网络，其驱动系统如下：

$$ {}_{0}^{C}D_t^{2\alpha} \tilde{u}_i(t) = -a_i\, {}_{0}^{C}D_t^{\alpha} \tilde{u}_i(t) - b_i \tilde{u}_i(t) + \sum_{k=1}^{n} \xi_{ik} f_k\left(\tilde{u}_k(t)\right) + \sum_{k=1}^{n} \gamma_{ik} g_k\left(\tilde{u}_k\left(t-\tau_k(t)\right)\right) + I_i(t), \tag{1} $$

其中 $\alpha\in(0,1)$， $\tilde{u}_i(t)\in\mathbb{C}$ 是第 i 个神经元的状态变量， $i\in\mathbb{Z}^{+}$， a_i 和 b_i 是常数， n 代表神经网络中神经元的数量， $\xi_{ik}\in\mathbb{C}$ 和 $\gamma_{ik}\in\mathbb{C}$ 表示神经元之间的链接权重， $f_k\left(\tilde{u}_k(t)\right)\in\mathbb{C}$ 和 $g_k\left(\tilde{u}_k\left(t-\tau_k(t)\right)\right)\in\mathbb{C}$ 分别表示不带有时滞和带有时滞的复值神经元的激活函数， $\tau_k(t)$ 是时滞， $I_i\in\mathbb{C}$ 是外部输入变量。系统(1)的初始条件是 $\tilde{u}_i(c)=\tilde{\varphi}_i(c)$ 和 ${}_{0}^{C}D_t^{\alpha}\tilde{u}_i(c)=\tilde{\psi}_i(c)$， $c\in(-\infty,0]$。

响应系统如下：

$$ \begin{aligned} {}_{0}^{C}D_t^{2\alpha} \hat{u}_i(t) = &-a_i\, {}_{0}^{C}D_t^{\alpha} \hat{u}_i(t) - b_i \hat{u}_i(t) + \sum_{k=1}^{n} \xi_{ik} f_k\left(\hat{u}_k(t)\right) \\ &+ \sum_{k=1}^{n} \gamma_{ik} g_k\left(\hat{u}_k\left(t-\tau_k(t)\right)\right) + I_i(t) + C_i(t), \end{aligned} \tag{2} $$

其中 C_i 是控制输入，系统(2)的初始值为 $\hat{u}_i(c)=\hat{\varphi}_i(c)$ 和 ${}_{0}^{C}D_t^{\alpha}\hat{u}_i(c)=\hat{\psi}_i(c)$， $c\in(-\infty,0]$。

定义同步误差 $u_i(t)=\hat{u}_i(t)-\tilde{u}_i(t)$，则根据系统(1)和系统(2)可得误差系统为：

$$ \begin{aligned} {}_{0}^{C}D_t^{2\alpha} u_i(t) = &-a_i\, {}_{0}^{C}D_t^{\alpha} u_i(t) - b_i u_i(t) \\ &+ \sum_{k=1}^{n} \xi_{ik} \left(F_k\left(u_k(t)\right)\right) + \sum_{k=1}^{n} \gamma_{ik} \left(G_k\left(u_k\left(t-\tau_k(t)\right)\right)\right) + C_i(t), \end{aligned} \tag{3} $$

其中 $F(u_k) = f_k\left(\hat{u}_k(t)\right) - f_k\left(\tilde{u}_k(t)\right)$， $G\left(u_k\left(t-\tau_k(t)\right)\right) = g_k\left(\hat{u}_k\left(t-\tau_k(t)\right)\right) - g_k\left(\tilde{u}_k\left(t-\tau_k(t)\right)\right)$。

对系统(3)做变换： $v_i(t) = u_i(t) + {}_{0}^{C}D_t^{\alpha} u_i(t) - C_{1i}(t)$ 得：

$$ \begin{cases} {}_{0}^{C}D_t^{\alpha} u_i(t) = v_i(t) - u_i(t) + C_{1i}(t), \\ {}_{0}^{C}D_t^{\alpha} v_i(t) = (1-a_i) v_i(t) - (1-a_i+b_i) u_i(t) + \sum_{k=1}^{n} \xi_{ik} \left(f_k\left(\hat{u}_k(t)\right) - f_k\left(\tilde{u}_k(t)\right)\right) \\ \qquad + \sum_{k=1}^{n} \gamma_{ik} \left(g_k\left(\hat{u}_k\left(t-\tau_k(t)\right)\right) - g_k\left(\tilde{u}_k\left(t-\tau_k(t)\right)\right)\right) + C_{2i}(t), \end{cases} \tag{4} $$

其中 $C_{2i}(t) + {}_{0}^{C}D_t^{\alpha} C_{1i}(t) - (1-a_i) C_{1i}(t) = C_i(t)$。

假设 1 [10]：对于任意的复值 u_i, v_i，可以表示为 $u_i = x_i + jy_i$， $v_i = z_i + jw_i$， j 为虚数单位，则激活函数可以表示为：

$$ \begin{cases} f_i(u_i) = f_i^{R}(x_i, y_i) + j f_i^{I}(x_i, y_i), \\ g_i(u_i) = g_i^{R}(x_i, y_i) + j g_i^{I}(x_i, y_i), \end{cases} $$

并且存在常数 $L_i^{RR}, L_i^{RI}, L_i^{IR}, L_i^{II}, M_i^{RR}, M_i^{RI}, M_i^{IR}, M_i^{II}$，满足

$$\begin{cases}\left|f_i^R(\hat{x}_i,\hat{y}_i)-f_i^R(\tilde{x}_i,\tilde{y}_i)\right|\le L_i^{RR}\left|\hat{x}_i-\tilde{x}_i\right|+L_i^{RI}\left|\hat{y}_i-\tilde{y}_i\right|,\\ \left|f_i^I(\hat{x}_i,\hat{y}_i)-f_i^I(\tilde{x}_i,\tilde{y}_i)\right|\le L_i^{IR}\left|\hat{x}_i-\tilde{x}_i\right|+L_i^{II}\left|\hat{y}_i-\tilde{y}_i\right|,\end{cases}$$

和

$$\begin{cases}\left|g_i^R(\hat{x}_i,\hat{y}_i)-g_i^R(\tilde{x}_i,\tilde{y}_i)\right|\le M_i^{RR}\left|\hat{x}_i-\tilde{x}_i\right|+M_i^{RI}\left|\hat{y}_i-\tilde{y}_i\right|,\\ \left|g_i^I(\hat{x}_i,\hat{y}_i)-g_i^I(\tilde{x}_i,\tilde{y}_i)\right|\le M_i^{IR}\left|\hat{x}_i-\tilde{x}_i\right|+M_i^{II}\left|\hat{y}_i-\tilde{y}_i\right|。\end{cases}$$

在假设 1 下，误差系统(4)可以表示如下：

$$\begin{cases}{}_0^CD_t^\alpha x_i(t)=z_i(t)-x_i(t)+C_{1i}^R(t),\\ {}_0^CD_t^\alpha y_i(t)=w_i(t)-y_i(t)+C_{1i}^I(t),\\ {}_0^CD_t^\alpha z_i(t)=(1-a_i)z_i(t)-(1-a_i+b_i)x_i(t)+\sum_{k=1}^n\xi_{ik}^R\left(F^R(u_k(t))\right)-\sum_{k=1}^n\xi_{ik}^I\left(F^I(u_k(t))\right)\\ \qquad+\sum_{k=1}^n\gamma_{ik}^R\left(G_k^R\left(u_k(t-\tau_k(t))\right)\right)-\sum_{k=1}^n\gamma_{ik}^I\left(G_k^I\left(u_k(t-\tau_k(t))\right)\right)+C_{2i}^R(t),\\ {}_0^CD_t^\alpha w_i(t)=(1-a_i)w_i(t)-(1-a_i+b_i)y_i(t)+\sum_{k=1}^n\xi_{ik}^R\left(F_k^I(u_k(t))\right)+\sum_{k=1}^n\xi_{ik}^I\left(F_k^R(u_k(t))\right)\\ \qquad+\sum_{k=1}^n\gamma_{ik}^R\left(G_k^I\left(u_k(t-\tau_k(t))\right)\right)+\sum_{k=1}^n\gamma_{ik}^I\left(G_k^R\left(u_k(t-\tau_k(t))\right)\right)+C_{2i}^I(t)。\end{cases}\tag{5}$$

下面介绍有限时间同步的定义。

定义 3：如果存在依赖误差系统(3)的初始值的时间 $T(u(0))\in[0,+\infty)$，使得 $\lim_{t\to T(u(0))}u_i(t)=0$，并且当 $t>T(u(0))$ 时，有 $u_i(t)=0$，则驱动系统(1)和响应系统(2)可以实现有限时间同步，$T(u(0))$ 称为沉降时间。

2.3. 相关理论

引理 1 [11]：对于任意的 $\alpha\in(0,1)$, $r\in\mathbb{R}$，有：

$${}_0^CD_t^\alpha v^r(t)=\frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)}v^{r-\alpha}(t)\,{}_0^CD_t^\alpha v(t)。$$

引理 2 [11]：如果 ${}_0^CD_t^\alpha v(t)$ 是可积的，则：

$${}_{t_0}I_t^\alpha\,{}_{t_0}^CD_t^\alpha v(t)=v(t)-\sum_{l=0}^{n-1}\frac{f^{(l)}(t_0)}{l!}(t-t_0)^l。$$

特别地，当 $t_0=0$，$\alpha\in(0,1)$ 时，${}_0I_t^\alpha\,{}_0^CD_t^\alpha v(t)=v(t)-v(0)$。

引理 3：对于任意的时间 $t\ge T$，存在关于 $x(t)$ 是连续的、正定的和径向无界的函数 $v(x(t))$，满足：

$${}_0^CD_t^\alpha v(x(t))\le\lambda_1v^\eta(x(t))-\lambda_2v^\beta(x(t))-\lambda_3v^\gamma(x(t)),$$

其中 $\alpha\in(0,1)$，$\lambda_2,\lambda_3>0$，$\lambda_1<\min\{\lambda_2,\lambda_3\}$，$0<\beta<\eta<\gamma<\alpha<1$，则分数阶微分系统实现有限时间稳定，沉降时间估计为：

$$T=\left[\frac{\Gamma(1-\gamma)\Gamma(1+\alpha)}{(\lambda_3-\lambda_1)\Gamma(1+\alpha-\gamma)}\left(v^{\alpha-\gamma}(x(0))-1\right)\right]^{\frac{1}{\alpha}}+\left[\frac{\Gamma(1-\beta)\Gamma(1+\alpha)}{(\lambda_2-\lambda_1)\Gamma(1+\alpha-\beta)}\right]^{\frac{1}{\alpha}}。$$

证明：当 $v(x(t))\ge 1$ 时，根据条件可以得 $v^{\beta}(x(t))\le v^{\eta}(x(t))\le v^{\gamma}(x(t))$，有：

$$\begin{aligned}{}_{0}^{C}D_{t}^{\alpha}v(x(t))&\le\lambda_1 v^{\eta}(x(t))-\lambda_2 v^{\beta}(x(t))-\lambda_3 v^{\gamma}(x(t))\\&\le\lambda_1 v^{\eta}(x(t))-\lambda_3 v^{\gamma}(x(t))\\&\le-(\lambda_3-\lambda_1)v^{\gamma}(x(t))。\end{aligned}\tag{6}$$

当 $0<v(x(t))<1$ 时，根据条件可以得 $v^{\gamma}(x(t))\le v^{\eta}(x(t))\le v^{\beta}(x(t))$，有：

$$\begin{aligned}{}_{0}^{C}D_{t}^{\alpha}v(x(t))&\le\lambda_1 v^{\eta}(x(t))-\lambda_2 v^{\beta}(x(t))-\lambda_3 v^{\gamma}(x(t))\\&\le\lambda_1 v^{\eta}(x(t))-\lambda_2 v^{\beta}(x(t))\\&\le-(\lambda_2-\lambda_1)v^{\beta}(x(t))。\end{aligned}\tag{7}$$

假设初始值 $v(x(0))\ge 1$，满足(6)式条件，在(6)式两边同乘 $\frac{\Gamma(1+\alpha-\gamma)}{\Gamma(1-\gamma)}v^{-\gamma}(x(t))$ 可以得：

$$\frac{\Gamma(1+\alpha-\gamma)}{\Gamma(1-\gamma)}v^{-\gamma}(x(t))\,{}_{0}^{C}D_{t}^{\alpha}v(x(t))\le-\frac{(\lambda_3-\lambda_1)\Gamma(1+\alpha-\gamma)}{\Gamma(1-\gamma)},$$

根据引理 1，有

$${}_{0}^{C}D_{t}^{\alpha}v^{\alpha-\gamma}(x(t))\le-\frac{(\lambda_3-\lambda_1)\Gamma(1+\alpha-\gamma)}{\Gamma(1-\gamma)},$$

根据引理 2，有

$$v^{\alpha-\gamma}(x(t))-v^{\alpha-\gamma}(x(0))\le-\frac{(\lambda_3-\lambda_1)\Gamma(1+\alpha-\gamma)}{\Gamma(1-\gamma)\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\,\mathrm{d}\tau,$$

$$v(x(t))\le\left[-\frac{(\lambda_3-\lambda_1)\Gamma(1+\alpha-\gamma)t^{\alpha}}{\Gamma(1-\gamma)\Gamma(1+\alpha)}+v^{\alpha-\gamma}(x(0))\right]^{\frac{1}{\alpha-\gamma}}。$$

从(6)式可以得到，$v(x(t))$ 是关于时间 t 单调递减的，于是存在时间 $\tilde{T}$，有 $v(x(\tilde{T}))=1$。令

$$v(x(\tilde{T}))=1=\left[-\frac{(\lambda_3-\lambda_1)\Gamma(1+\alpha-\gamma)\tilde{T}^{\alpha}}{\Gamma(1-\gamma)\Gamma(1+\alpha)}+v^{\alpha-\gamma}(x(0))\right]^{\frac{1}{\alpha-\gamma}},$$

则 $\tilde{T}=\left[\frac{\Gamma(1-\gamma)\Gamma(1+\alpha)}{(\lambda_3-\lambda_1)\Gamma(1+\alpha-\gamma)}\left(v^{\alpha-\gamma}(x(0))-1\right)\right]^{\frac{1}{\alpha}}$。

当到达时刻 $\tilde{T}$ 时，$v(x(\tilde{T}))=1$。随时间增加，$v(x(t))$ 的值会继续减小，满足(7)式的条件。在(7)式两边同乘 $\frac{\Gamma(1+\alpha-\beta)}{\Gamma(1-\beta)}v^{-\beta}(x(t))$ 可以得：

$$\frac{\Gamma(1+\alpha-\beta)}{\Gamma(1-\beta)}v^{-\beta}(x(t))\,{}_{0}^{C}D_{t}^{\alpha}v(x(t))\le-\frac{(\lambda_2-\lambda_1)\Gamma(1+\alpha-\beta)}{\Gamma(1-\beta)},$$

同理，根据引理 1 和引理 2 可得：

$$v^{\alpha-\beta}(x(t))-v^{\alpha-\beta}(x(\tilde{T}))\le-\frac{(\lambda_2-\lambda_1)\Gamma(1+\alpha-\beta)}{\Gamma(1-\beta)\Gamma(\alpha)}\int_{\tilde{T}}^{t}(t-\tau)^{\alpha-1}\,\mathrm{d}\tau,$$

$$v(x(t)) \le \left[-\frac{(\lambda_2-\lambda_1)\Gamma(1+\alpha-\beta)(t-\tilde{T})^\alpha}{\Gamma(1-\beta)\Gamma(1+\alpha)} + v^{\alpha-\beta}(x(\tilde{T}))\right]^{\frac{1}{\alpha-\beta}},$$

与前文相同，$v(x(\hat{T})) = 0 = \left[-\frac{(\lambda_2-\lambda_1)\Gamma(1+\alpha-\beta)(\hat{T}-\tilde{T})^\alpha}{\Gamma(1-\beta)\Gamma(1+\alpha)} + v^{\alpha-\beta}(x(\tilde{T}))\right]^{\frac{1}{\alpha-\beta}}$，通过计算可得

$$\begin{aligned}\hat{T} &= \tilde{T} + \left[\frac{\Gamma(1-\beta)\Gamma(1+\alpha)}{(\lambda_2-\lambda_1)\Gamma(1+\alpha-\beta)}\right]^{\frac{1}{\alpha}} \\ &= \left[\frac{\Gamma(1-\gamma)\Gamma(1+\alpha)}{(\lambda_3-\lambda_1)\Gamma(1+\alpha-\gamma)}\left(v^{\alpha-\gamma}(x(0))-1\right)\right]^{\frac{1}{\alpha}} + \left[\frac{\Gamma(1-\beta)\Gamma(1+\alpha)}{(\lambda_2-\lambda_1)\Gamma(1+\alpha-\beta)}\right]^{\frac{1}{\alpha}}。\end{aligned}$$

综上可得：

$$T = \left[\frac{\Gamma(1-\gamma)\Gamma(1+\alpha)}{(\lambda_3-\lambda_1)\Gamma(1+\alpha-\gamma)}\left(v^{\alpha-\gamma}(x(0))-1\right)\right]^{\frac{1}{\alpha}} + \left[\frac{\Gamma(1-\beta)\Gamma(1+\alpha)}{(\lambda_2-\lambda_1)(1+\alpha-\beta)}\right]^{\frac{1}{\alpha}},$$

由于 $v(x(t))$ 是正定的，因此当 $t \ge T$ 时，有 $v(x(t)) = 0$ 且 $x(t) = 0$。

引理 4 [12]：若 $v(t)$ 是连续可微函数，则有：

$${}_0^C D_t^\alpha |v(t)| \le sign(v(t))\, {}_0^C D_t^\alpha v(t),$$

其中 $0 < \alpha \le 1$，$sign(\cdot)$ 表示符号函数。

引理 5 [12]：如果 $x_i \ge 0 (i = 1,2,\cdots,n)$，$0 < p \le 1$，$1 < q < +\infty$，则有

$$n\left(\sum_{i=1}^n x_i\right)^p \ge \sum_{i=1}^n x_i^p \ge \left(\sum_{i=1}^n x_i\right)^p,\quad \sum_{i=1}^n x_i^q \ge n^{1-q}\left(\sum_{i=1}^n x_i\right)^q。$$

引理 6：如果 $0 < x_1 < x_2 < y_1 < y_2$，则对于任意的正定函数 $f(t)$ 有

$$f^{x_1}(t) + f^{y_2}(t) \ge \frac{1}{2}\left(f^{x_2}(t) + f^{y_1}(t)\right)。$$

证明：当 $0 \le f(t) < 1$ 时，由于 $f^{x_1}(t) \ge f^{x_2}(t) \ge f^{y_1}(t) \ge f^{y_2}(t)$，于是有

$$2\left(f^{x_1}(t) + f^{y_2}(t)\right) \ge 2f^{x_1}(t) \ge f^{x_2}(t) + f^{y_1}(t),$$

当 $f(t) \ge 1$ 时，由 $f^{x_1}(t) \le f^{x_2}(t) \le f^{y_1}(t) \le f^{y_2}(t)$，可得

$$2\left(f^{x_1}(t) + f^{y_2}(t)\right) > 2f^{y_2}(t) \ge f^{x_2}(t) + f^{y_1}(t)。$$

引理 7：如果 $0 < x_1 < x_2 < y_1 < y_2 < z_1 < z_2$，则对于任意的正定函数 $f(t)$ 有

$$2f^{x_1}(t) - \frac{1}{2}\left(f^{y_1}(t) + f^{z_1}(t)\right) \ge f^{x_2}(t) - f^{y_2}(t) - f^{z_2}(t)。$$

证明：根据引理 6，有

$$f^{x_1}(t) + f^{y_2}(t) \ge \frac{1}{2}\left(f^{x_2}(t) + f^{y_1}(t)\right)。$$

同理

$$f^{x_1}(t)+f^{z_2}(t)\geq\frac{1}{2}\left(f^{x_2}(t)+f^{z_1}(t)\right)。$$

因此

$$2f^{x_1}(t)-\frac{1}{2}\left(f^{z_1}(t)+f^{y_1}(t)\right)\geq f^{x_2}(t)-f^{y_2}(t)-f^{z_2}(t)。$$

3. 主要结果

在这一部分，将系统(1)和系统(2)的同步问题转化为误差系统(3)的稳定性问题。通过设置控制器，使用引理 3，确定误差系统的稳定性，并且给出沉降时间。

定理 1：在假设 1 的条件下，通过设计如下的控制器：

$$\begin{cases} C_{1i}^{R}=-\theta_{1i}x_i(t)+sign\left(x_i(t)\right)\left(d_{1i}\left|x_i(t)\right|^{o_1}-h_{1i}\left|x_i(t)\right|^{\varpi_1}-l_{1i}\left|x_i(t)\right|^{\sigma_1}\right), \\ C_{1i}^{I}=-\theta_{2i}y_i(t)+sign\left(y_i(t)\right)(d_{2i}\left|y_i(t)\right|^{o_2}-h_{2i}\left|y_i(t)\right|^{\varpi_2}-l_{2i}\left|y_i(t)\right|^{\sigma_2} \\ \qquad -e_{1i}\left(\sum_{k=1}^{n}\left|x_k\left(t-\tau_k(t)\right)\right|\right), \\ C_{2i}^{R}=-\theta_{3i}z_i(t)+sign\left(z_i(t)\right)(d_{3i}\left|z_i(t)\right|^{o_3}-h_{3i}\left|z_i(t)\right|^{\varpi_3}-l_{3i}\left|z_i(t)\right|^{\sigma_3}), \\ C_{2i}^{I}=-\theta_{4i}w_i(t)+sign\left(w_i(t)\right)\left(d_{4i}\left|w_i(t)\right|^{o_4}-h_{4i}\left|w_i(t)\right|^{\varpi_4}-l_{4i}\left|w_i(t)\right|^{\sigma_4}\right) \\ \qquad -e_{2i}\left(\sum_{k=1}^{n}\left|y_k\left(t-\tau_k(t)\right)\right|\right), \end{cases} \tag{8}$$

满足如下条件：

$$\begin{aligned} &\theta_{1i}>\left|1-a_i+b_i\right|+\sum_{k=1}^{n}\left(\left|\xi_{ki}^{R}\right|+\left|\xi_{ki}^{I}\right|\right)\left(L_i^{RR}+L_i^{IR}\right)-1, \\ &\theta_{2i}>\left|1-a_i+b_i\right|+\sum_{k=1}^{n}\left(\left|\xi_{ki}^{R}\right|+\left|\xi_{ki}^{I}\right|\right)\left(L_i^{RI}+L_i^{II}\right)-1, \\ &\theta_{3i}>2-a_i,\theta_{4i}>2-a_i,e_{1i}>\sum_{k=1}^{n}\left(\left|\gamma_{ki}^{R}\right|+\left|\gamma_{ki}^{I}\right|\right)\left(M_i^{RR}+M_i^{IR}\right), \\ &e_{2i}>\sum_{k=1}^{n}\left(\left|\gamma_{ki}^{R}\right|+\left|\gamma_{ki}^{I}\right|\right)\left(M_i^{RI}+M_i^{II}\right),0\leq 8\left(nd_{si}\right)^{\frac{o^*}{o_s}}<\min\left\{\frac{1}{2}l_{si}^{\frac{\sigma^*}{\sigma_s}},\frac{1}{2}h_{si}^{\frac{\varpi^*}{\varpi_s}}\right\}, \\ &\sigma^*=\min\left\{\sigma_s\right\}<\alpha<1,o^*=\min\left\{o_s\right\}<\sigma^*,\varpi^*=\min\left\{\varpi_s\right\}<o^*,s\in\mathbb{S}。 \end{aligned} \tag{9}$$

则误差系统(5)可以实现有限时间稳定，沉降时间估计为：

$$T=\left[\frac{\Gamma\left(1-\sigma^*\right)\Gamma\left(1+\alpha\right)}{\left(\frac{1}{2}\overline{l}-8\overline{d}\right)\Gamma\left(1+\alpha-\sigma^*\right)}\left(v^{\alpha-\sigma^*}\left(x(0)-1\right)\right)\right]^{\frac{1}{\alpha}}+\left[\frac{\Gamma\left(1-\varpi^*\right)\Gamma\left(1+\alpha\right)}{\left(\frac{1}{2}\overline{h}-8\overline{d}\right)\Gamma\left(1+\alpha-\varpi^*\right)}\right]^{\frac{1}{\alpha}} \tag{10}$$

其中 $\overline{d}=\max_{s\in\mathbb{S}}\left\{\left(nd_{si}\right)^{\frac{o^*}{o_s}}\right\}$，$\overline{h}=\min_{s\in\mathbb{S}}\left\{h_{si}^{\frac{\varpi^*}{\varpi_s}}\right\}$，$\overline{l}=\min_{s\in\mathbb{S}}\left\{l_{si}^{\frac{\sigma^*}{\sigma_s}}\right\}$。

证明：构造如下的李亚普洛夫函数：

$$v(t)=\sum_{i=1}^{n}\left(\left|x_i(t)\right|+\left|y_i(t)\right|+\left|z_i(t)\right|+\left|w_i(t)\right|\right), \tag{11}$$

沿着误差系统(5)的轨迹，对上式李亚普洛夫函数求α阶导数，根据引理4可得：

$$\begin{aligned}
{}_0^CD_t^\alpha v(t)\le&\sum_{i=1}^{n}\left(sign\left(x_i(t)\right){}_0^CD_t^\alpha x_i(t)+sign\left(y_i(t)\right){}_0^CD_t^\alpha y_i(t)+sign\left(z_i(t)\right){}_0^CD_t^\alpha z_i(t)+sign\left(w_i(t)\right){}_0^CD_t^\alpha w_i(t)\right)\\
\le&\sum_{i=1}^{n}sign\left(x_i(t)\right)\left(z_i(t)-x_i(t)-\theta_{1i}x_i(t)+sign\left(x_i(t)\right)\left(d_{1i}\left|x_i(t)\right|^{o_1}-h_{1i}\left|x_i(t)\right|^{\varpi_1}-l_{1i}\left|x_i(t)\right|^{\sigma_1}\right)\right)\\
&+\sum_{i=1}^{n}sign\left(y_i(t)\right)\Bigg(w_i(t)-y_i(t)-\theta_{2i}y_i(t)+sign\left(y_i(t)\right)\\
&\left(d_{2i}\left|y_i(t)\right|^{o_2}-h_{2i}\left|y_i(t)\right|^{\varpi_2}-l_{2i}\left|y_i(t)\right|^{\sigma_2}-e_{1i}\sum_{k=1}^{n}\left|x_k\left(t-\tau_k(t)\right)\right|\right)\Bigg)\\
&+\sum_{i=1}^{n}sign\left(z_i(t)\right)\Bigg(\left(1-a_i\right)z_i(t)-\left(1-a_i+b_i\right)x_i(t)+\sum_{k=1}^{n}\xi_{ik}^R\left(F_k^R\left(u_k(t)\right)\right)\\
&-\sum_{k=1}^{n}\xi_{ik}^I\left(F_k^I\left(u_k(t)\right)\right)+\sum_{k=1}^{n}\gamma_{ik}^R\left(G_k^R\left(u_k\left(t-\tau_k(t)\right)\right)\right)-\sum_{k=1}^{n}\gamma_{ik}^I\left(G_k^I\left(u_k\left(t-\tau_k(t)\right)\right)\right)-\theta_{3i}z_i(t)\\
&+sign\left(z_i(t)\right)\left(d_{3i}\left|z_i(t)\right|^{o_3}-h_{3i}\left|z_i(t)\right|^{\varpi_3}-l_{3i}\left|z_i(t)\right|^{\sigma_3}\right)\Bigg)\\
&+\sum_{i=1}^{n}sign\left(w_i(t)\right)\Bigg(\left(1-a_i\right)w_i(t)-\left(1-a_i+b_i\right)y_i(t)+\sum_{k=1}^{n}\xi_{ik}^R\left(F_k^I\left(u_k(t)\right)\right)\\
&+\sum_{k=1}^{n}\xi_{ik}^I\left(F_k^R\left(u_k(t)\right)\right)+\sum_{k=1}^{n}\gamma_{ik}^R\left(G_k^I\left(u_k\left(t-\tau_k(t)\right)\right)\right)+\sum_{k=1}^{n}\gamma_{ik}^I\left(G_k^R\left(u_k\left(t-\tau_k(t)\right)\right)\right)-\theta_{4i}w_i(t)\\
&+sign\left(w_i(t)\right)\left(d_{4i}\left|w_i(t)\right|^{o_4}-h_{4i}\left|w_i(t)\right|^{\varpi_4}-l_{4i}\left|w_i(t)\right|^{\sigma_4}-e_{2i}\sum_{k=1}^{n}\left|y_k\left(t-\tau_k(t)\right)\right|\right)\Bigg)。
\end{aligned}$$

根据假设1有：

$$\begin{aligned}
{}_0^CD_t^\alpha v(t)\le&\sum_{i=1}^{n}sign\left(x_i(t)\right)\left(z_i(t)-x_i(t)-\theta_{1i}x_i(t)+sign\left(x_i(t)\right)\left(d_{1i}\left|x_i(t)\right|^{o_1}-h_{1i}\left|x_i(t)\right|^{\varpi_1}-l_{1i}\left|x_i(t)\right|^{\sigma_1}\right)\right)\\
&+\sum_{i=1}^{n}sign\left(y_i(t)\right)\Bigg(w_i(t)-y_i(t)-\theta_{2i}y_i(t)+sign\left(y_i(t)\right)\\
&\left(d_{2i}\left|y_i(t)\right|^{o_2}-h_{2i}\left|y_i(t)\right|^{\varpi_2}-l_{2i}\left|y_i(t)\right|^{\sigma_2}-e_{1i}\sum_{k=1}^{n}\left|x_k\left(t-\tau_k(t)\right)\right|\right)\Bigg)\\
&+\sum_{i=1}^{n}sign\left(z_i(t)\right)\Bigg(\left(1-a_i\right)z_i(t)-\left(1-a_i+b_i\right)x_i(t)+\sum_{k=1}^{n}\left|\xi_{ki}^R\right|\left(L_i^{RR}\left|x_i(t)\right|+L_i^{RI}\left|y_k(t)\right|\right)\\
&+\sum_{k=1}^{n}\left|\xi_{ki}^I\right|\left(L_i^{IR}\left|x_i(t)\right|+L_i^{II}\left|y_i(t)\right|\right)+\sum_{k=1}^{n}\left|\gamma_{ki}^R\right|\left(M_i^{RR}\left|x_i\left(t-\tau_i(t)\right)\right|+M_i^{RI}\left|y_i\left(t-\tau_i(t)\right)\right|\right)\\
&+\sum_{k=1}^{n}\left|\gamma_{ki}^I\right|\left(M_i^{IR}\left|x_i\left(t-\tau_i(t)\right)\right|+M_i^{II}\left|y_i\left(t-\tau_i(t)\right)\right|\right)-\theta_{3i}z_i(t)\\
&+sign\left(z_i(t)\right)\left(d_{3i}\left|z_i(t)\right|^{o_3}-h_{3i}\left|z_i(t)\right|^{\varpi_3}-l_{3i}\left|z_i(t)\right|^{\sigma_3}\right)\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}sign\left(w_i(t)\right)\Bigg(\left(1-a_i\right)w_i(t)-\left(1-a_i+b_i\right)y_i(t)+\sum_{k=1}^{n}\left|\xi_{ki}^{R}\right|\left(L_i^{IR}\left|x_i(t)\right|+L_i^{II}\left|y_i(t)\right|\right)\\
&+\sum_{k=1}^{n}\left|\xi_{ki}^{I}\right|\left(L_i^{RR}\left|x_i(t)\right|+L_i^{RI}\left|y_i(t)\right|\right)+\sum_{k=1}^{n}\left|\gamma_{ki}^{R}\right|\left(M_i^{IR}\left|x_i\left(t-\tau_i(t)\right)\right|+M_i^{II}\left|y_i\left(t-\tau_i(t)\right)\right|\right)\\
&+\sum_{k=1}^{n}\left|\gamma_{ki}^{I}\right|\left(M_i^{RR}\left|x_i\left(t-\tau_i(t)\right)\right|+M_i^{RI}\left|y_i\left(t-\tau_i(t)\right)\right|\right)-\theta_{4i}w_i(t)+sign\left(w_i(t)\right)\\
&\left(d_{4i}\left|w_i(t)\right|^{o_4}-h_{4i}\left|w_i(t)\right|^{\varpi_4}-l_{4i}\left|w_i(t)\right|^{\sigma_4}-e_{2i}\sum_{k=1}^{n}\left|y_k\left(t-\tau_k(t)\right)\right|\right)\Bigg)\\
&\le\sum_{i=1}^{n}\Bigg(\left(-1+\left|1-a_i+b_i\right|+\sum_{k=1}^{n}\left|\xi_{ki}^{R}\right|L_i^{RR}+\sum_{k=1}^{n}\left|\xi_{ki}^{I}\right|L_i^{IR}+\sum_{k=1}^{n}\left|\xi_{ki}^{R}\right|L_i^{IR}+\sum_{k=1}^{n}\left|\xi_{ki}^{I}\right|L_i^{RR}-\theta_{1i}\right)\left|x_i(t)\right|\\
&+\left(d_{1i}\left|x_i(t)\right|^{o_1}-h_{1i}\left|x_i(t)\right|^{\varpi_1}-l_{1i}\left|x_i(t)\right|^{\sigma_1}\right)\\
&+\left(-1+\left|1-a_i+b_i\right|+\sum_{k=1}^{n}\left|\xi_{ki}^{R}\right|L_i^{II}+\sum_{k=1}^{n}\left|\xi_{ki}^{I}\right|L_i^{RI}+\sum_{k=1}^{n}\left|\xi_{ki}^{R}\right|L_i^{RI}+\sum_{k=1}^{n}\left|\xi_{ki}^{I}\right|L_i^{II}-\theta_{2i}\right)\left|y_i(t)\right|\\
&+\left(d_{2i}\left|y_i(t)\right|^{o_2}-h_{2i}\left|y_i(t)\right|^{\varpi_2}-l_{2i}\left|y_i(t)\right|^{\sigma_2}\right)\\
&+\left(2-a_i-\theta_{3i}\right)\left|z_i(t)\right|+\left(d_{3i}\left|z_i(t)\right|^{o_3}-h_{3i}\left|z_i(t)\right|^{\varpi_3}-l_{3i}\left|z_i(t)\right|^{\sigma_3}\right)\\
&+\left(2-a_i-\theta_{4i}\right)\left|w_i(t)\right|+\left(d_{4i}\left|w_i(t)\right|^{o_4}-h_{4i}\left|w_i(t)\right|^{\varpi_4}-l_{4i}\left|w_i(t)\right|^{\sigma_4}\right)\\
&+\left(\sum_{k=1}^{n}\left|\gamma_{ki}^{R}\right|M_i^{RR}+\sum_{k=1}^{n}\left|\gamma_{ki}^{I}\right|M_i^{IR}+\sum_{k=1}^{n}\left|\gamma_{ki}^{R}\right|M_i^{IR}+\sum_{k=1}^{n}\left|\gamma_{ki}^{I}\right|M_i^{RR}-\sum_{k=1}^{n}e_{1k}\right)\left|x_i\left(t-\tau_i(t)\right)\right|\\
&+\left(\sum_{k=1}^{n}\left|\gamma_{ki}^{R}\right|M_i^{RI}+\sum_{k=1}^{n}\left|\gamma_{ki}^{I}\right|M_i^{II}+\sum_{k=1}^{n}\left|\gamma_{ki}^{R}\right|M_i^{II}+\sum_{k=1}^{n}\left|\gamma_{ki}^{I}\right|M_i^{RI}-\sum_{k=1}^{n}e_{2k}\right)\left|y_i\left(t-\tau_i(t)\right)\right|\Bigg)。
\end{aligned}
$$

由(9)式可得：

$$
\begin{aligned}
{}_0^CD_t^{\alpha}v(t)\le&\sum_{i=1}^{n}\left(d_{1i}\left|x_i(t)\right|^{o_1}+d_{2i}\left|y_i(t)\right|^{o_2}+d_{3i}\left|z_i(t)\right|^{o_3}+d_{4i}\left|w_i(t)\right|^{o_4}\right)\\
&+\sum_{i=1}^{n}\left(-h_{1i}\left|x_i(t)\right|^{\varpi_1}-h_{2i}\left|y_i(t)\right|^{\varpi_2}-h_{3i}\left|z_i(t)\right|^{\varpi_3}-h_{4i}\left|w_i(t)\right|^{\varpi_4}\right)\\
&+\sum_{i=1}^{n}\left(-l_{1i}\left|x_i(t)\right|^{\sigma_1}-l_{2i}\left|y_i(t)\right|^{\sigma_2}-l_{3i}\left|z_i(t)\right|^{\sigma_3}-l_{4i}\left|w_i(t)\right|^{\sigma_4}\right)。
\end{aligned}
$$

根据引理 5 有：

$$
\begin{aligned}
{}_0^CD_t^{\alpha}v(t)\le&\left(\left(\sum_{i=1}^{n}\left(nd_{1i}\right)^{\frac{1}{o_1}}\left|x_i(t)\right|\right)^{o_1}+\left(\sum_{i=1}^{n}\left(nd_{2i}\right)^{\frac{1}{o_2}}\left|y_i(t)\right|\right)^{o_2}+\left(\sum_{i=1}^{n}\left(nd_{3i}\right)^{\frac{1}{o_3}}\left|z_i(t)\right|\right)^{o_3}+\left(\sum_{i=1}^{n}\left(nd_{4i}\right)^{\frac{1}{o_4}}\left|w_i(t)\right|\right)^{o_4}\right)\\
&-\left(\left(\sum_{i=1}^{n}h_{1i}^{\frac{1}{\varpi_1}}\left|x_i(t)\right|\right)^{\varpi_1}+\left(\sum_{i=1}^{n}h_{2i}^{\frac{1}{\varpi_2}}\left|y_i(t)\right|\right)^{\varpi_2}+\left(\sum_{i=1}^{n}h_{3i}^{\frac{1}{\varpi_3}}\left|z_i(t)\right|\right)^{\varpi_3}+\left(\sum_{i=1}^{n}h_{4i}^{\frac{1}{\varpi_4}}\left|w_i(t)\right|\right)^{\varpi_4}\right)\\
&-\left(\left(\sum_{i=1}^{n}l_{1i}^{\frac{1}{\sigma_1}}\left|x_i(t)\right|\right)^{\sigma_1}+\left(\sum_{i=1}^{n}l_{2i}^{\frac{1}{\sigma_2}}\left|y_i(t)\right|\right)^{\sigma_2}+\left(\sum_{i=1}^{n}l_{3i}^{\frac{1}{\sigma_3}}\left|z_i(t)\right|\right)^{\sigma_3}+\left(\sum_{i=1}^{n}l_{4i}^{\frac{1}{\sigma_4}}\left|w_i(t)\right|\right)^{\sigma_4}\right)。
\end{aligned}
$$

令 $\sigma^*=\min\{\sigma_s\}$，$o^*=\min\{o_s\}$，$\varpi^*=\min\{\varpi_s\}$，$s\in\mathbb{S}$。根据引理 7，有

$$\begin{aligned}
{}_0^C D_t^\alpha v(t) \le\ & 2\left(\left(\sum_{i=1}^n (nd_{1i})^{\frac{1}{o_1}}|x_i(t)|\right)^{o^*}+\left(\sum_{i=1}^n (nd_{2i})^{\frac{1}{o_2}}|y_i(t)|\right)^{o^*}+\left(\sum_{i=1}^n (nd_{3i})^{\frac{1}{o_3}}|z_i(t)|\right)^{o^*}+\left(\sum_{i=1}^n (nd_{4i})^{\frac{1}{o_4}}|w_i(t)|\right)^{o^*}\right)\\
&-\frac{1}{2}\left(\left(\sum_{i=1}^n h_{1i}^{\frac{1}{\varpi_1}}|x_i(t)|\right)^{\varpi^*}+\left(\sum_{i=1}^n h_{2i}^{\frac{1}{\varpi_2}}|y_i(t)|\right)^{\varpi^*}+\left(\sum_{i=1}^n h_{3i}^{\frac{1}{\varpi_3}}|z_i(t)|\right)^{\varpi^*}+\left(\sum_{i=1}^n h_{4i}^{\frac{1}{\varpi_4}}|w_i(t)|\right)^{\varpi^*}\right)\\
&-\frac{1}{2}\left(\left(\sum_{i=1}^n l_{1i}^{\frac{1}{\sigma_1}}|x_i(t)|\right)^{\sigma^*}+\left(\sum_{i=1}^n l_{i}^{\frac{1}{\sigma_2}}|y_i(t)|\right)^{\sigma^*}+\left(\sum_{i=1}^n l_{3i}^{\frac{1}{\sigma_3}}|z_i(t)|\right)^{\sigma^*}+\left(\sum_{i=1}^n l_{4i}^{\frac{1}{\sigma_4}}|w_i(t)|\right)^{\sigma^*}\right)\\
\le\ & 2\bar{d}\left(\left(\sum_{i=1}^n |x_i(t)|\right)^{o^*}+\left(\sum_{i=1}^n |y_i(t)|\right)^{o^*}+\left(\sum_{i=1}^n |z_i(t)|\right)^{o^*}+\left(\sum_{i=1}^n |w_i(t)|\right)^{o^*}\right)\\
&-\frac{1}{2}\bar{h}\left(\left(\sum_{i=1}^n |x_i(t)|\right)^{\varpi^*}+\left(\sum_{i=1}^n |y_i(t)|\right)^{\varpi^*}+\left(\sum_{i=1}^n |z_i(t)|\right)^{\varpi^*}+\left(\sum_{i=1}^n |w_i(t)|\right)^{\varpi^*}\right)\\
&-\frac{1}{2}\bar{l}\left(\left(\sum_{i=1}^n |x_i(t)|\right)^{\sigma^*}+\left(\sum_{i=1}^n |y_i(t)|\right)^{\sigma^*}+\left(\sum_{i=1}^n |z_i(t)|\right)^{\sigma^*}+\left(\sum_{i=1}^n |w_i(t)|\right)^{\sigma^*}\right)\\
\le\ & 8\bar{d}\left(\left(\sum_{i=1}^n |x_i(t)|\right)+\left(\sum_{i=1}^n |y_i(t)|\right)+\left(\sum_{i=1}^n |z_i(t)|\right)+\left(\sum_{i=1}^n |w_i(t)|\right)\right)^{o^*}\\
&-\frac{1}{2}\bar{h}\left(\left(\sum_{i=1}^n |x_i(t)|\right)+\left(\sum_{i=1}^n |y_i(t)|\right)+\left(\sum_{i=1}^n |z_i(t)|\right)+\left(\sum_{i=1}^n |w_i(t)|\right)\right)^{\varpi^*}\\
&-\frac{1}{2}\bar{l}\left(\left(\sum_{i=1}^n |x_i(t)|\right)+\left(\sum_{i=1}^n |y_i(t)|\right)+\left(\sum_{i=1}^n |z_i(t)|\right)+\left(\sum_{i=1}^n |w_i(t)|\right)\right)^{\sigma^*}\\
=\ & 8\bar{d}\left(v(t)\right)^{o^*}-\frac{1}{2}\bar{h}\left(v(t)\right)^{\varpi^*}-\frac{1}{2}\bar{l}\left(v(t)\right)^{\sigma^*},
\end{aligned}$$

根据引理 3 可知误差系统(5)可以实现有限时间稳定的，即系统(1)与系统(2)可以实现有限时间同步。其沉降时间估计为：

$$T=\left[\frac{\Gamma(1-\sigma^*)\Gamma(1+\alpha)}{\left(\frac{1}{2}\bar{l}-8\bar{d}\right)\Gamma(1+\alpha-\sigma^*)}\left(v^{\alpha-\sigma^*}(x(0))-1\right)\right]^{\frac{1}{\alpha}}+\left[\frac{\Gamma(1-\varpi^*)\Gamma(1+\alpha)}{\left(\frac{1}{2}\bar{h}-8\bar{d}\right)\Gamma(1+\alpha-\varpi^*)}\right]^{\frac{1}{\alpha}}。$$

4. 数值仿真

在这一部分，给出数值仿真去验证所得结论的正确性。

例 4.1：考虑一个 2 维系统，参数如下：

$$\alpha = 0.9, a_1 = 0.02, a_2 = -0.01, b_1 = 0.81, b_2 = 0.89,$$

$$\xi_{11} = 0.08 + 0.07j, \gamma_{11} = 0.12 + 0.18j, \xi_{12} = 0.18 + 0.05j, \gamma_{12} = 0.12 + 0.1j,$$

$$\xi_{21} = 0.12 + 0.18j, \gamma_{21} = 0.08 + 0.07j, \xi_{22} = 0.12 + 0.1j, \gamma_{22} = 0.18 + 0.05j,$$

$$I_1 = 0.13\sin\left(\frac{\pi t}{3}\right) + 0.15\cos\left(\frac{\pi t}{3}\right) + \left(0.1\sin\left(\frac{\pi t}{3}\right) + 0.1\cos\left(\frac{\pi t}{3}\right)\right)j,$$

$$I_2 = 0.15\sin\left(\frac{\pi t}{3}\right) + 0.1\cos\left(\frac{\pi t}{3}\right) + \left(0.2\sin\left(\frac{\pi t}{3}\right) + 0.3\cos\left(\frac{\pi t}{3}\right)\right)j,$$

$$f_k(\tilde{u}_k) = g_k(\tilde{u}_k) = \frac{1}{2}\left(\left|\mathrm{Re}(\tilde{u}_k)+1\right| - \left|\mathrm{Re}(\tilde{u}_k)-1\right|\right) + \frac{1}{2}\left(\left|\mathrm{Im}(\tilde{u}_k)+1\right| - \left|\mathrm{Im}(\tilde{u}_k)-1\right|\right)j,$$

$$L_i^{RR} = M_i^{RR} = L_i^{II} = M_i^{II} = 1, L_i^{RI} = M_i^{RI} = L_i^{IR} = M_i^{IR} = 0,$$

$$\tau_1(t) = 0.01\sin(t),\quad \tau_2(t) = 0.01\cos(t)。$$

初始值为

$$\tilde{u}_1(c) = -0.8 - 0.3j, \tilde{u}_2(c) = 1 - 0.9j,$$
$$\tilde{v}_1(c) = 0.2 - 0.2j, \tilde{v}_2(c) = -0.6 - 0.5j, c \in (-\infty, 0]。$$

2 维驱动系统的状态轨迹如图 1 所示。从图 1 可以得知，驱动系统明显是不稳定的。

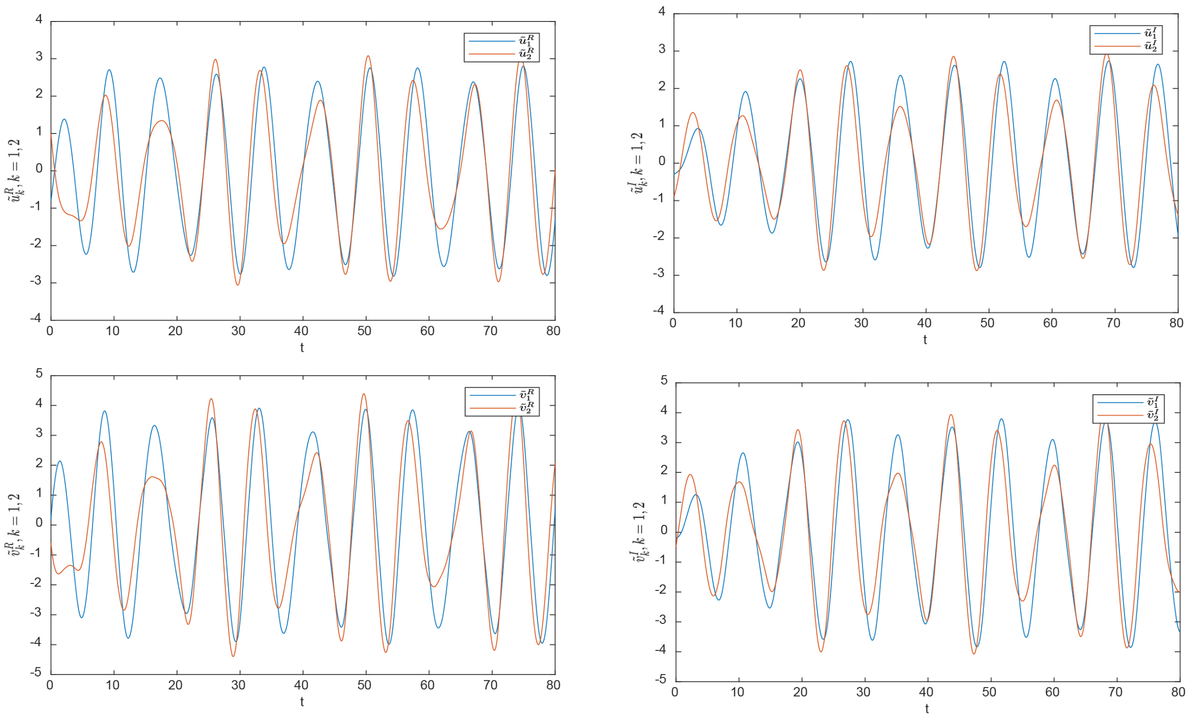

Figure 1. Drive system state trajectories for different initial values
图 1. 驱动系统在不同初始值下的状态轨迹

选择满足(9)式的控制器参数：

$$\theta_{11}=\theta_{21}=1.24,\theta_{12}=\theta_{22}=1.26,\theta_{31}=\theta_{41}=1.98,\theta_{32}=\theta_{42}=2.01,e_{11}=e_{21}=0.26,$$
$$e_{21}=e_{22}=0.14,d_{11}=d_{21}=d_{31}=d_{41}=d_{12}=d_{22}=d_{32}=d_{42}=0,h_{11}=h_{21}=h_{31}=h_{41}=1.$$
$$h_{12}=h_{22}=h_{32}=h_{42}=1,l_{11}=l_{21}=l_{31}=l_{41}=l_{12}=l_{22}=l_{32}=l_{42}=1,o_{1}=o_{2}=o_{3}=o_{4}=0.5,$$
$$\varpi_{1}=\varpi_{2}=\varpi_{3}=\varpi_{4}=0.4,\sigma_{1}=\sigma_{2}=\sigma_{3}=\sigma_{4}=0.7。$$

根据(10)式可得沉降时间为$T=1.8751$，误差系统的运动轨迹如图 2 所示。从图 2 可以得知，根据(8)式设计的控制器可使误差系统在有限时间达到稳定，这验证了定理 1 的正确性。

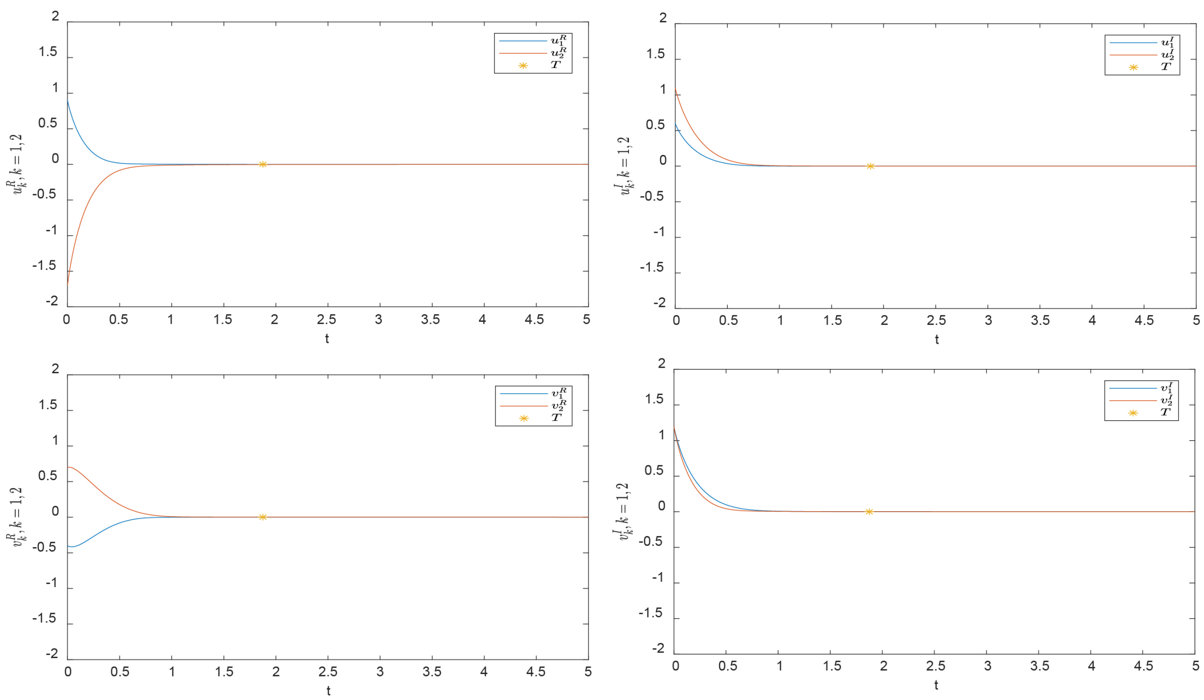

Figure 2. Motion state trajectory of the error system under the design controller
图 2. 在设计的控制器下误差系统运动状态轨迹

5. 结论

本文研究了时延分数阶复值惯性神经网络的有限时间控制问题。首先将复值高阶系统转化为多个低阶实值系统，然后设计了一个非线性控制器，根据新提出的分数阶稳定性引理，构造了李亚普洛夫函数和借助一些不等式，得出了驱动和响应系统达到有限时间同步的一个新的充分条件以及沉降时间。由于在有限时间控制中的沉降时间受到系统初始值的影响而限制了其实际应用，在将来我们计划研究时延分数阶复值惯性神经网络的固定时间控制问题，其沉降时间不再受系统初始值的影响。

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