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弹性梁边界斜置式惯性减振器

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摘　要

本文提出了一种在弹性梁两边加载斜置式惯性边界进行减振的方法。在谐波激励下，对具有斜置式惯性边界的弹性梁进行动力学分析。首先建立弹性梁的动力学控制方程。采用Galerkin方法离散连续偏微分方程，将谐波平衡法与伪弧长算法相结合得到梁的非线性稳态动态响应，并采用Runge-Kutta方法进行数值验证。基于最大幅值减振百分比展示了斜置式惯性边界对弹性梁振动的减振性能。结果表明，斜置式惯性边界在保证承载力的同时又有很好的减振效果。另外，讨论了斜置式惯性边界各项参数变化对其减振性能及系统响应非线性特性的影响。随着惯性增大，系统幅频特性曲线由右侧弯曲向左侧弯曲发生转变，使弹性梁的硬特性转化为软特性。

关键词

斜置式惯性边界，减振，软特性，谐波平衡法

Inclined Boundary Inertial Vibration Absorber for Elastic Beam

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Abstract

In this paper, a method of damping an elastic beam with diagonally placed inertial boundaries loaded on both sides of the beam is proposed. The dynamics of the elastic beam with oblique inertia boundaries will be analysed under harmonic excitation. The controlling equations for the dynamics of the elastic beam are first established. The Galerkin method is used to discretize the continuous partial differential equations, and the nonlinear steady-state dynamic response of the beam is obtained by combining the harmonic balance method with the pseudo arc-length algorithm, and numerically verified by the Runge-Kutta method. The damping performance of the inclined placed

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inertia boundary for elastic beam vibration is demonstrated based on the maximum magnitude damping percentage. The results show that the inclined inertia boundary has a good vibration damping effect while ensuring the load carrying capacity. In addition, the effects of the variation of the parameters of the inclined inertia boundary on the vibration damping performance and the nonlinear characteristics of the system response are discussed. With the increase of inertia, the amplitude-frequency characteristic curve of the system is transformed from the right side bending to the left side bending, so that the hard characteristic of the elastic beam is transformed to the soft characteristic.

Keywords

Inclined Inertial Boundary, Damping, Soft Characteristics, Harmonic Balance Method

1. 引言

梁结构在建筑[1]、航空[2]、航天[3]、船舶[4]等工程领域被广泛应用，这些结构总是受到各种外部激励的影响，如果不实施有效的控制，其结构稳定性与工作精度等都会受到振动的过度影响[5] [6]，所以为了尽可能减少及避免这些问题的出现，对机械进行有效的振动结构抑制是十分有必要的。因此，对连续体结构的非线性振动抑制的研究具有重要的理论和实际意义。

随着时代的发展，人们对振动给生活带来的影响有了更加深刻的认识，针对减振的研究也逐步深入。惯容器是一种优秀的减振机械装置，它具有小质量提供大惯性的特点，具有简单、可靠和高效的减振性能。近年来，惯容器被应用于车辆悬挂系统[7]、飞机起落架[8]、卫星[9]、工程建筑[10]、电缆[11]、桥梁[12]等领域。大量学者对使用惯容器应用于减振进行了深入研究，常见的研究主要是将惯容器作用在离散体上进行减振，如 Barredo 等人创造了一种基于惯容器的非传统动态吸振器，具有很好的减振性能[13]。Zhang 等人设计了一种具有可变负刚度的曲柄惯性隔振器[14]。Shi 等人研究了在菱形连杆机构中产生几何非线性的惯性隔振器[15]。Yang 等人提出了一种带有对角惯容器的悬挂隔振器[16]。

对于将惯容器应用于连续体中的研究，主要以惯性非线性能量汇的形式作用于连续体边界，如 Zhang 等人通过在弹性梁边界加载惯性非线性能量汇来抑制弹性梁的振动[17]。Sun 等人利用惯性非线性能量汇加载于弹性梁，减少了弹性梁在移动载荷作用下的非线性振动[18]。Yu 等人研究了惯性非线性能量汇与连续梁结构之间的连接位置对减振性能的影响[19]。

本文提出了一种新颖的惯性边界条件，即在弹性梁的边界上水平安装阻尼和惯容器，形成斜置式惯性边界。采用广义哈密顿原理建立弹性梁加载斜置式惯性减振器的横向振动控制方程和边界控制方程。根据线性派生系统动力学方程和分离变量法求解模态振型和固有频率。结合 Galerkin 截断法、谐波平衡法和伪弧长算法求解了弹性梁的横向振动稳态响应。探讨了斜置式惯性、阻尼及惯容器长度对弹性梁的稳态振幅频率响应的影响。最后，总结了本文的主要研究成果。这项研究将促进和拓宽惯容器在实际工程中的应用，并为弹性梁的横向振动减振提供一种新的方法。

2. 理论模型

本文研究在弹性梁两边加载斜置式惯性边界进行减振的方法。弹性梁结构的力学模型如图 1 所示。

弹性梁可以在垂直方向上自由振动，边界的转角受限制，无法转动。K_L 与 K_R 分别为两端弹簧的刚度系数；C_L 与 C_R 分别为垂直方向阻尼的阻尼系数；C_{HL} 与 C_{HR} 分别为斜置式阻尼的阻尼系数；B_L 与 B_R 分别为斜置式惯容器的惯容系数；l 为斜置式惯容器的长度；本文所研究的梁结构模型为 Euler-Bernoulli 梁，$W(X, T)$为横向振动位移；$U_L = U_R = U(X, T)$为基础激励。

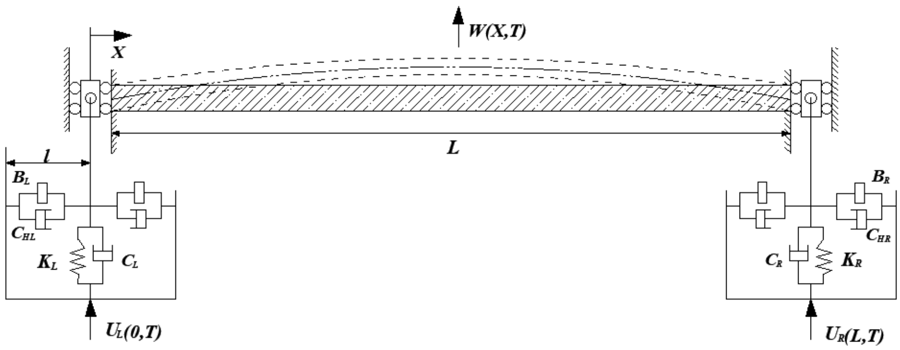


Figure 1. Physical modelling of axially loaded beam structures with elastic supports
图 1. 具有粘弹性支撑的轴向载荷梁结构物理模型

应用广义 Hamilton 原理、变分法以及泰勒级数展开推导系统的非线性偏微分控制方程为

$$\begin{aligned}
&\int_0^L \rho A\frac{\partial^2 W}{\partial T^2}+\left(EI+\Lambda I\frac{\mathrm{d}}{\mathrm{d}T}\right)\frac{\partial^4 W}{\partial X^4}-\frac{3}{2}EA\left(\frac{\partial W}{\partial X}\right)^2\frac{\partial^2 W}{\partial X^2}\mathrm{d}X+K_L U_L\delta(X)+K_R U_R\delta(X-L)\\
&+\left(C_L+2C_{HL}\frac{(W_L-U_L)^2}{l^2}\right)\left(\frac{\partial W_L}{\partial T}-\frac{\partial U_L}{\partial T}\right)\delta(X)+\left(C_R+2C_{HR}\frac{(W_R-U_R)^2}{l^2}\right)\left(\frac{\partial W_R}{\partial T}-\frac{\partial U_R}{\partial T}\right)\delta(X-L)\\
&+2B_L\left[\frac{W_L-U_L}{l^2}\left(\frac{\partial W_L}{\partial T}-\frac{\partial U_L}{\partial T}\right)^2+\frac{(W_L-U_L)^2}{l^2}\left(\frac{\partial^2 W_L}{\partial T^2}-\frac{\partial^2 U_L}{\partial T^2}\right)\right]\delta(X)\\
&+2B_R\left[\frac{W_R-U_R}{l^2}\left(\frac{\partial W_R}{\partial T}-\frac{\partial U_R}{\partial T}\right)^2+\frac{(W_R-U_R)^2}{l^2}\left(\frac{\partial^2 W_R}{\partial T^2}-\frac{\partial^2 U_R}{\partial T^2}\right)\right]\delta(X-L)=0
\end{aligned} \tag{1}$$

弹性梁的线弹性边界条件为

$$EI\frac{\partial W_L}{\partial X}=0,\ EI\frac{\partial W_R}{\partial X}=0,\ EI\frac{\partial^3 W_L}{\partial X^3}+K_L W_L=0,\ EI\frac{\partial^3 W_R}{\partial X^3}-K_R W_R=0 \tag{1}$$

对控制方程(1)进行无量纲化处理，得到控制方程为

$$\begin{aligned}
&\int_0^1\frac{\partial^2 w(x,t)}{\partial t^2}+\kappa^2\frac{\partial^4 w(x,t)}{\partial x^4}+\kappa^2\lambda\frac{\partial^5 w(x,t)}{\partial x^4\partial t}-\frac{3}{2}\left(\frac{\partial w(x,t)}{\partial x}\right)^2\frac{\partial^2 w(x,t)}{\partial x^2}\mathrm{d}x+\kappa_L\kappa^2 u_L\delta(x)+\kappa_R\kappa^2 u_R\delta(x-1)\\
&+\left(\zeta_L+2\zeta_{HL}\frac{(w_L-u_L)^2}{l_1^2}\right)\left(\frac{\partial w_L}{\partial t}-\frac{\partial u_L}{\partial t}\right)\delta(x)+\left(\zeta_R+2\zeta_{HR}\frac{(w_R-u_R)^2}{l_1^2}\right)\left(\frac{\partial w_R}{\partial t}-\frac{\partial u_R}{\partial t}\right)\delta(x-1)\\
&+2\mu_L\left(\frac{w_L-u_L}{l_1^2}\left(\frac{\partial w_L}{\partial t}-\frac{\partial u_L}{\partial t}\right)^2+\frac{(w_L-u_L)^2}{l_1^2}\left(\frac{\partial^2 w_L}{\partial t^2}-\frac{\partial^2 u_L}{\partial t^2}\right)\right)\delta(x)\\
&+2\mu_R\left(\frac{w_R-u_R}{l_1^2}\left(\frac{\partial w_R}{\partial t}-\frac{\partial u_R}{\partial t}\right)^2+\frac{(w_R-u_R)^2}{l_1^2}\left(\frac{\partial^2 w_R}{\partial t^2}-\frac{\partial^2 u_R}{\partial t^2}\right)\right)\delta(x-1)=0
\end{aligned} \tag{3}$$

无量纲边界条件为

$$\frac{\partial w_L}{\partial x}=0,\ \frac{\partial w_R}{\partial x}=0,\ \frac{\partial^3 w_L}{\partial x^3}+\kappa_L w_L=0,\ \frac{\partial^3 w_R}{\partial x^3}-\kappa_R w_R=0 \tag{4}$$

式中

$$\begin{aligned}&w=\frac{W}{L},u=\frac{U}{L},x=\frac{X}{L},l_1=\frac{l}{L},t=\frac{T}{L}\sqrt{\frac{E}{\rho}},\omega=\Omega L\sqrt{\frac{\rho}{E}},\kappa=\frac{1}{L}\sqrt{\frac{I}{A}},\lambda=\frac{\Lambda}{L}\sqrt{\frac{1}{E\rho}},\\&\kappa_L=\frac{K_L L^3}{EI},\kappa_R=\frac{K_R L^3}{EI},\mu_L=\frac{B_L}{\rho AL},\mu_R=\frac{B_R}{\rho AL},\zeta_L=\frac{c_L}{A}\sqrt{\frac{1}{E\rho}},\zeta_R=\frac{c_R}{A}\sqrt{\frac{1}{E\rho}}\end{aligned} \tag{5}$$

3. 固有频率和模态函数

忽略方程(3)和(4)中的阻尼项和非线性项，线性派生系统动力学方程及其边界条件如下

$$\frac{\partial^2 w(x,t)}{\partial t^2}+\kappa^2\frac{\partial^4 w(x,t)}{\partial x^4}=0 \tag{6}$$

$$\frac{\partial w_L}{\partial x}=0,\ \frac{\partial w_R}{\partial x}=0,\ \frac{\partial^3 w_L}{\partial x^3}+\kappa_L w_L=0,\ \frac{\partial^3 w_R}{\partial x^3}-\kappa_R w_R=0 \tag{7}$$

假设弹性梁横向自由振动位移解的形式为

$$W(X,T)=\Psi(X)Q(T) \tag{8}$$

式中 $\phi(x)$和 $q(t)$分别为弹性梁的模态函数及其相应的广义坐标。

模态函数可以写作如下形式

$$\Psi(X)=C_1\cos\beta X+C_2\sin\beta X+C_3\mathrm{ch}\beta X+C_4\mathrm{sh}\beta X \tag{9}$$

式中待定系数 C_j (j = 1, 2, 3, 4)和特征值 β 可通过梁的边界条件求得。特征值 β 与固有频率 ω 存在如下关系式

$$\beta^4=\frac{\omega^2}{\kappa^2} \tag{10}$$

边界条件(7)演化为

$$\Psi'(0)=0,\ \Psi'(L)=0,\ EI\Psi'''(0)+K_L\Psi(0)=0,\ EI\Psi'''(L)-K_R\Psi(L)=0 \tag{11}$$

将方程(9)代入边界条件(11)中，整理得到

$$\begin{aligned}&C_2\beta+\beta C_4=0,\\&-C_1\beta\sin\beta L+\beta C_2\cos\beta L+\beta C_3\mathrm{sh}\beta L+\beta C_4\mathrm{ch}\beta L=0,\\&EI\left(-\beta^3C_2+\beta^3C_4\right)+K_L\left(C_1+C_3\right)=0,\\&EI\left(C_1\beta^3\sin\beta L-\beta^3C_2\cos\beta L+\beta^3C_3\mathrm{sh}\beta L+\beta^3C_4\mathrm{ch}\beta L\right)\\&\quad-K_R\left(C_1\cos\beta L+C_2\sin\beta L+C_3\mathrm{ch}\beta L+C_4\mathrm{sh}\beta L\right)=0\end{aligned} \tag{12}$$

由方程(12)整理可得

$$C_2=-C_4,C_1=\frac{K_L\cos\beta L-K_L\mathrm{ch}\beta L+2EI\beta^3\mathrm{sh}\beta L}{K_L\left(\sin\beta L+\mathrm{sh}\beta L\right)}C_2,C_3=\frac{K_L\mathrm{ch}\beta L-K_L\cos\beta L+2EI\beta^3\sin\beta L}{K_L\left(\sin\beta L+\mathrm{sh}\beta L\right)}C_2 \tag{13}$$

方程(13)整理成矩阵形式为

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ K_L & -EI\beta^3 & K_L & EI\beta^3 \\ -s_{\mathrm{in}} & c_{\mathrm{os}} & s_{\mathrm{h}} & c_{\mathrm{h}} \\ EI\beta^3 s_{\mathrm{in}} - K_R c_{\mathrm{os}} & -EI\beta^3 c_{\mathrm{os}} - K_R s_{\mathrm{in}} & EI\beta^3 s_{\mathrm{h}} - K_R c_{\mathrm{h}} & EI\beta^3 c_{\mathrm{h}} - K_R s_{\mathrm{h}} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = 0 \tag{14}$$

式中 $c_{\mathrm{os}} = \cos\beta L, s_{\mathrm{in}} = \sin\beta L, c_{\mathrm{h}} = \mathrm{ch}\beta L, s_{\mathrm{h}} = \mathrm{sh}\beta L$ 。

Table 1. Physical and geometrical parameters of the beams
表 1. 梁的物理和几何参数

名称	符号	值
杨氏模量	E	68.9 GPa
密度	ρ	2800 kg/m^3
粘性阻尼	Λ	4×10^6 N·s/m^2
长	L	1 m
宽	b	0.02 m
高	h	0.01 m
横截面面积	A	2×10^{-4} m^2
截面惯性距	I	1.67×10^{-9} m^4
左端垂向弹簧刚度	K_L	5753.15 N/m
右端垂向弹簧刚度	K_R	5753.15 N/m
左端惯容器惯性	B_L	2.8 kg
右端惯容器惯性	B_R	2.8 kg
左端阻尼	C_L	0.5 N·s/m
右端阻尼	C_R	0.5 N·s/m
左端水平阻尼	C_{HL}	10 N·s/m
右端水平阻尼	C_{HR}	10 N·s/m
惯容器长度	l	0.25 m

弹性梁的材料选定为铝合金，其物理参数和几何参数如表 1 所示。弹性梁的两端考虑为对称的边界条件，垂向弹簧的刚度为 $K_L = K_R = 5753.15$ N/m。基于给定的系统参数，求得模态函数的特征值为 $\beta_1 = 3.0560$、$\beta_2 = 4.1315$、$\beta_3 = 6.4844$、$\beta_4 = 9.4851$。弹性梁的前四阶固有频率分别为 $f_1 = 21.3060$ Hz、$f_2 = 38.9419$ Hz、$f_3 = 95.9270$ Hz、$f_4 = 205.2483$ Hz。令 $C_1 = 1$，那么弹性梁的前四阶模态函数如式(15)所示。

图 2 给出了弹性梁的前四阶模态振型。如图所示，弹性梁第一阶模态具有整体位移，最大位移出现在梁的中点位置。后三阶模态均具有模态节点，最大位移在梁的边界上。此外还可以发现，弹性梁的第一、三阶模态振型关于梁的中点呈对称分布，而其第二、四阶模态振型相对于中点呈反对称分布。弹性梁的两端具有相同的位移幅值，因此以下的研究中，只给出弹性梁的左端点和中点的响应。

$$\begin{aligned} \phi_1(x) &= \cos(3.05x) + 23.35\sin(3.05x) + 25.66\cosh(3.05x) - 23.35\sinh(3.05x), \\ \phi_2(x) &= \cos(4.13x) + 0.53\sin(4.13x) + 0.52\cosh(4.13x) - 0.53\sinh(4.13x), \\ \phi_3(x) &= \cos(6.48x) + 0.10\sin(6.48x) + 0.10\cosh(6.48x) - 0.10\sinh(6.48x), \\ \phi_4(x) &= \cos(9.48x) + 0.30\sin(9.48x) + 0.30\cosh(9.48x) - 0.30\sinh(9.48x) \end{aligned} \tag{15}$$

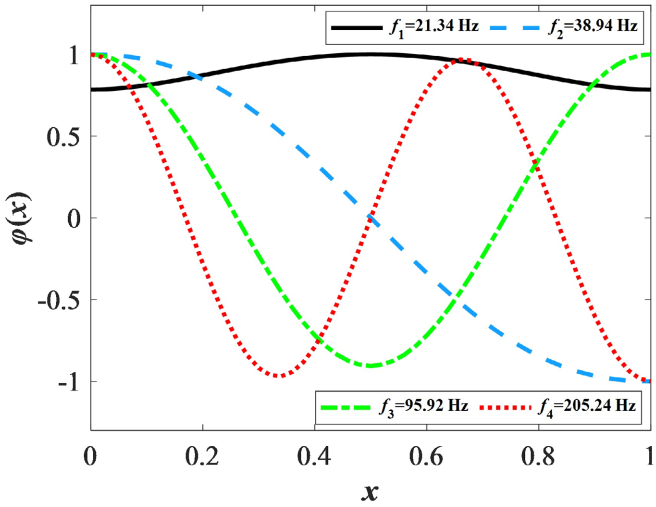


Figure 2. First-fourth order mode shapes of elastic beams
图 2. 弹性梁前四阶模态振型

4. 稳态幅频响应

4.1. Galerkin 截断方程及其收敛性判断

应用 Galerkin 方法将偏微分控制方程(3)截断为常微分程。然后利用四阶 Runge-Kutta 方法求解系统的时间历程。假设控制方程(3)中横向振动位移的近似解为

$$w(x,t)=\sum_{n=1}^{N}\phi_n(x)q_n(t) \tag{16}$$

式中，N 为大于等于 1 的整数，$\phi_n(x)$为线性梁的模态函数，$q_n(t)$为梁横向振动的广义位移。Galerkin 方法的势函数和权函数都选择为线性梁的模态函数如式(15)所示。偏微分方程(3)经过 Galerkin 截断处理后得

$$\int_0^1\sum_{n=1}^{N}\phi_n(x)\ddot{q}_n(t)\psi_m(x)\mathrm{d}x+\kappa^2\int_0^1\sum_{n=1}^{N}\phi_n^{(4)}(x)q_n(t)\psi_m(x)\mathrm{d}x+\kappa^2\lambda\int_0^1\sum_{n=1}^{N}\phi_n^{(4)}(x)\dot{q}_n(t)\psi_m(x)\mathrm{d}x$$

$$-\frac{3}{2}\int_0^1\left(\sum_{n=1}^{N}\dot{\phi}_n(x)q_n(t)\right)^2\left(\sum_{n=1}^{N}\ddot{\phi}_n(x)q_n(t)\right)\psi_m(x)\mathrm{d}x+\kappa_L\kappa^2u_L\cos(\omega\cdot t)\psi_m(0)+\kappa_R\kappa^2u_R\cos(\omega\cdot t)\psi_m(1)$$

$$+\left(\zeta_L+2\zeta_{HL}\frac{\left(\sum_{n=1}^{N}\phi_n(0)q_n(t)-u_L\cos(\omega\cdot t)\right)^2}{l_1^2}\right)\left(\sum_{n=1}^{N}\phi_n(0)\dot{q}_n(t)+\omega\cdot u_L\sin(\omega\cdot t)\right)\psi_m(0)$$

$$+\left(\zeta_R+2\zeta_{HR}\frac{\left(\sum_{n=1}^{N}\phi_n(1)q_n(t)-u_R\cos(\omega\cdot t)\right)^2}{l_1^2}\right)\left(\sum_{n=1}^{N}\phi_n(1)\dot{q}_n(t)+\omega\cdot u_R\sin(\omega\cdot t)\right)\psi_m(1)$$

$$+2\mu_L\left(\frac{\sum_{n=1}^{N}\phi_n(0)q_n(t)-u_L\cos(\omega\cdot t)}{l_1^2}\left(\sum_{n=1}^{N}\phi_n(0)\dot{q}_n(t)+\omega\cdot u_L\sin(\omega\cdot t)\right)^2\right.$$

$$
+\frac{\left(\sum_{n=1}^{N}\phi_n(0)q_n(t)-u_L\cos(\omega\cdot t)\right)^2}{l_1^2}\left(\sum_{n=1}^{N}\phi_n(0)\ddot{q}_n(t)+\omega^2\cdot u_L\cos(\omega\cdot t)\right)\Bigg)\psi_m(0)
$$

$$
+2\mu_R\left(\frac{\sum_{n=1}^{N}\phi_n(1)q_n(t)-u_R\cos(\omega\cdot t)}{l_1^2}\left(\sum_{n=1}^{N}\phi_n(1)\dot{q}_n(t)+\omega\cdot u_R\sin(\omega\cdot t)\right)^2\right. \tag{17}
$$

$$
\left.+\frac{\left(\sum_{n=1}^{N}\phi_n(1)q_n(t)-u_R\cos(\omega\cdot t)\right)^2}{l_1^2}\left(\sum_{n=1}^{N}\phi_n(1)\ddot{q}_n(t)+\omega^2\cdot u_R\cos(\omega\cdot t)\right)\right)\psi_m(1)=0
$$

式中 $m=1,2,\cdots,N$ 。

$$
\int_0^1\phi_n(x)\psi_m(x)\mathrm{d}x=0,n\neq m,\ \int_0^1\phi_n^{(4)}(x)\psi_m(x)\mathrm{d}x=0,n\neq m \tag{18}
$$

方程(17)可以写作

$$
M_m\ddot{q}_m(t)+\kappa^2K_mq_m(t)+\lambda\kappa^2K_m\dot{q}_m(t)-\frac{3}{2}D_mq_m(t)^3+\kappa_L\kappa^2u_L\cos(\omega\cdot t)\psi_m(0)+\kappa_R\kappa^2u_R\cos(\omega\cdot t)\psi_m(1)
$$

$$
+\left(\zeta_L+2\zeta_{HL}\frac{\left(\sum_{n=1}^{N}\phi_n(0)q_n(t)-u_L\cos(\omega\cdot t)\right)^2}{l_1^2}\right)\left(\sum_{n=1}^{N}\phi_n(0)\dot{q}_n(t)+\omega\cdot u_L\sin(\omega\cdot t)\right)\psi_m(0)
$$

$$
+\left(\zeta_R+2\zeta_{HR}\frac{\left(\sum_{n=1}^{N}\phi_n(1)q_n(t)-u_R\cos(\omega\cdot t)\right)^2}{l_1^2}\right)\left(\sum_{n=1}^{N}\phi_n(1)\dot{q}_n(t)+\omega\cdot u_R\sin(\omega\cdot t)\right)\psi_m(1)
$$

$$
+2\mu_L\left(\frac{\sum_{n=1}^{N}\phi_n(0)q_n(t)-u_L\cos(\omega\cdot t)}{l_1^2}\left(\sum_{n=1}^{N}\phi_n(0)\dot{q}_n(t)+\omega\cdot u_L\sin(\omega\cdot t)\right)^2\right.
$$

$$
\left.+\frac{\left(\sum_{n=1}^{N}\phi_n(0)q_n(t)-u_L\cos(\omega\cdot t)\right)^2}{l_1^2}\left(\sum_{n=1}^{N}\phi_n(0)\ddot{q}_n(t)+\omega^2\cdot u_L\cos(\omega\cdot t)\right)\right)\psi_m(0)
$$

$$
+2\mu_R\left(\frac{\sum_{n=1}^{N}\phi_n(1)q_n(t)-u_R\cos(\omega\cdot t)}{l_1^2}\left(\sum_{n=1}^{N}\phi_n(1)\dot{q}_n(t)+\omega\cdot u_R\sin(\omega\cdot t)\right)^2\right. \tag{19}
$$

$$
\left.+\frac{\left(\sum_{n=1}^{N}\phi_n(1)q_n(t)-u_R\cos(\omega\cdot t)\right)^2}{l_1^2}\left(\sum_{n=1}^{N}\phi_n(1)\ddot{q}_n(t)+\omega^2\cdot u_R\cos(\omega\cdot t)\right)\right)\psi_m(1)=0
$$

式中

$$M_m=\int_0^1\phi_m(x)\psi_m(x)\mathrm{d}x,\ K_m=\int_0^1\phi_m^{(4)}(x)\psi_m(x)\mathrm{d}x,\ D_m=\int_0^1(\dot{\phi}_m(x))^2\ddot{\phi}_m(x)\psi_m(x)\mathrm{d}x,\ F_m=\int_0^1\psi_m(x)\mathrm{d}x \tag{20}$$

常微分方程(19)为弹性梁结构在 Galerkin 截断处理下的动力学方程。通过 Runge-Kutta 方法可以对方程进行数值求解。首先分析 Galerkin 截断阶数对系统自由振动响应的影响，系统初始条件设置如下

$$q_1=0.05, \dot{q}_1=0, q_j=0, \dot{q}_j=0, j=2,\cdots,N \tag{21}$$

图 3 为 Galerkin 截断分别取 2、4、6 阶时系统的自由衰减振动时间历程。图中 4、6 阶截断曲线是重合的，而 2 阶截断曲线却有些偏差。由此得出，4、6 阶 Galerkin 截断比 2 阶能更好地满足收敛条件。

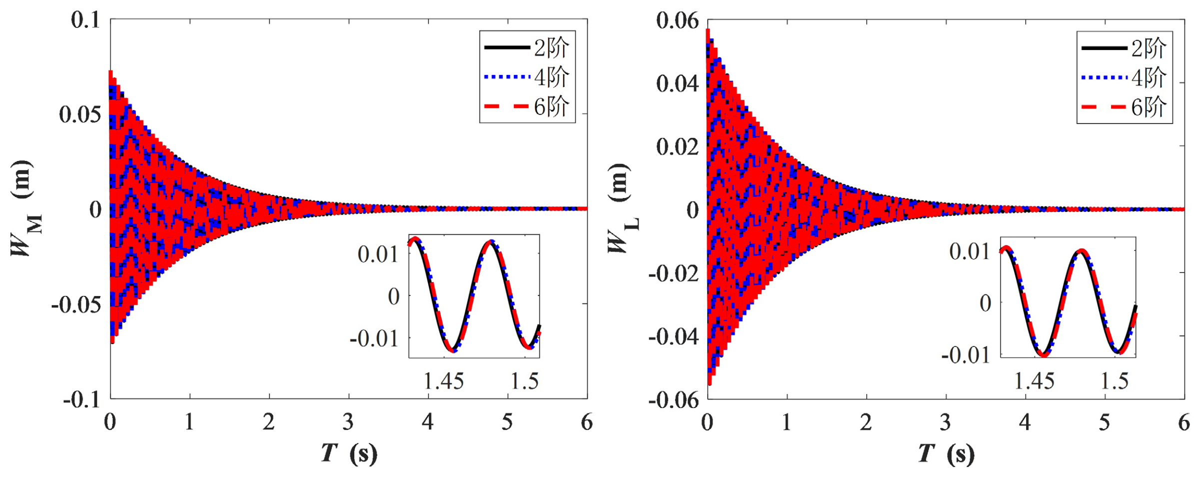


Figure 3. Free decay vibration time histories at different Galerkin truncation orders: (a) left end point of the beam; (b) middle point of the beam

图 3. 不同 Galerkin 截断阶数下的自由衰减振动时间历程：(a) 梁左端点；(b) 梁中点

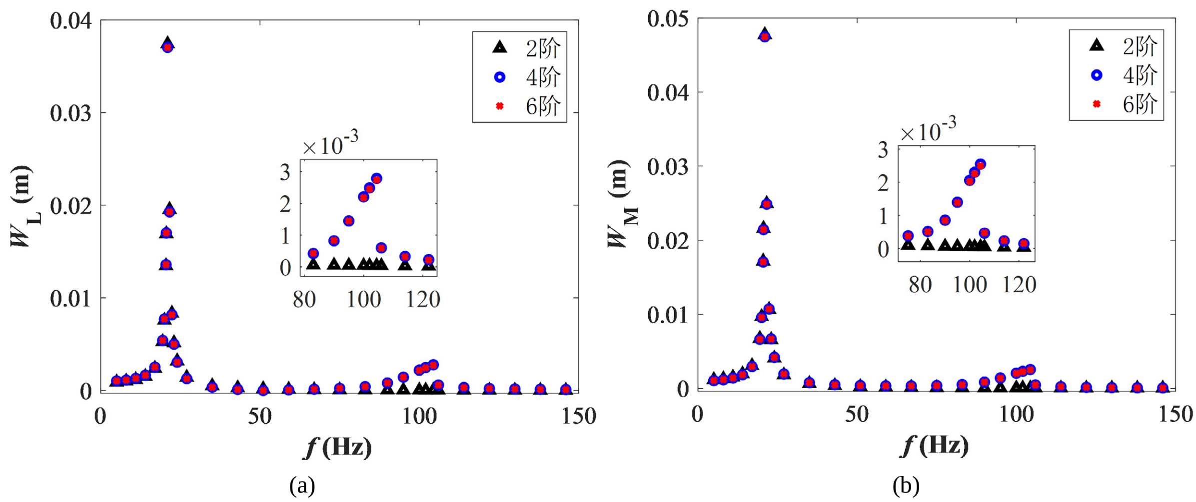


Figure 4. Convergence of Galerkin's truncation order for (a) the left endpoint of the beam; (b) the midpoint of the beam of the beam

图 4. Galerkin 截断阶数的收敛性：(a) 梁左端点；(b) 梁中点

将式(21)中系统的初值全部设为 0，基础激励幅值 $U=0.001$ m。由不同的激励频率下，系统受迫振动

的时间历程中提取出系统在稳定状态下的响应幅值，可得到系统的幅频响应曲线。在均布简谐力激励下，系统的幅频响应曲线只有两个共振峰，其共振频率分别对应着系统的第一阶和第三阶固有频率。图 4 中，第一阶共振峰处各阶 Galerkin 截断的结果近乎相同。而在三阶峰处 2 阶 Galerkin 截断却无法正常显示。综合自由衰减振动时间历程的结果，使用 4 阶 Galerkin 截断完全可以满足计算系统稳态幅频响应的收敛性要求。因此在本文仿真计算中，均采用 4 阶 Galerkin 截断方程。

4.2. 谐波平衡方法求解及其数值验证

常微分方程(19)通过谐波平衡法进行近似解析求解。假设常微分方程的解为

$$q_m = \sum_{i=0}^{h} \left\{ a_{m,2i+1} \sin\left[(2i+1)\omega t\right] + b_{m,2i+1} \cos\left[(2i+1)\omega t\right] \right\} \tag{22}$$

式中，m 为 Galerkin 截断阶数，$2i+1$ 为谐波阶数。

以一阶 Galerkin 截断方程和一阶谐波假设解为例给出谐波平衡法的推导过程。Galerkin 截断取一阶时，弹性梁系统的控制方程为

$$\begin{aligned}
&M_1\ddot{q}_1 + \kappa^2 K_1 q_1 + \lambda\kappa^2 K_1 \dot{q}_1 - \frac{3}{2} D_1 q_1^3 + \kappa_L \kappa^2 u_L \cos(\omega\cdot t)\psi_{L,1} + \kappa_R \kappa^2 u_R \cos(\omega\cdot t)\psi_{R,1} \\
&+ \zeta_L\left(\phi_{L,1}\dot{q}_1 + \omega\cdot u_L \sin(\omega\cdot t)\right)\psi_{L,1} + 2\zeta_{HL}\frac{\left(\phi_{L,1}q_1 - u_L\cos(\omega\cdot t)\right)^2}{l_1^2}\left(\phi_{L,1}\dot{q}_1 + \omega\cdot u_L\sin(\omega\cdot t)\right)\psi_{L,1} \\
&+ \zeta_R\left(\phi_{R,1}\dot{q}_1 + \omega\cdot u_R \sin(\omega\cdot t)\right)\psi_{R,1} + 2\zeta_{HR}\frac{\left(\phi_{R,1}q_1 - u_R\cos(\omega\cdot t)\right)^2}{l_1^2}\left(\phi_{R,1}\dot{q}_1 + \omega\cdot u_R\sin(\omega\cdot t)\right)\psi_{R,1} \\
&+ 2\mu_L\left(\frac{\phi_{L,1}q_1 - u_L\cos(\omega\cdot t)}{l_1^2}\left(\phi_{L,1}\dot{q}_1 + \omega\cdot u_L\sin(\omega\cdot t)\right)^2\right. \\
&\left.+\frac{\left(\phi_{L,1}q_1 - u_L\cos(\omega\cdot t)\right)^2}{l_1^2}\left(\phi_{L,1}\ddot{q}_1 + \omega^2\cdot u_L\cos(\omega\cdot t)\right)\right)\psi_{L,1} \\
&+ 2\mu_R\left(\frac{\phi_{R,1}q_1 - u_R\cos(\omega\cdot t)}{l_1^2}\left(\phi_{R,1}\dot{q}_1 + \omega\cdot u_R\sin(\omega\cdot t)\right)^2\right. \\
&\left.+\frac{\left(\phi_{R,1}q_1 - u_R\cos(\omega\cdot t)\right)^2}{l_1^2}\left(\phi_{R,1}\ddot{q}_1 + \omega^2\cdot u_R\cos(\omega\cdot t)\right)\right)\psi_{R,1} = 0
\end{aligned} \tag{23}$$

式中，$\phi_{L,1} = \phi_1(0), \psi_{L,1} = \psi_1(0), \phi_{R,1} = \phi_1(1), \psi_{R,1} = \psi_1(1)$。

一阶谐波假设解为

$$q_1 = a_{1,1}\sin(\omega t) + b_{1,1}\cos(\omega t) \tag{24}$$

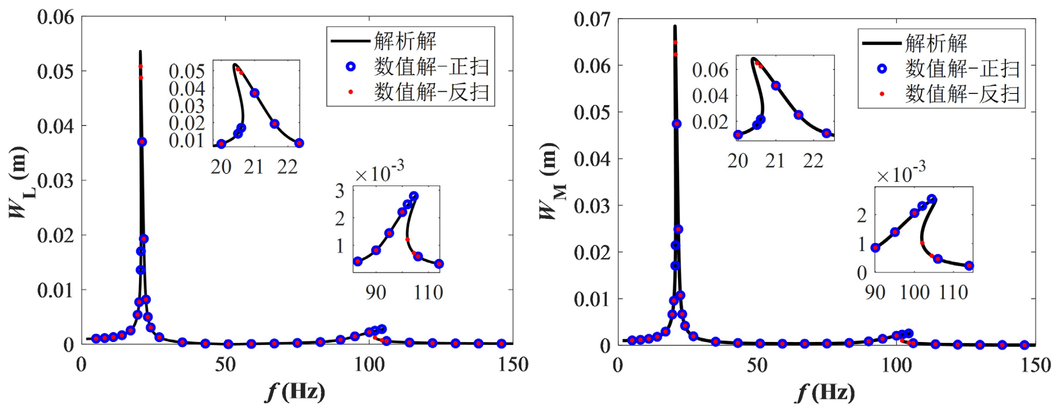
将式(24)带入到方程(23)中整理各阶谐波对应的系数方程得到如下所示代数方程

$$\begin{aligned}
&\frac{-9a_{1,1}^3 D_1}{8} + \frac{\left(-8\omega^2 M_1 - 9b_{1,1}^2 D_1 + 8K_1\kappa^2\right)a_{1,1}}{8} + \omega b_{1,1} K_1 \kappa^2 \lambda \\
&+ \frac{1}{l_1^2}\left(\psi_{L,1}\left(\frac{\left(-3\mu_{HL} + \mu_L\right)\left(a_{1,1}\psi_{L,1} - U_L\right)\left(\left(a_{1,1}^2 + b_{1,1}^2\right)\psi_{L,1}^2 - 2a_{1,1}U_L\psi_{L,1} + U_L^2\right)\omega^2}{2}\right.\right. \\
&\left.\left.+\psi_{L,1}\left(\frac{\zeta_{HL}\left(a_{1,1}^2 + b_{1,1}^2\right)\psi_{L,1}^2}{2} - U_L\zeta_{HL}\psi_{L,1}a_{1,1} + l_1^2\zeta_L + \frac{U_L^2\zeta_{HL}}{2}\right)\omega b_{1,1} + \kappa^2 l_1^2 U_L \kappa_L\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{l_1^2}\Bigg(\psi_{R,1}\Bigg(\frac{\left(-3\mu_{HR}+\mu_R\right)\left(a_{1,1}\psi_{R,1}-U_R\right)\left(\left(a_{1,1}^2+b_{1,1}^2\right)\psi_{R,1}^2-2a_{1,1}U_R\psi_{R,1}+U_R^2\right)\omega^2}{2}\\
&+\psi_{R,1}\left(\frac{\zeta_{HR}\left(a_{1,1}^2+b_{1,1}^2\right)\psi_{R,1}^2}{2}-U_R\zeta_{HR}\psi_{R,1}a_{1,1}+l_1^2\zeta_R+\frac{U_R^2\zeta_{HR}}{2}\right)\omega b_{1,1}+\kappa^2 l_1^2 U_R\kappa_R\Bigg)\Bigg)=0,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{-9b_{1,1}^3D_1}{8}+\frac{\left(-8\omega^2M_1-9a_{1,1}^2D_1+8K_1\kappa^2\right)b_{1,1}}{8}-\omega a_{1,1}K_1\kappa^2\lambda\\
&+\frac{1}{l_1^2}\Bigg(\psi_{L,1}\omega\Bigg(\frac{\left(-\omega\left(-3\mu_{HL}+\mu_L\right)b_{1,1}+a_{1,1}\zeta_{HL}\right)\left(a_{1,1}^2+b_{1,1}^2\right)\psi_{L,1}^3}{2}\\
&-\frac{3U_L\left(\frac{b_{1,1}^2\zeta_{HL}}{3}-\frac{2a_{1,1}\omega\left(-3\mu_{HL}+\mu_L\right)b_{1,1}}{3}+a_{1,1}^2\zeta_{HL}\right)\psi_{L,1}^2}{2}\\
&+\left(\frac{-U_L^2\omega\left(-3\mu_{HL}+\mu_L\right)b_{1,1}}{2}+a_{1,1}\left(l_1^2\zeta_L+\frac{3U_L^2\zeta_{HL}}{2}\right)\right)\psi_{L,1}-U_Ll_1^2\zeta_L-\frac{U_L^3\zeta_{HL}}{2}\Bigg)\Bigg)\\
&+\frac{1}{l_1^2}\Bigg(\psi_{R,1}\omega\Bigg(\frac{\left(-\omega\left(-3\mu_{HR}+\mu_R\right)b_{1,1}+a_{1,1}\zeta_{HR}\right)\left(a_{1,1}^2+b_{1,1}^2\right)\psi_{R,1}^3}{2}\\
&-\frac{3U_R\left(\frac{b_{1,1}^2\zeta_{HR}}{3}-\frac{2a_{1,1}\omega\left(-3\mu_{HR}+\mu_R\right)b_{1,1}}{3}+a_{1,1}^2\zeta_{HR}\right)\psi_{R,1}^2}{2}\\
&+\left(\frac{-U_R^2\omega\left(-3\mu_{HR}+\mu_R\right)b_{1,1}}{2}+a_{1,1}\left(l_1^2\zeta_R+\frac{3U_R^2\zeta_{HR}}{2}\right)\right)\psi_{R,1}-U_Rl_1^2\zeta_R-\frac{U_R^3\zeta_{HR}}{2}\Bigg)\Bigg)=0
\end{aligned}
\tag{25}
$$

采用弧长法对方程求解得到系统的幅频响应曲线。



(a) (b)

Figure 5. Comparison of the analytical solution of the harmonic balance method with the numerical solution of the Runge-Kutta method for (a) the left end point of the beam; (b) the middle point of the beam

图 5. 谐波平衡方法解析解与 Runge-Kutta 方法数值解比较：(a) 梁左端点；(b) 梁中点

前一节已经讨论过，Galerkin 截断取到四阶较为合适。图 5 中给出了对比由谐波平衡近似解析方法和 Runge-Kutta 数值方法得到的系统稳态幅频响应曲线。两种方法的结果相比较可以看出，两种结果重合

度很高，可以充分表明解析结果是可靠的。在对比图中，第一阶共振峰向左弯曲，出现了非线性的软特性，第三阶共振峰向右弯曲，出现了非线性的硬特性。Runge-Kutta 方法的正扫与反扫对比在共振峰处表现出了明显的非线性跳跃现象。

5. 减振效果分析

通过最大幅值减少百分比评价斜置式惯容系统对稳态响应的减振性能。未加斜置式惯容系统的主共振最大幅值记为 A_u，斜置式惯容系统的主共振最大幅值记为 A_i，则斜置式惯容系统的减振效果百分比为

$$R_A = \frac{A_u - A_i}{A_u} \cdot 100\% \tag{26}$$

给予基础激励幅值 U = 0.001 m，未控系统和斜置式惯容系统的稳态幅频响应曲线如图 6 所示。显然，斜置式惯容系统对一阶主共振具有显著的减振效果。梁左端点第一阶主共振的减振百分比为 53.4%。梁中点第一阶主共振的减振百分比为 53.1%。通过比较，斜置式惯容系统对第一阶主共振的减振效果优异，但是对第三阶主共振的减振效果不大，对端点处的减振效果优于对中点的减振效果。

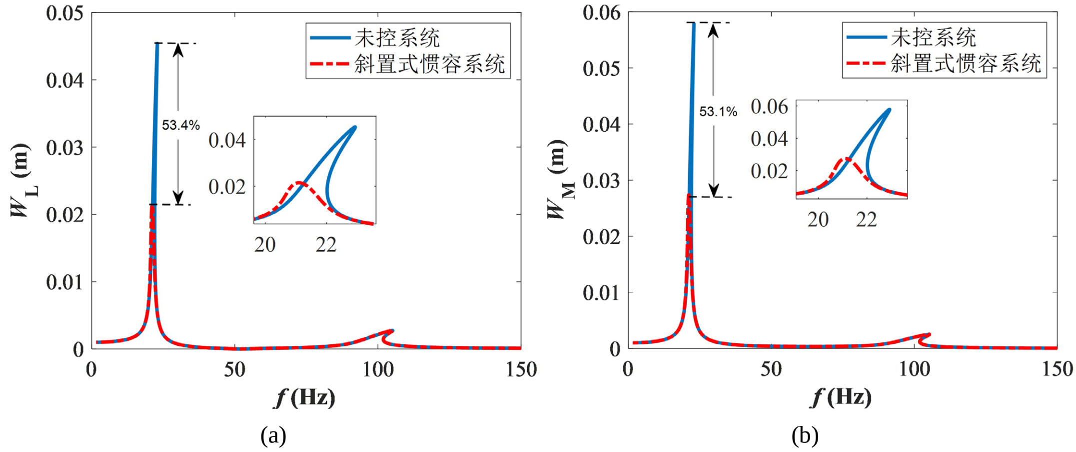


Figure 6. Vibration damping effect U = 0.001 m for the inclined inertial capacity system: (a) left end point of the beam; (b) middle point of the beam
图 6. 斜置式惯容系统的减振效果 U = 0.001 m：(a) 梁左端点；(b) 梁中点

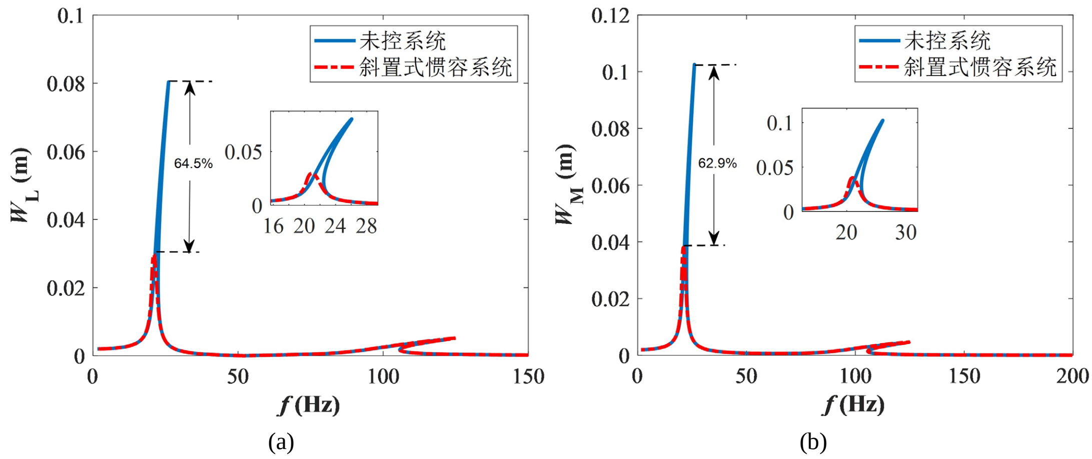
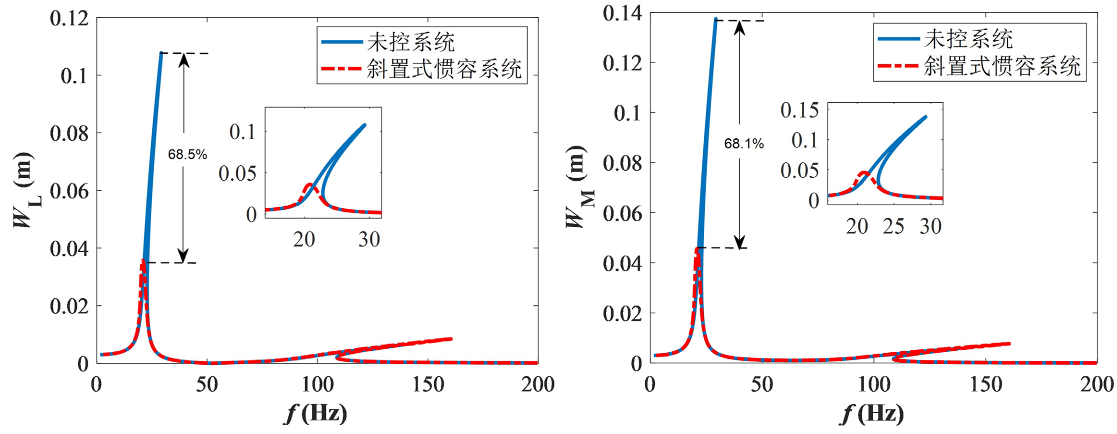


Figure 7. Vibration damping effect U = 0.002 m for the inclined inertial capacity system: (a) left end point of the beam; (b) middle point of the beam
图 7. 斜置式惯容系统的减振效果 U = 0.002 m：(a) 梁左端点；(b) 梁中点



(a) (b)

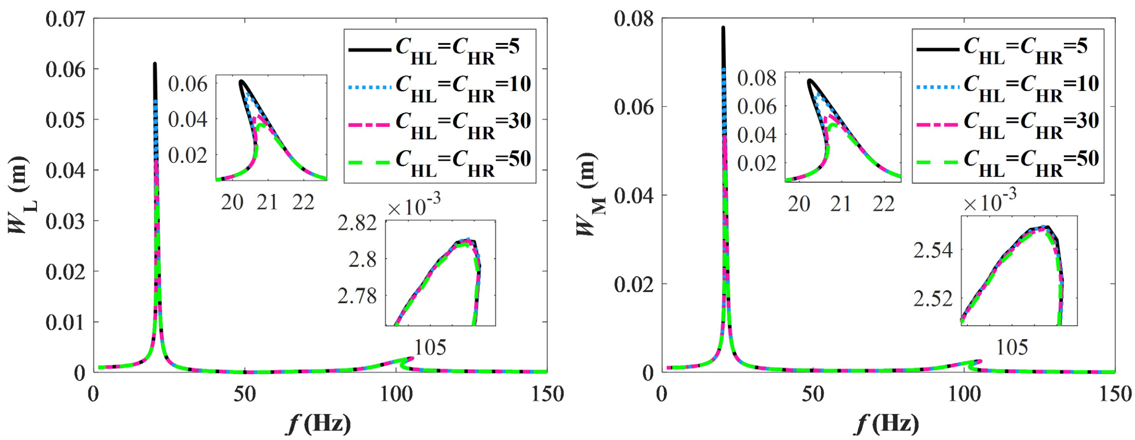
Figure 8. Vibration damping effect U = 0.003 m for the inclined inertial capacity system: (a) left end point of the beam; (b) middle point of the beam
图 8. 斜置式惯容系统的减振效果 U = 0.003 m：(a) 梁左端点；(b) 梁中点

接下来讨论基础激励幅值对斜置式惯容系统减振效果的影响。基础激励幅值 U = 0.002 m 和 U = 0.003 m 时，斜置式惯容系统的减振效果分别如图 7 和图 8 所示。比较三种基础激励幅值下斜置式惯容系统的减振效果，激励幅值对第一阶主共振的减振效果影响显著。例如，左端点第一阶主共振在小激励幅值 U = 0.001 m 时斜置式惯容系统的减振百分比为 53.4%，在中等激励 U = 0.002 m 时和大激励幅值 U = 0.003 m 时的减振百分比分别为 64.5%和 68.5%，相比于小激励幅值，减振百分比分别增加了 11.1%和 15.1%。可以看出，基础激励幅值增大，斜置式惯容系统对第一阶主共振的抑制效果越好，非线性硬化现象被削弱的效果越强。

6. 参数影响

斜置式惯容系统三个主要参数是斜置式阻尼、惯容器惯性和长度。参数的选择会直接影响到斜置式惯容系统的减振性能。因此，要实现高效的减振效果，合理地选择三个参数是十分必要的。参数研究基于稳态响应来展开，斜置式惯容系统的参数基于表 1，基础激励幅值设定为 U = 0.001 m。

6.1. 斜置式阻尼的影响



(a) (b)

Figure 9. Effect of slant-mounted damping parameters on the amplitude-frequency response curves for (a) left end point of the beam; (b) middle point of the beam
图 9. 斜置式阻尼参数对幅频响应曲线的影响：(a) 梁左端点；(b) 梁中点

斜置式阻尼的变化对幅频响应曲线的影响如图 9 所示。斜置式阻尼参数的变化对第一阶主共振影响明显，但是对第三阶主共振没有明显的影响。此外，还可以发现，第一阶主共振有明显的非线性软化现象，系统响应存在多解情况，随着斜置式阻尼的增大，非线性现象逐渐消失了，单一激励频率下，系统响应只有一组解。比较图 9(a)和图 9(b)发现，左端点和中点的第一阶主共振随斜置式阻尼变化的趋势是相同的。

6.2. 斜置式惯容系统的影响

图 10 为斜置式惯容器系统惯性对系统幅频响应曲线的影响。左端点和中点的幅频响应曲线有着相同的变化趋势。对于第一阶主共振响应，随着惯性参数的增大，共振峰值向左上方移动，将向右的非线性硬化特性转变为线性，继续增大，将转变为向左的非线性软化特性，而第三阶主共振几乎没有变化。

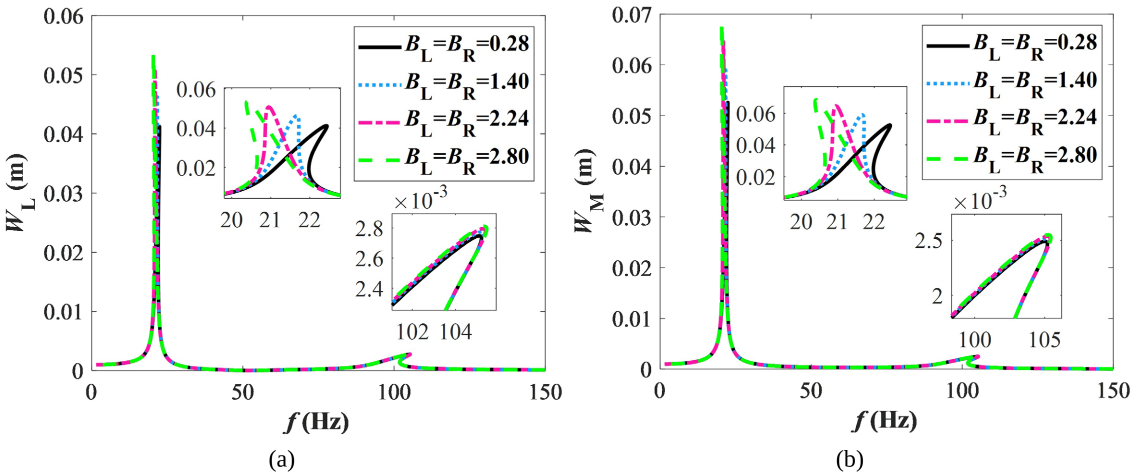


Figure 10. Influence of the oblique-mounted inertial capacity system parameters on the amplitude-frequency response curves for (a) left end point of the beam; (b) middle point of the beam

图 10. 斜置式惯容参数对幅频响应曲线的影响：(a) 梁左端点；(b) 梁中点

6.3. 斜置式惯容器长度的影响

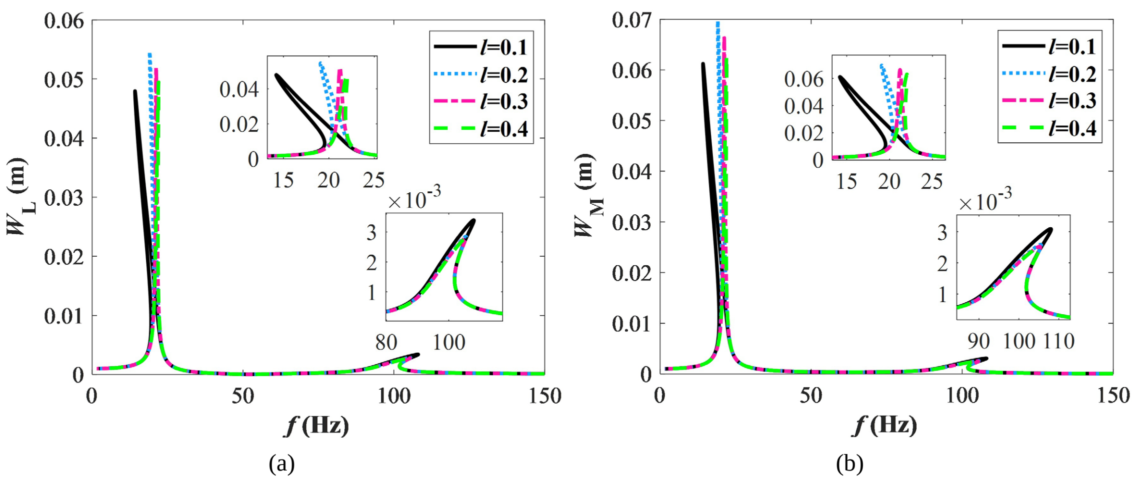


Figure 11. Effect of the length parameter of the inclined inertia vessel on the amplitude-frequency response curves of (a) left end point of the beam; (b) middle point of the beam

图 11. 斜置式惯容器长度参数对幅频响应曲线的影响：(a) 梁左端点；(b) 梁中点

图 11 为斜置式惯容器长度变化对系统幅频响应曲线的影响。随着斜置式惯容器长度的增加，第一阶主共振的峰值向右移动，曲线呈现出由非线性软化特性转化为线性，而后又继续向右移动形成非线性硬化特性，共振频带也随之先变窄而后变宽，共振峰值也随之先增大后减小。在第三阶主共振处，无论是左端点还是中点，在 0.1 m 到 0.2 m 时，系统幅频响应曲线下降很明显，而在继续增加斜置式惯容器长度时，不同长度下的系统幅频响应曲线几乎是完全重合的。

7. 结论

本文研究了一种在弹性梁两边加载斜置式惯性边界进行减振的方法。基于自由振动控制方程，揭示了斜置式惯容系统对弹性梁固有频率和模态振型的影响。采用 Galerkin 方法离散连续偏微分方程，将谐波平衡法与伪弧长算法相结合得到梁的非线性稳态响应，并采用 Runge-Kutta 方法进行数值验证。通过最大幅值减少百分比评价斜置式惯容系统的减振性能。基于稳态幅频响应曲线，论证了斜置式惯容系统中斜置式阻尼、惯容器惯性和惯容器长度三个主要参数对斜置式惯容减振器减振性能的影响。

(1) 通过与未控系统的稳态响应相比较，直观明显地表现出斜置式惯容系统在一阶共振峰处优异的减振性能。

(2) 斜置式惯容系统中斜置式阻尼、惯容器惯性和长度对低阶共振峰的影响远大于对高阶共振峰的影响。

(3) 斜置式惯容器的惯性随着参数的增大，共振峰值向左上方移动，将向右的非线性硬化特性转变为线性，继续增大，将转变为向左的非线性软化特性。因此，这项工作为弹性连续体的非线性消除提供了一种新方法。

(4) 随着斜置式惯容器的长度的增加，第一阶主共振的峰值向右移动，曲线呈现出由非线性软化特性转化为线性，而后又继续向右移动形成非线性硬化特性，共振频带也随之先变窄而后变宽。在小长度时，对幅频响应影响较明显。

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