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一类$Kirchhoff-Choquard$方程基态解的存在性

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摘 要

本文研究一类带有势函数对数非线性项的$Kirchhoff-Choquard$方程基态解的存在性, 通过$Ekeland$变分方法, 对数$Sobolev$不等式, $Hardy-Littlewood-Sobolev$ 不等式以及对对数非线性项的技巧性处理, 得到$\alpha \in (0,3)$, $\alpha = 0$以及带有非线性扰动项等三种情况下$Kirchhoff-Choquard$方程存在基态解的结论。

关键词

$Kirchhoff-Choquard$方程，$Ekeland$变分方法，对数$Sobolev$不等式，$Hardy-Littlewood-Sobolev$不等式，基态解

Existence of Groundstates for a Class of Kirchhoff-Choquard Equations

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Abstract

In this paper, we consider the existence of ground state solutions for a class of Kirchhoff-Choquard equations with logarithmic nonlinear terms of potential functions by Ekeland variational method logarithmic Sobolev inequality and Hardy-Littlewood-Sobolev inequality. It is concluded that the Kirchhoff-Choquard equation has ground state solutions in $\alpha \in (0,3)$, $\alpha = 0$ and with nonlinear perturbation term.

Keywords

Kirchhoff-Choquard Equation, Ekeland Variational Method, Logarithmic Sobolev Inequality, Hardy-Littlewood-Sobolev Inequality, Ground State Solution

1. 引言

本文主要研究下列Kirchhoff-Choquard方程基态解的存在性

$$\text{问题(1):}\quad \begin{cases} -(a+b\int_\Omega |\nabla u|^2)\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u + H(x)u\log|u|, & x\in\Omega, \\ u=0, & x\in\partial\Omega, \end{cases} \tag{1}$$

其中$\Omega\subset\mathbb{R}^3$是有界光滑区域, $p\in(2,3+\alpha)$, $\alpha\in(0,3)$, a, $b>0$, V, H 是势函数.

作为偏微分方程中的一类经典的方程, $Kirchhoff$型方程有着深厚的物理背景, 如弹性力学, 人口动力学等方面. 近些年, 关于$Kirchhoff$型方程已取得丰富的成果. 可参看文献 [1–5].

在方程中, 当$b=0$且无对数非线性项时, 上述方程简化为$Choquard-Pekar$方程, 用于描述极化子的一种量子理论模型. 自提出以后, 越来越多的学者对$Choquard$方程感兴趣并且在数学方面已取得丰富的成果. 在文献 [6]中, 作者研究了一类带有$Hartree$型非线性的$Kirchhoff$型系统非平凡非负基态解的存在性和集中性. 文献 [7]考虑了带有不定势和临界指数增长的$Choquard$方程. 关于更多$Choquard$解的性质, 可参看文献 [8–15].

基于以上文献, 我们研究了一类带变号势和势函数对数非线性项$Kirchhoff-Choquard$方程基态解的存在性, 也就是问题(1)基态解的存在性.

假设问题(1)中势函数V, H满足:

(V): $V\in L^{3/2}(\Omega)$和$|V^-|_{3/2}<S$, 其中$V^+=\max\{V,0\}, V^-=\min\{V,0\}$和 $S=\inf_{u\in H_0^1(\Omega)\setminus\{0\}}\|u\|^2_{H_0^1}/|u|_6^2$;

(H): $H\in C(\overline{\Omega}),\ \mu:=\inf_\Omega H>0,\ M:=\max_{\overline{\Omega}} H$, 满足

$$M\leqslant 2\pi(1-S^{-1}|V^-|_{3/2})/e^{-8|\Omega|^{1/2}+2}.$$

我们有以下结论.

定理1 假设条件(V), (H)成立, 则问题(1)存在基态解.

当$\alpha=0$时, 则问题(1)转化为下列的方程:

$$\text{问题(2):}\quad \begin{cases} -(a+b\int_\Omega|\bigtriangledown u|^2)\Delta u+V(x)u=|u|^{2p-2}u+H(x)u\log|u|, & x\in\Omega,\\ u=0, & x\in\partial\Omega. \end{cases} \tag{2}$$

定理2 在定理1的条件下, 问题(2)存在基态解.

基于定理1, 定理2, 考虑下列带有$Hatree$多项式和对数非线性项的问题.

$$\text{问题(3):}\quad \begin{cases} -(a+b\int_\Omega|\bigtriangledown u|^2)\Delta u+V(x)u=\phi(x)u+H(x)u\log|u|+f(x,u), & x\in\Omega,\\ -\triangle\phi=u^2, & x\in\Omega,\\ u=\phi=0, & x\in\partial\Omega, \end{cases} \tag{3}$$

其中非线性项f满足如下条件$H(f)$:

(f_1) $f\in C(\bar{\Omega}\times\mathbb{R})$且存在$C>0, q\in(2,2^*), 2^*=\frac{2N}{N-2}$指的是$Sobolev$临界指数, 使得

$$|f(x,t)|\le C(1+|t|^{q-1}),\quad (x,t)\in\bar{\Omega}\times\mathbb{R};$$

(f_2) $lim\ sup_{t\to 0}f(x,t)/t=f_0$对$x\in\Omega$一致成立;

(f_3) 存在$\beta>4, R>0$, 使得

$$0<\beta F(x,t)\le tf(x,t),\quad x\in\bar{\Omega}, |t|\ge R,$$

其中$F(x,t)=\int_0^t f(x,s)ds, (x,t)\in\bar{\Omega}\times\mathbb{R}$.

针对问题(3)有以下结论.

定理3 假设条件(V), (H), $H(f)$成立, 则问题(3)存在基态解.

接下来, 我们简要的介绍本文的创新:

1) 据我们所知带有变号势函数和对数非线性项的$Kirchoff-Choquard$方程基态解的存在性目前没有研究过.

2) 对数非线性项不同于一般的多项式. 由于对数非线性项不满足单调性条件和AR条件, 导致对数非线性项比一般的多项式更加复杂.

3) 由于势函数V是变号的, 我们需要技巧性的处理$Hatree$非线性项和对数非线性项.

4) 本文通过渐近行为考虑了带有多项式和对数非线性的$Kirchhoff$方程解的存在性. 值得注意的是: 我们采用问题(1)的证明方法去证明问题(2) 的基态解的存在性.

5) 基于$Hatree$非线性项, 我们研究了$p=2$的情形也就是$Possion$项, 由于能量泛函的最高次数是4, 导致不满足山路结构. 为了克服困难, 我们需要加扰动项f.

接下来, 我们简单的介绍一下本文的结构. 第二部分, 给出本文所需的一些定义和命题. 第三部分, 主要证明定理1通过扰动的方法和满足山路结构和(PS)条件. 第四部分, 讨论了带有多项式对数非线性项基态解的存在性. 最后一部分主要证明定理带有扰动非线性f的$Kirchhoff-Possion$ 方程基态解的存在性.

2. 准备工作

设$H_0^1(\Omega)$是通常的$Sobolev$空间, 其范数是

$$||u||_{H_0^1}=\left(\int_\Omega|\nabla u|^2+u^2\right)^{1/2},\quad u\in H_0^1(\Omega).$$

根据V的条件, 定义范数:

$$||u||=\left(\int_\Omega(|\nabla u|^2+Vu^2)\right)^{1/2},\quad u\in H_0^1(\Omega).$$

则$||\cdot||$与$||\cdot||_{H_0^1}$是等价的. 对任意的$p\in[1,+\infty)$, $L^p(\Omega)$是通常的$Lebesgue$空间, 其标准范数为

$$|u|_p=\left(\int_\Omega|u|^p\right)^{1/p},\quad u\in L^p(\Omega).$$

用$|\cdot|$表示$Lebesgue$测度, $o_n(1)$表示无穷小量. 根据文献 [16–19], 我们有下面的对数$Sobolev$不等式.

命题1(对数$Sobolev$不等式) 设$\alpha>0$, 则对于所有的$u\in H^1(\mathbb{R}^3)\backslash\{0\}$, 有

$$2\int_{\mathbb{R}^3}u^2\log\frac{|u|}{|u|_2}+3(1+\log a)|u|_2^2\le\frac{a^2}{\pi}\int_{\mathbb{R}^3}|\nabla u|^2.$$

对于所有的$u \in H_0^1(\Omega)\backslash\{0\}$, 当$x \in \mathbb{R}^3\backslash\Omega$时, 定义$u(x)=0$, 根据命题1, 有

$$\int_\Omega |u^2| \log \frac{|u|}{|u|_2} \leq \frac{a^2}{2\pi}\int_\Omega |\nabla u|^2 - \frac{3}{2}(1+\log a)|u|_2^2.$$

命题2($Hardy-Littlewood-Sobolev$不等式) 设$s,t>1, \alpha\in(0,3)$ 且$1/s+1/r=1+\alpha/3$. 存在常数$C(\alpha,s,r)>0$ 使得对任意$g\in L^s(\Omega)$ 和$h\in L^r(\Omega)$, 有

$$\int_\Omega\int_\Omega \frac{g(x)h(y)}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \leqslant C(\alpha,s,r)|g|_s|h|_r.$$

根据$Hardy-Littlewood-Sobolev$不等式, 有

$$\int_\Omega (I_\alpha * |u|^p)|u|^p\mathrm{d}x \leqslant C_1|u|_{pr}^{2p},$$

其中$r=6/(3+\alpha)$. 关于更多$Hardy-Littlewood-Sobolev$不等式, 可参看文献 [11–14].

为了完成定理1的证明, 我们需要给出下列重要的命题.

命题3 定义下列的能量泛函族$\{J_\lambda\}_{\lambda\in\Lambda}\subset C^1(H_0^1(\Omega),\mathbb{R})$,

$$J_\lambda(u)=A(u)-\lambda B(u),\quad u\in H_0^1(\Omega),\lambda\in\Lambda,$$

其中 $B(u)\geq 0, u\in H_0^1(\Omega)$和$\lambda\in[\delta,1], \delta\in(0,1)$, 而且

$$\lim_{\|u\|\to\infty} A(u)=\infty,$$

或者

$$\lim_{\|u\|\to\infty} B(u)=\infty.$$

对所有的$\lambda\in\Lambda$, 令

$$\Gamma_\lambda=\{\gamma\in C([0,1],H_0^1(\Omega)):\gamma(0)=0, J_\lambda(\gamma(1))<0\}. \tag{4}$$

如果对每一个$\lambda\in\Lambda$, Γ_λ非空, 并且都有

$$c_\lambda=\inf_{\gamma\in\Gamma_\lambda}\max_{t\in[0,1]} J_\lambda(\gamma(t))>0,$$

那么, 对几乎处处的每一个$\lambda\in[\delta,1]$, 存在序列$\{u_n\}\subset H_0^1(\Omega)$, 使得

(i) $\{u_n\}$在$H_0^1(\Omega)$中是有界的;

(ii) $J_\lambda(u_n)\to c_\lambda$;

(iii) 在$H_0^1(\Omega)$中, $J'_\lambda(u_n)\to 0$.

3. 主要结果的证明

根据命题3, 定义下面的能量泛函族:

$$A(u)=\frac{1}{2}\int_{\Omega}(a|\nabla u|^2+Vu^2)+\frac{b}{4}||u||^4-\frac{1}{2}\int_{\Omega}Hu^2\log|u|+\frac{1}{4}\int_{\Omega}Hu^2,$$

显然, $\lim_{\|u\|\to\infty}A(u)=\infty$ 和

$$B(u)=\frac{1}{2p}\int_{\Omega}(I_\alpha*|u|^p)|u|^p\mathrm{d}x,\quad B(u)\geqslant 0.$$

有

$$\begin{aligned}J_\lambda(u)\quad&=\quad\frac{1}{2}\int_{\Omega}(a|\nabla u|^2+Vu^2)+\frac{b}{4}||u||^4-\frac{1}{2}\int_{\Omega}Hu^2ln|u|+\frac{1}{4}\int_{\Omega}Hu^2\\&-\quad\lambda\frac{1}{2p}\int_{\Omega}(I_\alpha*|u|^p)|u|^p\mathrm{d}x,\ \ u\in H_0^1(\Omega).\end{aligned}$$

则$J_\lambda\in C^1(H_0^1(\Omega),\mathbb{R})$且

$$\begin{aligned}\langle J_\lambda'(u),v\rangle\quad&=\quad\int_{\Omega}(a\nabla u\cdot\nabla v+Vuv)+b||u||^2\int_{\Omega}(u,v)-\int_{\Omega}Huv\log|u|\\&-\quad\lambda\int_{\Omega}(I_\alpha*|u|^p)|u|^{p-2}uv\mathrm{d}x,\ \ u,v\in H_0^1(\Omega).\end{aligned}$$

接下来, 我们给出四个重要的引理.

引理1 对于所有的$\lambda\in[\delta,1]$, $\Gamma_\lambda\neq\emptyset$.

证明 取$u\in H_0^1(\Omega)\backslash\{0\}$, 则对于所有的$\lambda\in[\delta,1]$, 有

$$\begin{aligned}J_\lambda(tu)\quad&=\quad\frac{t^2}{2}||u||^2+\frac{t^2}{2}\int_{\Omega}Vu^2+\frac{bt^4}{4}||u||^4-\frac{t^2}{2}\int_{\Omega}Hu^2\log|tu|+\frac{t^2}{4}\int_{\Omega}Hu^2-\frac{\lambda t^{2p}}{2p}\int_{\Omega}(I_\alpha*|u|^p)|u|^p\mathrm{d}x\\&\leq\quad\frac{t^2}{2}||u||^2+\frac{t^2}{2}\int_{\Omega}Vu^2+\frac{bt^4}{4}||u||^4-\frac{t^2\log t}{2}\int_{\Omega}H|u|_2^2-\frac{t^2}{2}\int_{\Omega}Hu^2\log|u|\\&+\quad\frac{t^2}{4}\int_{\Omega}Hu^2-\frac{\lambda t^{2p}}{2p}\int_{\Omega}(I_\alpha*|u|^p)|u|^p\mathrm{d}x,\quad t>0.\end{aligned}$$

注意到$p\in(2,3+\alpha)$, $\alpha\in(0,3)$. 因此, 当$t\to\infty$时, 有$J_\lambda(tu)\to-\infty$. 不难发现, 存在足够大的t_0, 对于$u\in H_0^1(\Omega)\backslash\{0\}$, 使得$J_\lambda(t_0u)<0$. □

接下来, 给出带有势函数H对数非线性项的一个重要不等式. 设$\Omega_1=\{x\in\Omega:|u(x)|/|u|_2<1\}$,

对于所有的$u \in H_0^1(\Omega) \setminus \{0\}$, 根据Hölder不等式和命题1, 我们有

$$\begin{aligned}
\int_\Omega Hu^2 \log \frac{|u|}{|u|_2} &= \int_{\Omega_1} Hu^2 \log \frac{|u|}{|u|_2} + \int_{\Omega \setminus \Omega_1} Hu^2 \log \frac{|u|}{|u|_2} \\
&\leqslant M \int_{\Omega_1} u^2 \log \frac{|u|}{|u|_2} + M \left(\int_\Omega u^2 \log \frac{|u|}{|u|_2} - \int_{\Omega_1} u^2 \log \frac{|u|}{|u|_2} \right) \\
&\leqslant M|\Omega|^{1/2}|u|_2^2 + M\Big(\frac{a^2}{2\pi} \int_\Omega |\nabla u|^2 - \frac{3(1+\log a)}{2}|u|_2^2 \\
&\quad +|\Omega|^{1/2}|u|_2^2 \Big) \\
&\leqslant M \left(\frac{a^2}{2\pi} \int_\Omega |\nabla u|^2 - \frac{3(1+\log a)}{2}|u|_2^2 + 2|\Omega|^{1/2}|u|_2^2 \right).
\end{aligned}$$

对于所有的$u \in H_0^1(\Omega) \setminus \{0\}$, 有

$$\begin{aligned}
J(u) &\geqslant \frac{1}{2} \int_\Omega \mu |\nabla u|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{2} \int_\Omega Hu^2 \log \frac{|u|}{|u|_2} - \frac{1}{2} \int_\Omega Hu^2 \log |u|_2 \\
&\quad -C_1 k |u|_{pr}^{2p} \\
&\geqslant \frac{1}{4} \left(2\mu - \frac{Ma^2}{\pi} \right) \int_\Omega |\nabla u|^2 + \frac{b}{4}\|u\|^4 + \frac{M}{4}(3 + 3\log a - 4|\Omega|^{1/2} - 2\log |u|_2)|u|_2^2 \\
&\quad -C_2 k \|u\|^{2p}.
\end{aligned}$$

其中$\mu = 1 - S^{-1}|V^-|_{3/2}$. 当$|u|_2 < a^{\frac{3}{2}} e^{\frac{3-4|\Omega|^{1/2}}{2}}$, 有

$$3 + 3\log a - 4|\Omega|^{1/2} - 2\log |u|_2 > 0.$$

另一方面, 根据a的任意性, 存在$a > 0$使得

$$2\mu - \frac{Ma^2}{\pi} > 0.$$

取$\|u\| = \rho > 0$足够小, 存在$c > 0$, 使得对所有的$\lambda \in [\delta, 1]$, 有$J_\lambda(u) \geq c$. 因此, $\Gamma_\lambda \neq \emptyset$, 所以存在$\gamma \in \Gamma_\lambda$. 于是, 固定$\lambda \in \Lambda$, 根据$\Gamma_\lambda$的定义, 我们就可得到$||\gamma(1)||_{H_0^1(\Omega)} > r$. 利用连续函数介值性定理, 那么就存在$t_0 \in (0, 1)$, 使得$||\gamma(t_0)||_{H_0^1(\Omega)} = r$. 因此, 有

$$c_\lambda \geq \inf_{\gamma \in \Gamma_\lambda} J_\lambda(\gamma(t_0)) \geq c > 0.$$

引理2 对于所有的$\lambda \in [\delta, 1]$, J_λ的每一个有界PS序列都有收敛子列.

证明 设$\{u_n\} \subset H_0^1(\Omega)$是$J_\lambda$的一个有界$PS$序列, 则存在$C > 0$, 使得对所有的$n$, $|J_\lambda(u_n)| \leq C$;

当$n \to \infty$时, $J'_\lambda(u_n) \to 0$. 因为$\{u_n\}$ 在$H_0^1(\Omega)$中有界, 则存在弱收敛的子序列$\{u_n\}$, 在$H_0^1(\Omega)$中有$u_n \rightharpoonup u$.

$$\begin{aligned} o_n(1) &= \langle J'_\lambda(u_n) - J'_\lambda(u), u_n - u\rangle \\ &= ||u_n - u||^2 + \int_\Omega V(u_n - u)^2 + b[||u_n||^4 - ||u_n||^2 \int_\Omega (u_n, u) - ||u||^2 \int_\Omega (u_n, u) + ||u||^4] \\ &- \int_\Omega (Hu_n \log|u_n| - Hu\log|u|)(u_n - u) \\ &- \int_\Omega [(I_\alpha * |u_n|^p)|u_n|^{p-2}u_n - (I_\alpha * |u|^p)|u|^{p-2}u](u_n - u)\mathrm{d}x. \end{aligned}$$

根据$Sobolev$嵌入定理和$Hardy-Littlewood-Sobolev$不等式, 有

$$\left|\int_\Omega (Hu_n \log|u_n| - Hu\log|u|)(u_n - u)\mathrm{d}x\right| \to 0.$$

和

$$\left|\int_\Omega [(I_\alpha * |u_n|^p)|u_n|^{p-2}u_n - (I_\alpha * |u|^p)|u|^{p-2}u](u_n - u)\mathrm{d}x\right| \to 0.$$

此外, 因为$V \in L^{3/2}$, 则有

$$\int_\Omega V(u_n - u)^2\mathrm{d}x \to 0.$$

$$o_n(1) = ||u_n - u||^2 + b[||u_n||^4 - ||u_n||^2 \int_\Omega (u_n, u) - ||u||^2 \int_\Omega (u_n, u) + ||u||^4],$$

注意到

$$\begin{aligned} &||u_n||^4 - ||u_n||^2 \int_\Omega (u_n, u) - ||u||^2 \int_\Omega (u_n, u) + ||u||^4 \\ &\geq ||u_n||^4 - ||u_n||^3||u|| - ||u||^3||u_n|| + ||u||^4 \\ &= (||u_n||^3 - ||u||^3)(||u_n|| - ||u||) \geq 0. \end{aligned}$$

因此, 有$||u_n - u||_{H_0^1(\Omega)} \to 0$. □

引理3 对几乎处处的每一个$\lambda \in H_0^1(\Omega)$, 存在$u^\lambda \in H_0^1(\Omega)\backslash\{0\}$, 使得$J_\lambda(u^\lambda) = c_\lambda$且$J'_\lambda(u^\lambda) = 0$ 成立.

证明 根据引理2和$\lambda_n \in C^1(H_0^1, \mathbb{R})$, 我们容易得出引理3的结论. □

引理4 设$\{u_n\}$是J_{λ_n}在水平c_{λ_n}的临界点序列, 则对于所有的$n \in \mathbb{N}$, 序列$\{u_n\}$是有界的.

证明 因为$\{u_n\}$是J_{λ_n}在水平c_{λ_n}的临界点序列, 则存在$C > 0$, 有$|J_{\lambda_n}(u_n)| \leq C$ 且$J'_{\lambda_n}(u_n) = 0$.

则

$$\begin{aligned}
C &\geq J_{\lambda_n}(u_n)-\frac{1}{2p}\langle J'_{\lambda_n}(u_n),u_n\rangle\\
&= \left(\frac{1}{2}-\frac{1}{2p}\right)||u_n||^2+\left(\frac{1}{2}-\frac{1}{2p}\right)\int_\Omega Vu_n^2+\left(\frac{1}{4}-\frac{1}{2p}\right)\|u_n\|^4\\
&+ \left(\frac{1}{2}-\frac{1}{2p}\right)\int_\Omega Hu^2-\left(\frac{1}{2}-\frac{1}{2p}\right)\int_\Omega Hu^2\log|u|_2\\
&\geq \left(\frac{1}{2}-\frac{1}{2p}\right)(\mu-M\frac{a^2}{2\pi})||u_n||^2+\left(\frac{1}{2}-\frac{1}{2p}\right)M\left(\frac{3(1+\log a)}{2}|u_n|_2^2-2|\Omega|^{1/2}|u_n|_2^2\right)\\
&- \left(\frac{1}{2}-\frac{1}{2p}\right)(M+1)\log|u|_2|u|_2^2+\left(\frac{1}{4}-\frac{1}{2p}\right)\|u_n\|^4.
\end{aligned}$$

根据对数$Sobolev$不等式, 存在$a>0$, 使得$\left(\frac{3(1+\log a)}{2}-2|\Omega|^{1/2}\right)>0$和$\mu-M\frac{a^2}{2\pi}>0$, 从而有

$$\left(\frac{1}{2}-\frac{1}{2p}\right)(\mu-M\frac{a^2}{2\pi})||u_n||^2+\left(\frac{1}{4}-\frac{1}{2p}\right)\|u_n\|^4+\left(\frac{1}{2}-\frac{1}{2p}\right)(M+1)\|u_n\|^3<C.$$

则$\{u_n\}$在$H_0^1(\Omega)$中有界. □

定理1的证明 因为$\lambda_n\to 1^-$, 所以我们易知$\{u_n\}$是J的一个PS序列. 事实上, 序列$\{u_n\}$的有界性暗示了$\{J(u_n)\}$ 的有界性. 另外, 因为

$$\langle J'(u_n),v\rangle=\langle J'_{\lambda_n}(u_n),v\rangle+(\lambda_n-1)\int_\Omega(I_\alpha*|u_n|^p)|u_n|^{p-2}u_nv\mathrm{d}x\quad v\in H_0^1(\Omega),$$

所以$J'_{\lambda_n}\to 0$, 这就说明$\{u_n\}$是J的一个有界PS序列. 于是, 根据引理2, 则$\{u_n\}$ 有一个收敛子列. 不妨我们仍记为$\{u_n\}$, 则有$u_n\to u$, 进而$J'(u)=0$. 再根据引理3, 有

$$J(u)=\lim_{n\to\infty}J(u_n)=\lim_{n\to\infty}J_{\lambda_n}(u_n)\geq c>0.$$

因此, 问题(1)有一个非平凡解. 设

$$K=\{u\in H_0^1(\Omega)\backslash\{0\}:J'(u)=0\}.$$

显然, K是非空的. 于是我们断言, 存在$\rho>0$, 使得对所有的$u\in K$, 都有$||u||_{H_0^1(\Omega)}\geq\rho$成立. 对所有的$u\in K$, 有

$$0=\langle J'(u),u\rangle=||u||^2+\int_\Omega Vu^2+b||u||^4-\int_\Omega Hu^2\log|u|-\int_\Omega(I_\alpha*|u|^p)|u|^p\mathrm{d}x.$$

根据对数不等式和$Hardy-Littlewood-Sobolev$不等式, 存在$C_2,C_4>0$使得

$$\begin{aligned}
\|u_n\|^2+\|u_n\|^4 &= k\int_\Omega(I_\alpha*|u_n|^p)|u_n|^p\mathrm{d}x+\int_\Omega Hu_n^2\log|u_n|\\
&\leqslant C_2\|u_n\|^{2p}+C_4\|u_n\|^4.
\end{aligned}$$

则

$$\|u\|^2 \leqslant C_2\|u\|^{2p} + C_4\|u\|^4.$$

进一步, 我们推断出存在$c > 0$, 使得$\|u\| > c$. 对所有的$u \in K$, 有

$$\begin{aligned}
&J(u) - \frac{1}{2p}\langle J'(u), u\rangle \\
&= \left(\frac{1}{2} - \frac{1}{2p}\right)||u||^2 + \left(\frac{1}{2} - \frac{1}{2p}\right)\int_\Omega Vu_n^2 + \left(\frac{1}{4} - \frac{1}{2p}\right)\|u\|^4 \\
&+ \left(\frac{1}{2} - \frac{1}{2p}\right)\int_\Omega Hu^2 - \left(\frac{1}{2} - \frac{1}{2p}\right)\int_\Omega Hu^2\log|u|_2 \\
&\geq \left(\frac{1}{2} - \frac{1}{2p}\right)(\mu - M\frac{a^2}{2\pi})||u||^2 + \left(\frac{1}{2} - \frac{1}{2p}\right)M\left(\frac{3(1+\log a)}{2}|u|_2^2 - 2|\Omega|^{1/2}|u|_2^2\right) \\
&- \left(\frac{1}{2} - \frac{1}{2p}\right)(M+1)\log|u|_2|u|_2^2 + \left(\frac{1}{4} - \frac{1}{2p}\right)\|u\|^4 \\
&\geq \left(\frac{1}{2} - \frac{1}{2p}\right)(\mu - M\frac{a^2}{2\pi})||u||^2 + \left(\frac{1}{2} - \frac{1}{2p}\right)M\left(\frac{3(1+\log a)}{2}|u|_2^2 - 2|\Omega|^{1/2}|u|_2^2\right) \\
&- \left(\frac{1}{2} - \frac{1}{2p}\right)(M+1)|u|_2^3 + \left(\frac{1}{4} - \frac{1}{2p}\right)\|u\|^4.
\end{aligned}$$

则J在K上是强制的, 进而J在K上有下界. 因此, 我们可以定义

$$c_0 := \inf_K J.$$

最后, 设$\{u_n\} \subset K$是J的一个极小化序列, 即

$$J(u_n) \to c_0, J'(u_n) = 0.$$

类似于引理2和引理3的证明, 可知序列$\{u_n\}$在$H_0^1(\Omega)$中是有界, 进而存在$u \in H_0^1(\Omega)$, 使得$u_n \to u$, $n \to \infty$,所以$J(u) = c_0$且$J'(u_n) = 0$. 又有$u \neq 0$, 因此, $u \in H_0^1(\Omega)$是问题(1)的一个基态解. □

定理2的证明 在$H_0^1(\Omega)$上, 定义关于问题(2)的能量泛函J_1

$$J_1(u) = \frac{1}{2}\int_\Omega(|\nabla u|^2 + Vu^2) + \frac{b}{4}||u||^4 - \frac{1}{2p}\int_\Omega |u|^{2p} - \frac{1}{2}\int_\Omega Hu^2\log|u| + \frac{1}{4}\int_\Omega Hu^2,$$

对于所有的$u \in H_0^1(\Omega)$. 通过计算, 有$J_1 \in C^1(H_0^1(\Omega), \mathbb{R})$. 此外,

$$\langle J_1'(u), v\rangle = \int_\Omega(\nabla u \cdot \nabla v + Vuv) + b||u||^2(u,v) - \int_\Omega |u|^{p-2}uv - \int_\Omega Huv\log|u|,$$

对于所有的$v \in H_0^1(\Omega)$.

类似于问题(1)的证明方法, 易证, 问题(2)存在非平凡解. 这里我们忽略这部分的证明.

接下来, 我们证明问题(2)存在基态解. 设

$$K_1 = \{u \in H_0^1(\Omega)\backslash\{0\} : J'(u) = 0\}.$$

显然, K_1是非空的. 于是, 我们断言, 存在$\rho > 0$, 使得对所有的$u \in K_1$, 都有$||u||_{H_0^1(\Omega)} \geq \rho$成立且

$$0 = \langle J'(u), u\rangle = ||u||^2 + \int_\Omega Vu^2 + b||u||^4 - \int_\Omega Hu^2 \log|u| - \int_\Omega (I_\alpha * |u|^p)|u|^p dx.$$

根据对数不等式和$Hardy - Littlewood - Sobolev$不等式, 有

$$\|u_n\|^2 + \|u_n\|^4 \leqslant C_2\|u_n\|^{2p} + C_4\|u_n\|^4.$$

则

$$\|u\|^2 \leqslant C_2\|u\|^{2p} + C_4\|u\|^4.$$

进一步, 我们推断出存在$c > 0$, 使得$\|u\| > c$. 对所有的$u \in K_1$, 有

$$\begin{aligned}
&J(u) - \frac{1}{2p}\langle J'(u), u\rangle \\
&= \left(\frac{1}{2} - \frac{1}{2p}\right)||u||^2 + \left(\frac{1}{2} - \frac{1}{2p}\right)\int_\Omega Vu_n^2 \\
&+ \left(\frac{1}{2} - \frac{1}{2p}\right)\int_\Omega Hu^2 - \left(\frac{1}{2} - \frac{1}{2p}\right)\int_\Omega Hu^2 \log|u|_2 \\
&\geq \left(\frac{1}{2} - \frac{1}{2p}\right)(\mu - M\frac{a^2}{2\pi})||u||^2 + \left(\frac{1}{2} - \frac{1}{2p}\right)M\left(\frac{3(1+\log a)}{2}|u|_2^2 - 2|\Omega|^{1/2}|u|_2^2\right) \\
&- \left(\frac{1}{2} - \frac{1}{2p}\right)(M+1)\log|u|_2|u|_2^2 + \left(\frac{1}{4} - \frac{1}{2p}\right)\|u\|^4 \\
&\geq \left(\frac{1}{2} - \frac{1}{2p}\right)(\mu - M\frac{a^2}{2\pi})||u||^2 + \left(\frac{1}{2} - \frac{1}{2p}\right)M\left(\frac{3(1+\log a)}{2}|u|_2^2 - 2|\Omega|^{1/2}|u|_2^2\right) \\
&- \left(\frac{1}{2} - \frac{1}{2p}\right)(M+1)|u|_2^3 + \left(\frac{1}{4} - \frac{1}{2p}\right)\|u\|^4.
\end{aligned}$$

则J在K上强制, 进而J在K上是有下界. 因此, 我们可以定义

$$c_1 := \inf_K J.$$

最后, 设$\{u_n\}\subset K$是J的一个极小化序列, 即

$$J(u_n)\to c_1, J'(u_n)=0.$$

类似于引理2和引理3的证明，序列$\{u_n\}$在$H_0^1(\Omega)$中有界，进而存在$u\in H_0^1(\Omega)$，使得$u_n\to u$, $n\to\infty$,所以$J(u)=c_1$ 且$J'(u_n)=0$. 又有$u\neq 0$, 因此, $u\in H_0^1(\Omega)$ 是问题(2)的一个基态解. □

定理3的证明 首先, 定义问题(3)的能量泛函族:

$$\begin{aligned} J_{2\lambda}(u) &= \frac{1}{2}\int_\Omega(a|\nabla u|^2+Vu^2)+\frac{b}{4}||u||^4-\frac{1}{2}\int_\Omega Hu^2ln|u|+\frac{1}{4}\int_\Omega Hu^2 \\ &- \frac{1}{4}\int_\Omega \phi_u u^2-\lambda\int_\Omega F(x,u), u\in H_0^1(\Omega). \end{aligned}$$

则$J_{2\lambda}\in C^1(H_0^1(\Omega),\mathbb{R})$且

$$\begin{aligned} \langle J'_{2\lambda}(u),v\rangle &= \int_\Omega(a\nabla u\cdot\nabla v+Vuv)+b||u||^2(u,v)-\int_\Omega Huv\log|u| \\ &- \frac{1}{4}\int_\Omega \phi_u uv-\lambda\int_\Omega f(x,u)v, u,v\in H_0^1(\Omega). \end{aligned}$$

类似于定理1的证明, 根据$Hatree$非线性项的性质和命题, 我们可知问题(3)存在非平凡解. 由于大部分证明类似, 我们在这里忽略, 值得注意的是我们取$B(u)=\lambda\int_\Omega F(x,u)$.

设

$$K_2=\{u\in H_0^1(\Omega)/\{0\}: J_2'(u)=0, \lambda=1\}.$$

显然, K_2是非空的. 于是, 我们断言存在$\rho>0$, 使得对所有的$u\in K_2$, 都有$||u||_{H_0^1(\Omega)}\geq\rho$成立. 则对所有的$u\in K_2$, 有

$$0=\langle J'(u),u\rangle=||u||^2+\int_\Omega Vu^2+b||u||^4-\int_\Omega Hu^2\log|u|-\frac{1}{4}\int_\Omega\phi_u u^2-\int_\Omega f(x,u)u.$$

根据文献 [20–22], 我们推出$\int_R\phi u^2\leq 4||u||^4$. 根据条件$H(f)$, 有

$$\begin{aligned} &\|u\|^2+\|u\|^4+\frac{M}{4}(3+3\log a-4|\Omega|^{1/2}-2\log|u|_2)|u|_2^2 \\ =\ & \int_\Omega Hu^2\log|u|+\frac{1}{4}\int_\Omega\phi_u u^2+\int_\Omega f(x,u)u \\ \leqslant\ & C_2\|u\|^{2p}+C_4\|u\|^4+\frac{f_0+1}{2}|u|_2^2+C\|u\|^q. \end{aligned}$$

根据对数$Sobolev$不等式, 有

$$\|u\|^2\leqslant C_4\|u\|^4+C\|u\|^q.$$

进一步, 我们推断出存在$c>0$, 使得$\|u\|>c$. 对所有的$u\in K_2$, 有

$$
\begin{aligned}
&J_2(u)-\frac{1}{4}\langle J_2'(u),u\rangle\\
&=\left(\frac{1}{2}-\frac{1}{4}\right)||u||^2+\left(\frac{1}{2}-\frac{1}{4}\right)\int_\Omega Vu_n^2+\left(\frac{1}{4}-\frac{1}{4}\right)\|u\|^4+\int_\Omega(\frac{1}{4}f(x,u)u-F(x,u))\\
&+\left(\frac{1}{2}-\frac{1}{4}\right)\int_\Omega Hu^2-\left(\frac{1}{2}-\frac{1}{4}\right)\int_\Omega Hu^2\log|u|_2+(\frac{1}{4}-\frac{1}{4})\int_\Omega\phi_u u^2\\
&\geq\left(\frac{1}{2}-\frac{1}{2p}\right)(\mu-M\frac{a^2}{2\pi})||u||^2+\left(\frac{1}{2}-\frac{1}{2p}\right)M\left(\frac{3(1+\log a)}{2}|u|_2^2-2|\Omega|^{1/2}|u|_2^2\right)\\
&-\left(\frac{1}{2}-\frac{1}{2p}\right)(M+1)\log|u|_2|u|_2^2+\left(\frac{1}{4}-\frac{1}{2p}\right)\|u\|^4\\
&\geq\left(\frac{1}{2}-\frac{1}{2p}\right)(\mu-M\frac{a^2}{2\pi})||u||^2+\left(\frac{1}{2}-\frac{1}{2p}\right)M\left(\frac{3(1+\log a)}{2}|u|_2^2-2|\Omega|^{1/2}|u|_2^2\right)\\
&-\left(\frac{1}{2}-\frac{1}{2p}\right)(M+1)|u|_2^3+\left(\frac{1}{4}-\frac{1}{2p}\right)\|u\|^4.
\end{aligned}
$$

则J在K上强制, 进而J在K上是有下界. 因此, 我们可以定义

$$c_2:=\inf_K J.$$

最后, 设$\{u_n\}\subset K$是J的一个极小化序列, 即

$$J(u_n)\to c_2, J'(u_n)=0.$$

类似于引理2和引理3的证明, 序列$\{u_n\}$在$H_0^1(\Omega)$中是有界的, 进而存在$u\in H_0^1(\Omega)$, 使得$u_n\to u$, $n\to\infty$,所以$J(u)=c_2$且$J'(u_n)=0$. 又有$u\neq 0$, 因此, $u\in H_0^1(\Omega)$是问题(3)的一个基态解. □

4. 总结

本文利用单调性技巧, 对数$Sobolev$不等式和$Hardy-Littlewood-Sobolev$不等式研究一类带有对数非线性项和$Hatree$非线性项$Kirchhoff$方程基态解的存在性. 值得注意的是势函数V是变号的, 对数非线性项含有势函数. 另外, 为研究$Hatree$非线性项参数趋于N和0打下一定的基础, 我们猜测带有对数非线性项和$Hatree$非线性项的$Kirchhoff$方程的基态解趋于带有对数非线性项和多项式的$Kirchhoff$方程的基态解.

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参考文献

[1] Zhou, J. and Wu, Y.S. (2021) Existence of Solutions for a Class of Kirchhoff-Type Equations with Indefinite Potential. *Boundary Value Problems*, **12**, Article No. 74. https://doi.org/10.1186/s13661-021-01550-5

[2] Zhou, F. and Yang, M.B. (2021) Solutions for a Kirchhoff Type Problem with Critical Exponent in $\mathbb{R}^N$. *Journal of Mathematical Analysis and Applications*, **494**, Article ID: 124638.

[3] Vicente, A. (2022) Well-Posedness and Stability for Kirchhoff Equation with Non-Porous Acoustic Boundary Conditions. *Journal of Differential Equations*, **313**, 25-38. https://doi.org/10.1016/j.jde.2022.01.002

[4] Zhou, L. and Zhu, C.X. (2022) Ground State Solution for a Class of Kirchhoff-Type Equation with General Convolution Nonlinearity. *Zeitschrift für angewandte Mathematik und Physik*, **73**, Article No. 75. https://doi.org/10.1007/s00033-022-01712-0

[5] Gu, G.Z. and Yang, Z.P. (2022) On the Singularly Perturbation Fractional Kirchhoff Equations: Critical Case. *Advances in Nonlinear Analysis*, **11**, 1097-1116. https://doi.org/10.1515/anona-2022-0234

[6] Li, G. and Tang, C. (2018) Existence of a Ground State Solution for Choquard Equation with the Upper Critical Exponent. *Computers and Mathematics with Applications*, **76**, 2635-2647. https://doi.org/10.1016/j.camwa.2018.08.052

[7] Li, F., Gao, C. and Liang, Z. (2018) Existence and Concentration of Nontrivial Nonnegative Ground State Solutions to Kirchhoff-Type System with Hartree-Type Nonlinearity. *Zeitschrift für angewandte Mathematik und Physik*, **69**, Article No. 148. https://doi.org/10.1007/s00033-018-1043-5

[8] Li, G., Li, Y., Tang, C. and Yin, L. (2019) Existence and Concentrate Behavior of Ground State Solutions for Critical Choquard Equations. *Applied Mathematics Letters*, **96**, 101-107. https://doi.org/10.1016/j.aml.2019.04.020

[9] Gao, F.S. and Yang, M.B. (2017) On Nonlocal Choquard Equations with Hardy-Littlewood-Sobolev Critical Exponents. *Journal of Mathematical Analysis and Applications*, **448**, 1006-1041. https://doi.org/10.1016/j.jmaa.2016.11.015

[10] Li, F.Y., Gao, C.J. and Zhu, X.L. (2017) Existence and Concentration of Sign-Changing Solutions to Kirchhoff-Type System with Hartree-Type Nonlinearity. *Journal of Mathematical Analysis and Applications*, **448**, 60-80. https://doi.org/10.1016/j.jmaa.2016.10.069

[11] Liu, H., Liu, Z. and Xiao, Q. (2017) Ground State Solution for a Fourth-Order Nonlinear Elliptic Problem with Logarithmic Nonlinearity. *Applied Mathematics Letters*, **79**, 176-181. https://doi.org/10.1016/j.aml.2017.12.015

[12] Schaftingen, J.V. and Xia, J. (2018) Groundstates for a Local Nonlinear Perturbation of the Choquard Equations with Lower Critical Exponent. *Journal of Mathematical Analysis and Applications*, **464**, 1184-1202. https://doi.org/10.1016/j.jmaa.2018.04.047

[13] Tian, S. (2017) Multiple Solutions for the Semilinear Elliptic Equations with the Sign-Changing Logarithmic Nonlinearity. *Journal of Mathematical Analysis and Applications*, **454**, 816-828. https://doi.org/10.1016/j.jmaa.2017.05.015

[14] Squassina, M. and Szulkin, A. (2015) Multiple Solutions to Logarithmic Schrödinger Equations with Periodic Potential. *Calculus of Variations and Partial Differential Equations*, **54**, 585-597. https://doi.org/10.1007/s00526-014-0796-8

[15] Gao, F. and Yang, M. (2018) On the Brezis-Nirenberg Type Critical Problem for Nonlinear Choquard Equation. *Science China Mathematics*, **61**, 1219-1242. https://doi.org/10.1007/s11425-016-9067-5

[16] Le, X.T. (2019) The Nehari Manifold for Fractional p-Laplacian Equation with Logarithmic Nonlinearity on Whole Space. *Computers and Mathematics with Applications*, **78**, 3931-3940. https://doi.org/10.1016/j.camwa.2019.06.024

[17] Alves, C.O. and de Morais Filho, D.C. (2018) Existence and Concentration of Positive Solutions for a Schrödinger Logarithmic Equation. *Zeitschrift für angewandte Mathematik und Physik*, **69**, Article No. 144. https://doi.org/10.1007/s00033-018-1038-2

[18] Chen, H. and Tian, S. (2015) Initial Boundary Value Problem for a Class of Semilinear Pseudo-Parabolic Equations with Logarithmic Nonlinearity. *Journal of Differential Equations*, **258**, 4424-4442. https://doi.org/10.1016/j.jde.2015.01.038

[19] Ji, C. and Szulkin, A. (2016) A Logarithmic Schrödinger Equation with Asymptotic Conditions on the Potential. *Journal of Mathematical Analysis and Applications*, **437**, 241-254. https://doi.org/10.1016/j.jmaa.2015.11.071

[20] Wang, J., Tian, L.X., Xu, J.X., *et al.* (2013) Erratum to: Existence and Concentration of Positive Solutions for Semilinear Schrödinger-Poisson Systems in R^3. *Calculus of Variations and Partial Differential Equations*, **48**, 275-276. https://doi.org/10.1007/s00526-012-0573-5

[21] Li, Y.H., Li, F.Y. and Shi, J.P. (2017) Existence and Multiplicity of Positive Solutions to Schrödinger-Poisson Type Systems with Critical Nonlocal Term. *Calculus of Variations and Partial Differential Equations*, **56**, Article No. 134. https://doi.org/10.1007/s00526-017-1229-2

[22] Wang, Z.P. and Zhou, H.-S. (2015) Sign-Changing Solutions for Nonlinear Schrödinger-Poisson System in R^3. *Calculus of Variations and Partial Differential Equations*, **52**, 927-943. https://doi.org/10.1007/s00526-014-0738-5