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两个二维多参数绝对值优化问题解的等价性

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摘 要

对带稀疏惩罚的最小一乘问题，采用MCP函数来连续松弛ℓ_0函数，在二维情况下研究由此得到的两个多参数绝对值优化问题解的等价性。在简单条件下证明了两个问题具有相同全局最优解和最优值，为进一步研究相应高维问题提供了参考。

关键词

最小一乘问题，ℓ_0 范数，MCP函数，多参数绝对值优化问题，解的等价性

Equivalence of Solutions for Two Two-Dimensional Multi-Parameter Absolute Value Optimization Problems

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Abstract

In this paper, the MCP function is used to continuously relax the ℓ_0 function for the least absolute deviation with sparse penalty, and the equivalence of the solutions of the two multi-parameter absolute value optimization problems is studied in two-dimensional space. Under simple conditions, it is proved that the two problems have the same global optimal solution and optimal value, which provides a reference for further study of the corresponding high-dimensional problems.

Keywords

Least Absolute Deviation, ℓ_0 Norm, MCP Function, Multi-Parameter Absolute Value Optimization Problem, Equivalence of Solutions

1. 引言

稀疏约束优化问题 [1–3]是指带有稀疏约束的一般非线性优化问题. 这类问题广泛应用在金融、计量经济学、图像处理、模式识别、变量选择、变量空间降维、稀疏重构、投资组合等多个领域 [4–7]. 成为近十年最优化及其相关领域的一个热点研究课题. 一般来讲, 连续优化的理论通常不能用于处理该类问题, 但是稀疏约束集合的特殊结构也为我们的研究提供了一个新的机遇. 稀疏优化目的是寻找一个欠定线性或非线性方程的稀疏解, 即要求非零分量个数尽可能少, 其优化模型如下:

$$\min_{x\in\mathbb{R}^n} G(x) := f(x) + \lambda\|x\|_0, \tag{1}$$

其中, $f(x)$表示损失函数, 例如最小一乘、最小二乘、Huber 函数等. $\|x\|_0$ 表示向量x 非零分量的个数, λ 是一个正的常数.

因为带有ℓ_0正则的优化问题是非凸、非光滑、非Lipschitz的, 且文献 [8, 9]也指出求解问题(1)是NP难的. 因此Donoho-Candés-Tao 等人将ℓ_0范数松弛为ℓ_1范数 [10–12], 其模型为:

$$\min_{x\in\mathbb{R}^n} G(x) := f(x) + \lambda\|x\|_1,$$

其中, $\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$, $\lambda > 0$.

研究表明, ℓ_1松弛模型与ℓ_0惩罚模型相比分析和求解难度大大降低 [13], 且ℓ_1 模型的解容易通过现有的优化算法得到. 考虑到ℓ_1 松弛经常导致过度正则化, 即得到的结果是有偏估计量 [14], 并且该模型的解缺少Oracle 性质 [15–17]. 因此, 部分研究者建议采用具有良好性质的折叠凹函数来松弛ℓ_0范数, 比如, Capped-ℓ_1, MCP, SCAD 等折叠凹罚函数, 这些非凸模型产生的估计量具有无偏性, 连续性, 稀疏性 [17] 等良好的性质, 值得关注的是, 在这些罚函数中, MCP 在松弛上不仅有Oracle 性质, 而且在理论以及数值上具有良好的表现 [18]. 因此, 本文考虑用MCP函数来松弛ℓ_0范数, 松弛模型如下:

$$\min_{x\in\mathbb{R}^n} G(x) := f(x) + \Phi(x),$$

其中$\Phi(x) = \sum_{i=1}^n \varphi(\theta, \lambda, x_i)$, 并且

$$\varphi(\theta,\lambda,x_i) = \begin{cases} \lambda|x_i| - \frac{x_i^2}{2\theta}, & |x_i| \le \theta\lambda, \\ \frac{1}{2}\theta\lambda^2, & |x_i| > \theta\lambda, \end{cases} \quad \lambda > 0, \theta > 1.$$

在应用统计领域, 最小一乘损失可以有效地处理离群值, 这使得最小一乘损失具有鲁棒性 [19]. 因此, 本文采用最小一乘损失函数.

一个重要且有趣的问题是原始问题与松弛问题解的关系问题 [18, 20–22].

本文在二维情形下讨论以下ℓ_0 罚问题

$$\min_{x_1,x_2\in\mathbb{R}} G(x) := |ax_1 + bx_2 - c| + \lambda\|x\|_0, \tag{2}$$

其中

$$\|x\|_0 = \begin{cases} 0, & x_1 = x_2 = 0, \\ 1, & x_1\text{与 } x_2\text{有且仅有一个为}0, \\ 2, & x_1\text{与 } x_2\text{均不为}0, \end{cases}$$

与其松弛问题

$$\min_{x_1,x_2\in\mathbb{R}} \tilde{G}(x) := |ax_1+bx_2-c| + \Phi(x), \tag{3}$$

其中$\Phi(x) = \sum_{i=1}^{2} \varphi(\theta,\lambda,x_i)$,

$$\varphi(\theta,\lambda,x_i) = \begin{cases} \lambda|x_i| - \frac{x_i^2}{2\theta}, & |x_i| \le \theta\lambda, \\ \frac{1}{2}\theta\lambda^2, & |x_i| > \theta\lambda, \end{cases} \quad \lambda > 0, \theta > 1,$$

之间解的等价性. 问题(2)与(3)都是含多参数的绝对值优化问题, 都是非凸非光滑的.

2. 主要结果

为讨论问题(2)与(3)解的等价性, 首先给出问题(2)的解.

定理 2.1 设$a \neq 0$, $b \neq 0$, $c \in \mathbb{R}$, $\lambda > 0$, $x \in \mathbb{R}^2$, 则问题(2) 的全局最优解x^*如下

$$x^* = \begin{cases} (0,0), & \lambda > |c|, \\ (0,0), (\frac{c}{a},0)\text{或}(0,\frac{c}{b}), & \lambda = |c|, \\ (\frac{c}{a},0)\text{或}(0,\frac{c}{b}), & \lambda < |c|. \end{cases}$$

证明. 当$x_1 = 0$, $x_2 = 0$时, $\|x\|_0 = 0$, 有

$$G_0(x) = |ax_1+bx_2-c| + \lambda\|x\|_0 = |c|;$$

当$x_1 \neq 0$, $x_2 = 0$时, $\|x\|_0 = 1$, 则

$$G_1(x) = (ax_1-c)\text{sign}(ax_1-c) + \lambda,$$

当$x_1 = 0, x_2 \neq 0$时, $\|x\|_0 = 1$, 则

$$G_2(x) = (bx_2-c)\text{sign}(bx_2-c) + \lambda,$$

当$x_1 \neq 0$,$x_2 \neq 0$时, $\|x\|_0 = 2$, 有

$$G_3(x) = (ax_1+bx_2-c)\text{sign}(ax_1+bx_2-c) + 2\lambda.$$

全局解分析:

若$G_0(x) < \min\{G_1(x), G_2(x), G_3(x)\}$, 即$\lambda > |c|$ 时, 有$x^* = 0$;

若$G_0(x)=G_1(x)=G_2(x)$, 即$\lambda=|c|$时, 恒有$G(0)<G_3(x)$, 故$x^*=0$, $(\frac{c}{a},0)$或$(0,\frac{c}{b})$;

若$G_1(x)=G_2(x)<\min\{G_0(x),G_3(x)\}$, 即$\lambda<|c|$时, 故$x^*=(\frac{c}{a},0)$ 或$(0,\frac{c}{b})$. □

关于二元函数的极值, 有以下引理.

引理 2.1 [23]设f 在(x_0,y_0) 附近具有二阶连续偏导数, (x_0,y_0) 为f 的驻点. 记

$$A=f_{xx}\left(x_0,y_0\right),\quad B=f_{xy}\left(x_0,y_0\right),\quad C=f_{yy}\left(x_0,y_0\right),$$

$$H=\begin{vmatrix} A & B \\ B & C \end{vmatrix}=AC-B^2.$$

(i). 若$H>0: A>0$ 时$f\left(x_0,y_0\right)$ 为极小值; $A<0$ 时$f\left(x_0,y_0\right)$ 为极大值;

(ii). 若$H<0: f\left(x_0,y_0\right)$ 不是极值;

(iii). 当$H=0$ 时, $f\left(x_0,y_0\right)$ 可能是极值, 也可能不是极值.

下面给出本文的主要结果.

定理 2.2 设$ab\neq 0$, $\lambda>0$, $\theta>0$, $\lambda\theta=2$, 则$\forall c\in\mathbb{R}$, 问题(2)与(3)具有相同的全局最优解和最优值.

证明. 下面主要通过讨论函数在定义域的不同区域内去绝对值和寻找极值点并比较大小的方法来确定问题(3)的最优解.

1^0 $x_1<-\lambda\theta, x_2<-\lambda\theta$, 且$ax_1+bx_2<c$时,

$$\tilde{G}_1(x)=c-ax_1-bx_2+\theta\lambda^2>\theta\lambda^2.$$

2^0 $x_1<-\lambda\theta, x_2<-\lambda\theta$, 且$ax_1+bx_2>c$时,

$$\tilde{G}_2(x)=ax_1+bx_2-c+\theta\lambda^2>\theta\lambda^2.$$

3^0 $x_1<-\lambda\theta, x_2<-\lambda\theta$, 且$ax_1+bx_2=c$时,

$$\tilde{G}_3(x)=\theta\lambda^2.$$

综合1^0-3^0, 得到在区域$S_1=\{x_1<-\lambda\theta, x_2<-\lambda\theta\}$上的最优解为: $x\in\{x\in\mathbb{R}^2: ax_1+bx_2=c, x_1<-\lambda\theta, x_2<-\lambda\theta\}$, 最优值为$\tilde{G}(x)=\theta\lambda^2$.

4^0 $x_1 < -\lambda\theta, -\lambda\theta \le x_2 < 0$, 且$a\lambda\theta > -c$, 则$ax_1 + bx_2 < c$,

$$\tilde{G}_4(x) = c - ax_1 - bx_2 + \frac{1}{2}\theta\lambda^2 - \lambda x_2 - \frac{x_2^2}{2\theta}$$

由于

(i) $\tilde{G}(-\lambda\theta, 0) = c + a\lambda\theta + \frac{1}{2}\theta\lambda^2 > \frac{1}{2}\theta\lambda^2 = \tilde{G}(\frac{c}{a}, 0)$.

(ii) $\tilde{G}(-\lambda\theta, -\lambda\theta) = |c + a\theta\lambda + b\theta\lambda| + \theta\lambda^2 > \theta\lambda^2 > \frac{1}{2}\theta\lambda^2 = \tilde{G}(\frac{c}{a}, 0)$.

且$(\frac{c}{a}, 0)$不在该区域, 也无其他极小值点, 故该区域无极小值.

5^0 $x_1 < -\lambda\theta, -\lambda\theta \le x_2 < 0$, 且$a\lambda\theta \le -c$, 则$ax_1 + bx_2 \ge c$,

$$\tilde{G}_5(x) = ax_1 + bx_2 - c + \frac{1}{2}\theta\lambda^2 - \lambda x_2 - \frac{x_2^2}{2\theta}$$

由于

(i) $\tilde{G}(-\lambda\theta, 0) = c + a\theta\lambda + \frac{1}{2}\theta\lambda^2 \ge \frac{1}{2}\theta\lambda^2 = \tilde{G}(\frac{c}{a}, 0)$.

(ii) $\tilde{G}(-\lambda\theta, -\lambda\theta) = |a\theta\lambda + b\theta\lambda + c| + \theta\lambda^2 \ge \theta\lambda^2 > \frac{1}{2}\theta\lambda^2 = \tilde{G}(\frac{c}{a}, 0)$.

且$(\frac{c}{a}, 0)$不在该区域, 也无其他极小值点, 故该区域无极小值.

6^0 $-\lambda\theta \le x_1 < 0, x_2 < -\lambda\theta$, 且当$b\lambda\theta < -c$时, $ax_1 + bx_2 > c$,

$$\tilde{G}_6(x) = ax_1 + bx_2 - c - \lambda x_1 - \frac{x_1^2}{2\theta} + \frac{1}{2}\theta\lambda^2$$

由于

(i) $\tilde{G}(0, -\lambda\theta) = b\theta\lambda + c + \frac{1}{2}\theta\lambda^2 > \frac{1}{2}\theta\lambda^2 = \tilde{G}(0, \frac{c}{b})$.

(ii) $\tilde{G}(-\lambda\theta, -\lambda\theta) = |a\theta\lambda + b\theta\lambda + c| + \theta\lambda^2 > \theta\lambda^2 > \frac{1}{2}\theta\lambda^2 = \tilde{G}(0, \frac{c}{b})$.

且$(0, \frac{c}{b})$不在此区域, 也无其他极小值点, 故在该区域无极小值.

7^0 $-\lambda\theta \le x_1 < 0, x_2 < -\lambda\theta$, 且当$b\lambda\theta \ge -c$时, $ax_1 + bx_2 \le c$,

$$\tilde{G}_7(x) = c - ax_1 - bx_2 - \lambda x_1 - \frac{x_1^2}{2\theta} + \frac{1}{2}\theta\lambda^2$$

由于

(i) $\tilde{G}(0, -\lambda\theta) = c + b\theta\lambda + \frac{1}{2}\theta\lambda^2 \ge \frac{1}{2}\theta\lambda^2 = \tilde{G}(0, \frac{c}{b})$.

(ii) $\tilde{G}(-\lambda\theta, -\lambda\theta) = |a\theta\lambda + b\theta\lambda + c| + \theta\lambda^2 \ge \theta\lambda^2 > \frac{1}{2}\theta\lambda^2 = \tilde{G}(0, \frac{c}{b})$.

且$(0, \frac{c}{b})$不在此区域, 也无其他极小值点, 故在该区域无极小值.

8^0 $-\lambda\theta \le x_1 < 0, -\lambda\theta \le x_2 < 0$, 且当$\lambda\theta \ge \frac{-c}{a+b}$时,$ax_1 + bx_2 \ge c$,

$$\tilde{G}_8(x) = ax_1 + bx_2 - c - \lambda x_1 - \frac{x_1^2}{2\theta} - \lambda x_2 - \frac{x_2^2}{2\theta}$$

由于

(i) $\tilde{G}(0,0) = -c$.

(ii) $\tilde{G}(-\lambda\theta, 0) = |c + a\theta\lambda| + \frac{1}{2}\theta\lambda^2 > \frac{1}{2}\theta\lambda^2 = \tilde{G}(\frac{c}{a}, 0)$.

(iii) $\tilde{G}(0, -\lambda\theta) = |c + b\theta\lambda| + \frac{1}{2}\theta\lambda^2 > \frac{1}{2}\theta\lambda^2 = \tilde{G}(0, \frac{c}{b})$.

(iv) $\tilde{G}(-\lambda\theta, -\lambda\theta) = a\theta\lambda + b\theta\lambda + c + \theta\lambda^2 \ge \theta\lambda^2$.

(v) $\tilde{G}_8(x)' = 0$时, 得到驻点$((a-\lambda)\theta, (b-\lambda)\theta)$.

所以

A. 若$\theta\lambda^2 > -2c$, $(0,0)$为极小值点, 但$(0,0)$ 不在此区域.

B. 若$\theta\lambda^2 < -2c$, $(0, \frac{c}{b}), (\frac{c}{a}, 0)$为极小值点, 但都不在此区域.

C. 若$\theta\lambda^2 = -2c$,$(0, \frac{c}{b}), (\frac{c}{a}, 0), (0,0), (\frac{c}{a+b}, \frac{c}{a+b})$ 为极小值点, 但$(0, \frac{c}{b}), (\frac{c}{a}, 0), (0,0)$ 不在此区域, 故该区域只有点$(\frac{c}{a+b}, \frac{c}{a+b})$存在极小值, 其值为$\theta\lambda^2$.

D. 根据引理2.1, $A = C = -\frac{1}{\theta} < 0, B = 0, H = \frac{1}{\theta^2} > 0$, 从而$\tilde{G}_8((a-\lambda)\theta, (b-\lambda)\theta)$, 是该区域的极大值点, 故不是该区域极小值点.

9^0 $-\lambda\theta \le x_1 < 0, -\lambda\theta \le x_2 < 0$, 且当$\lambda\theta < \frac{-c}{a+b}$时, $ax_1 + bx_2 < c$,

$$\tilde{G}_9(x) = c - ax_1 - bx_2 - \lambda x_1 - \frac{x_1^2}{2\theta} - \lambda x_2 - \frac{x_2^2}{2\theta}$$

由于

(i) $\tilde{G}(0,0) = c$.

(ii) $\tilde{G}(-\lambda\theta, -\lambda\theta) = c + a\theta\lambda + b\theta\lambda + \theta\lambda^2 > \theta\lambda^2$.

(iii) $\tilde{G}(-\lambda\theta, 0) = |c + a\theta\lambda| + \frac{1}{2}\theta\lambda^2 > \frac{1}{2}\theta\lambda^2 = \tilde{G}(\frac{c}{a}, 0)$.

(iv) $\tilde{G}(0, -\lambda\theta) = |c + b\theta\lambda| + \frac{1}{2}\theta\lambda^2 > \frac{1}{2}\theta\lambda^2 = \tilde{G}(0, \frac{c}{b})$.

(v) $\tilde{G}_9(x)' = 0$时, 得到驻点$((-a-\lambda)\theta, (-b-\lambda)\theta)$.

所以

A. 若$\theta\lambda^2 > 2c$时, $(0,0)$为极小值点, 但$(0,0)$不在此区域.

B. 若$\theta\lambda^2 = 2c$时, $(0,0)$,$(\frac{c}{a}, 0), (0, \frac{c}{b}), (\frac{c}{a+b}, \frac{c}{a+b})$ 为极小值点, 但不在此区域.

C. 若$\theta\lambda^2 < 2c$时, $(\frac{c}{a}, 0), (0, \frac{c}{b})$为极小值点, 但都不在此区域.

D. 根据引理2.1, $A = C = -\frac{1}{\theta} < 0, B = 0, H = \frac{1}{\theta^2} > 0$, 从而$\tilde{G}_9((-a-\lambda)\theta, (-b-\lambda)\theta)$, 是该区域的极大值点, 故不是该区间极小值点.

综合1^0-9^0, 在区域$S_2 = \{x_1 < 0, x_2 < 0\}$上存在最小值为$\theta\lambda^2$, 最优解为$x \in \{x \in \mathbb{R}^2 : ax_1 + bx_2 = c, x_1 < -\lambda\theta, x_2 < -\lambda\theta\} \cup \{(\frac{c}{a+b}, \frac{c}{a+b})\}$,

10^0 $x_1 < -\lambda\theta, x_2 > \lambda\theta$且$ax_1 + bx_2 > c$, 故

$$\tilde{G}_{10}(x) = ax_1 + bx_2 - c + \theta\lambda^2,$$

由于$\tilde{G}(x) > \theta\lambda^2$, 故在此区域内无最小值.

11^0 $x_1 < -\lambda\theta, x_2 > \lambda\theta$且$ax_1 + bx_2 < c$, 故

$$\tilde{G}_{11}(x) = c - ax_1 - bx_2 + \theta\lambda^2,$$

由于$\tilde{G}(x) > \theta\lambda^2$, 故在此区域内无最小值.

12^0 $x_1 < -\lambda\theta, x_2 > \lambda\theta$且$ax_1 + bx_2 = c$, 故

$$\tilde{G}_{12}(x) = \theta\lambda^2.$$

由于$\tilde{G}_{12}(x)$为常数, 故极小值依然为: $\tilde{G}(x) = \theta\lambda^2$

综合10^0-12^0, 在区域$S_3 = \{x_1 < -\lambda\theta, x_2 > \lambda\theta\}$上存在最小值为$\theta\lambda^2$, 最小点为$x \in \{x \in \mathbb{R}^2 : ax_1 + bx_2 = c, x_1 < -\lambda\theta, x_2 > \lambda\theta\}$.

13^0 $x_1 < -\lambda\theta, 0 \le x_2 < \lambda\theta$, 且当$a\lambda\theta < -c$时, $ax_1 + bx_2 < c$, 故

$$\tilde{G}_{13}(x) = c - ax_1 - bx_2 + \frac{1}{2}\theta\lambda^2 + \lambda x_2 - \frac{x_2^2}{2\theta}$$

由于

(i) $\tilde{G}(-\lambda\theta, 0) = c + a\theta\lambda + \frac{1}{2}\theta\lambda^2 > \frac{1}{2}\theta\lambda^2 = \tilde{G}(\frac{c}{a}, 0)$.

(ii) $\tilde{G}(-\lambda\theta, \lambda\theta) = |-a\theta\lambda + b\theta\lambda - c| + \theta\lambda^2 > \theta\lambda^2 > \tilde{G}(\frac{c}{a}, 0)$.

且$(\frac{c}{a}, 0)$不在该区域, 也无其他极小值点, 故在该区域无最优值.

14^0 $x_1 < -\lambda\theta, 0 \le x_2 < \lambda\theta$, 且当$a\lambda\theta \ge -c$时, $ax_1 + bx_2 \ge c$, 故

$$\tilde{G}_{14}(x) = ax_1 + bx_2 - c + \frac{1}{2}\theta\lambda^2 + \lambda x_2 - \frac{x_2^2}{2\theta}$$

由于

(i) $\tilde{G}(-\lambda\theta,0)=-c-a\theta\lambda+\frac{1}{2}\theta\lambda^2>\frac{1}{2}\theta\lambda^2=\tilde{G}(\frac{c}{a},0)$.

(ii) $\tilde{G}(-\lambda\theta,\lambda\theta)=|-a\theta\lambda+b\theta\lambda-c|+\theta\lambda^2>\theta\lambda^2>\tilde{G}(\frac{c}{a},0)$.

且$(\frac{c}{a},0)$不在该区域, 也无其他极小值点, 故在该区域无最优值.

15^0 $-\lambda\theta\le x_1<0, x_2>\lambda\theta$且当$b\lambda\theta>c$时, $ax_1+bx_2>c$, 故

$$\tilde{G}_{15}(x_1,x_2)=ax_1+bx_2-c-\lambda x_1-\frac{x_1^2}{2\theta}+\frac{1}{2}\theta\lambda^2$$

由于

(1) $\tilde{G}(0,\lambda\theta)=b\lambda\theta-c+\frac{1}{2}\theta\lambda^2>\frac{1}{2}\theta\lambda^2=\tilde{G}(0,\frac{c}{b})$.

(2) $\tilde{G}(-\lambda\theta,\lambda\theta)=|-a\theta\lambda+b\theta\lambda-c|+\theta\lambda^2>\theta\lambda^2>\tilde{G}(0,\frac{c}{b})$.

且$(0,\frac{c}{b})$不在此区域, 也无其他极小值点, 故不是该区域的最优解.

16^0 $-\lambda\theta\le x_1<0, x_2>\lambda\theta$且当$b\lambda\theta\le c$时, $ax_1+bx_2\le c$, 故

$$\tilde{G}(x_1,x_2)=c-ax_1-bx_2-\lambda x_1-\frac{x_1^2}{2\theta}+\frac{1}{2}\theta\lambda^2$$

由于

(1) $\tilde{G}(0,\lambda\theta)=-b\lambda\theta+c+\frac{1}{2}\theta\lambda^2>\frac{1}{2}\theta\lambda^2=\tilde{G}(0,\frac{c}{b})$.

(2) $\tilde{G}(-\lambda\theta,\lambda\theta)=|-a\theta\lambda+b\theta\lambda-c|+\theta\lambda^2>\theta\lambda^2>\tilde{G}(0,\frac{c}{b})$.

且$(0,\frac{c}{b})$不在此区域, 也无其他极小值点, 故不是该区域的最优解.

17^0 $-\lambda\theta\le x_1<0, 0\le x_2\le\lambda\theta$且当$\lambda\theta<\frac{c}{b}$或$\lambda\theta>\frac{-c}{a}$时, $ax_1+bx_2<c$, 故

$$\tilde{G}_{17}(x)=c-ax_1-bx_2-\lambda x_1-\frac{x_1^2}{2\theta}+\lambda x_2-\frac{x_2^2}{2\theta}$$

由于

(i) $\tilde{G}(0,0)=c$.

(ii) $\tilde{G}(-\lambda\theta,0)=c+a\theta\lambda+\frac{1}{2}\theta\lambda^2\ge\frac{1}{2}\theta\lambda^2=\tilde{G}(\frac{c}{a},0)$.

(iii) $\tilde{G}(0,\lambda\theta)=c-b\theta\lambda+\frac{1}{2}\theta\lambda^2>\frac{1}{2}\theta\lambda^2=\tilde{G}(0,\frac{c}{b})$.

(iv) $\tilde{G}(-\lambda\theta,\lambda\theta)=|-a\theta\lambda+b\theta\lambda-c|+\theta\lambda^2\ge\tilde{G}(\frac{c}{a+b},\frac{c}{a+b})=\theta\lambda^2>\tilde{G}(\frac{c}{a},0)$.

(v) $\tilde{G}(x)'=0$时, 得到稳定点$((-a-\lambda)\theta,(-b+\lambda)\theta)$.

所以

A. 当$\theta\lambda^2>2c$时, 又因为$(0,0)$不在此区域, 故无该点的极小值.

B. 当$\theta\lambda^2=2c$时, 由于$(0,\frac{c}{b}),(\frac{c}{a},0),(0,0),(-\frac{c}{a},\frac{c}{b})$不在此区域, 故此时无极小值.

C. 当$\theta\lambda^2 < 2c$时, 由于$(0,\frac{c}{b}),(\frac{c}{a},0)$不在此区域, 故此时无极小值.

D. 恒有$\tilde{G}((-a-\lambda)\theta,(-b+\lambda)\theta) > \tilde{G}(-\lambda\theta,\lambda\theta),\tilde{G}(-\lambda\theta,0),\tilde{G}(0,0),\tilde{G}(0,\lambda\theta)$ 故不是该区域极小值点.

18^0 $-\lambda\theta \le x_1 < 0, 0 \le x_2 \le \lambda\theta$且当$\lambda\theta \ge \frac{c}{b}$或$\lambda\theta \le \frac{-c}{a}$ 时, $ax_1+bx_2 \ge c$, 故

$$\tilde{G}_{18}(x) = ax_1+bx_2-c-\lambda x_1-\frac{x_1^2}{2\theta}+\lambda x_2-\frac{x_2^2}{2\theta}$$

由于

(i) $\tilde{G}(0,0) = -c$.

(ii) $\tilde{G}(-\lambda\theta,0) = -c-a\theta\lambda+\frac{1}{2}\theta\lambda^2 \ge \frac{1}{2}\theta\lambda^2 = \tilde{G}(\frac{c}{a},0)$.

(iii) $\tilde{G}(0,\lambda\theta) = -c+b\theta\lambda+\frac{1}{2}\theta\lambda^2 > \frac{1}{2}\theta\lambda^2 = \tilde{G}(0,\frac{c}{b})$.

(iv) $\tilde{G}(-\lambda\theta,\lambda\theta) = |-a\theta\lambda+b\theta\lambda-c|+\theta\lambda^2 \ge \tilde{G}(\frac{c}{a+b},\frac{c}{a+b}) = \theta\lambda^2 > \tilde{G}(\frac{c}{a},0)$.

(v) $\tilde{G}_{18}(x)' = 0$时, 得到稳定点$((a-\lambda)\theta,(b+\lambda)\theta)$.

所以

A. 当$\theta\lambda^2 > -2c$时, 因为$(0,0)$不在此区域, 故无该点的极小值.

B. 当$\theta\lambda^2 = -2c$时, 因为$(\frac{c}{a},0),(0,\frac{c}{b}),(-\frac{c}{a},\frac{c}{b})$ 不在此区域, 故无该点的极小值.

C. 当$\theta\lambda^2 < -2c$时, 因为$(0,\frac{c}{b})$不在此区域, 故只有点$(\frac{c}{a},0)$处存在极小值, 所以$(\frac{c}{a},0)$是该区域的最小值.

D. 根据引理2.1, $A = C = -\frac{1}{\theta} < 0, B = 0, H = \frac{1}{\theta^2} > 0$, 从而$\tilde{G}_{18}((a-\lambda)\theta,(b+\lambda)\theta)$, 是该区域的极大值点, 故不是该区间极小值点.

综合13^0-18^0, 在区域$S_4 = \{x_1 < 0, \lambda\theta \ge x_2\}$上存在最小值为$\frac{1}{2}\theta\lambda^2$, 最优解为$x = (\frac{c}{a},0)$.

19^0 $x_1 > \lambda\theta, x_2 < -\lambda\theta, ax_1+bx_2 < c$, 故

$$\tilde{G}_{19}(x_1,x_2) = c-ax_1-bx_2+\theta\lambda^2,$$

由于$\tilde{G}(x) > \theta\lambda^2$, 故无极小值.

20^0 $x_1 > \lambda\theta, x_2 < -\lambda\theta,\ ax_1+bx_2 > c$, 故

$$\tilde{G}_{20}(x) = ax_1+bx_2-c+\theta\lambda^2,$$

由于$\tilde{G}(x) > \theta\lambda^2$, 故无极小值.

21^0 $x_1 > \lambda\theta, x_2 < -\lambda\theta,\ ax_1+bx_2 = c$, 故

$$\tilde{G}_{21}(x) = \theta\lambda^2,$$

由于$\tilde{G}_{21}(x)$是常数, 故极小值在$ax_1+bx_2=c$处取得, 值为: $\theta\lambda^2$.

综合19^0-21^0, 在区域$S_5=\{x_1>\lambda\theta, x_2<-\lambda\theta\}$上存在最小值$\theta\lambda^2$, 最优解为$x\in\{x\in\mathbb{R}^2: ax_1+bx_2=c, x_1>\lambda\theta, x_2<-\lambda\theta\}$.

22^0 $0\le x_1\le\lambda\theta, x_2<-\lambda\theta$且当$b\lambda\theta\ge -c$时, $ax_1+bx_2\le c$, 故

$$\tilde{G}_{22}(x)=c-ax_1-bx_2+\lambda x_1-\frac{x_1^2}{2\theta}+\frac{1}{2}\theta\lambda^2,$$

由于

(i) $\tilde{G}(0,-\lambda\theta)=b\lambda\theta+c+\frac{1}{2}\theta\lambda^2>\frac{1}{2}\theta\lambda^2=\tilde{G}(0,\frac{c}{b})$.

(ii) $\tilde{G}(\lambda\theta,-\lambda\theta)=|a\theta\lambda-b\theta\lambda-c|+\theta\lambda^2>\theta\lambda^2>\tilde{G}(0,\frac{c}{b})$.

且$(0,\frac{c}{b})$不在此区域, 故无极小值.

23^0 $0\le x_1\le\lambda\theta, x_2<-\lambda\theta$且当$b\lambda\theta<-c$时, $ax_1+bx_2>c$, 故

$$\tilde{G}_{23}(x_1,x_2)=ax_1+bx_2-c+\lambda x_1-\frac{x_1^2}{2\theta}+\frac{1}{2}\theta\lambda^2$$

由于

(i) $\tilde{G}(0,-\lambda\theta)=-b\lambda\theta-c+\frac{1}{2}\theta\lambda^2>\frac{1}{2}\theta\lambda^2=\tilde{G}(0,\frac{c}{b})$.

(ii) $\tilde{G}(\lambda\theta,-\lambda\theta)=|a\theta\lambda-b\theta\lambda-c|+\theta\lambda^2>\theta\lambda^2>\tilde{G}(0,\frac{c}{b})$.

且$(0,\frac{c}{b})$不在此区域, 也无其他极小值点, 故不是该区域的最优解.

24^0 $x_1>\lambda\theta, -\lambda\theta\le x_2<0$且当$a\lambda\theta\ge c$时, $ax_1+bx_2\ge c$, 故

$$\tilde{G}_{24}(x_1,x_2)=ax_1+bx_2-c-\lambda x_2-\frac{x_2^2}{2\theta}+\frac{1}{2}\theta\lambda^2$$

由于

(i) $\tilde{G}(\lambda\theta,0)=a\lambda\theta-c+\frac{1}{2}\theta\lambda^2>\frac{1}{2}\theta\lambda^2=\tilde{G}(\frac{c}{a},0)$.

(ii) $\tilde{G}(\lambda\theta,-\lambda\theta)=|a\lambda\theta-b\lambda\theta-c|+\theta\lambda^2>\theta\lambda^2>\tilde{G}(\frac{c}{a},0)$.

但$(\frac{c}{a},0)$不在此区域, 也无其他极小值点, 故不是该区域的最优解.

25^0 $x_1>\lambda\theta, -\lambda\theta\le x_2<0$且当$a\lambda\theta<c$时, $ax_1+bx_2<c$, 故

$$\tilde{G}_{25}(x_1,x_2)=c-ax_1-bx_2-\lambda x_2-\frac{x_2^2}{2\theta}+\frac{1}{2}\theta\lambda^2$$

由于

(i) $\tilde{G}(\lambda\theta,0)=c-a\lambda\theta+\frac{1}{2}\theta\lambda^2>\frac{1}{2}\theta\lambda^2=\tilde{G}(\frac{c}{a},0)$.

(ii) $\tilde{G}(\lambda\theta, -\lambda\theta) = |a\lambda\theta - b\lambda\theta - c| + \theta\lambda^2 > \theta\lambda^2 > \tilde{G}(\frac{c}{a}, 0)$.

但$(\frac{c}{a}, 0)$不在此区域, 也无其他极小值点, 故不是该区域的最优解.

26^0 $0 \le x_1 \le \lambda\theta, -\lambda\theta \le x_2 < 0$ 当且$\lambda\theta \le \frac{c}{a}$或$\lambda\theta \ge \frac{c}{-b}$时, $ax_1 + bx_2 \le c$, 故

$$\tilde{G}_{26}(x_1, x_2) = c - ax_1 - bx_2 + \lambda x_1 - \frac{x_1^2}{2\theta} - \lambda x_2 - \frac{x_2^2}{2\theta}$$

由于

(i) $\tilde{G}(0, 0) = c$.

(ii) $\tilde{G}(0, -\lambda\theta) = c + b\lambda\theta + \frac{1}{2}\theta\lambda^2 \ge \frac{1}{2}\theta\lambda^2 = \tilde{G}(0, \frac{c}{b})$.

(iii) $\tilde{G}(\lambda\theta, 0) = c - a\lambda\theta + \frac{1}{2}\theta\lambda^2 \ge \frac{1}{2}\theta\lambda^2 = \tilde{G}(\frac{c}{a}, 0)$.

(iv) $\tilde{G}(\lambda\theta, -\lambda\theta) = |a\theta\lambda - b\theta\lambda - c| + \theta\lambda^2 \ge \theta\lambda^2 > \tilde{G}(0, \frac{c}{b})$.

(v) $\tilde{G}_{26}(x)' = 0$时, 得到稳定点$((\lambda - a)\theta, (-b - \lambda)\theta)$.

所以

A. 当$\frac{1}{2}\theta\lambda^2 > c$时, 由于$(0, 0)$点不在该区域, 故无该点的极小值.

B. 当$\frac{1}{2}\theta\lambda^2 = c$时, 只有$(0, 0)$点符合条件, 但$(0, 0)$点不在该区域, 故无该点的极小值.

C. 当$\frac{1}{2}\theta\lambda^2 < c$时, 由于点$(\frac{c}{a}, 0)$不在此区域, 故只在$(0, \frac{c}{b})$处存在极小值, 其值为$\frac{1}{2}\theta\lambda^2$.

D. 根据引理2.1, $A = C = -\frac{1}{\theta} < 0, B = 0, H = \frac{1}{\theta^2} > 0$, 从而$\tilde{G}_{26}((\lambda - a)\theta, (-b - \lambda)\theta)$, 是该区域的极大值点, 故不是该区间极小值点.

27^0 $0 \le x_1 \le \lambda\theta, -\lambda\theta \le x_2 < 0$ 当且$\lambda\theta > \frac{c}{a}$或者$\lambda\theta < \frac{c}{-b}$时, $ax_1 + bx_2 > c$, 故

$$\tilde{G_{27}}(x_1, x_2) = ax_1 + bx_2 - c + \lambda x_1 - \frac{x_1^2}{2\theta} - \lambda x_2 - \frac{x_2^2}{2\theta}$$

由于

(i) $\tilde{G}(0, 0) = -c$.

(ii) $\tilde{G}(0, -\lambda\theta) = -c - b\lambda\theta + \frac{1}{2}\theta\lambda^2 > \frac{1}{2}\theta\lambda^2 = \tilde{G}(0, \frac{c}{b})$.

(iii) $\tilde{G}(\lambda\theta, 0) = -c + a\lambda\theta + \frac{1}{2}\theta\lambda^2 > \frac{1}{2}\theta\lambda^2 = \tilde{G}(\frac{c}{a}, 0)$.

(iv) $\tilde{G}(\lambda\theta, -\lambda\theta) = |a\lambda\theta - b\lambda\theta - c| + \theta\lambda^2 > \theta\lambda^2 > \tilde{G}(0, \frac{c}{b})$.

(v) $\tilde{G}_{27}(x)' = 0$时, 得到稳定点$((\lambda + a)\theta, (b - \lambda)\theta)$.

所以

A. 当$\frac{1}{2}\theta\lambda^2 > -c$时, 由于$(0, 0)$点不在该区域, 故无该点的极小值.

B. 当$\frac{1}{2}\theta\lambda^2 = -c$时, 但在此区域无法取到$-c$值, 故在$(\frac{c}{a}, 0), (0, \frac{c}{b}), (\frac{c}{a}, -\frac{c}{b})$ 处无极值.

C. 当$\frac{1}{2}\theta\lambda^2 < -c$时, 由于点$(\frac{c}{a}, 0), (0, \frac{c}{b})$不在此区域, 故无极小值.

D. 根据引理2.1, $A = C = -\frac{1}{\theta} < 0, B = 0, H = \frac{1}{\theta^2} > 0$, 从而$\tilde{G}_{27}((\lambda + a)\theta, (b - \lambda)\theta)$, 是该区域的极大值点, 故不是该区间极小值点.

综合22^0-27^0, 在区域$S_6 = \{x_1 \geq 0, x_2 < 0\}$上存在最小值为$\frac{1}{2}\theta\lambda^2$, 最优解为$x = \{(0, \frac{c}{b})\}$.

28^0 $x_1 > \lambda\theta$, $x_2 > \lambda\theta$, 且$ax_1 + bx_2 > c$时,

$$\tilde{G}_{28}(x) = ax_1 + bx_2 - c + \theta\lambda^2$$

由于$\tilde{G}(x) > \theta\lambda^2$, 故无极小值.

29^0 $x_1 > \lambda\theta$, $x_2 > \lambda\theta$, 且当$ax_1 + bx_2 < c$时,

$$\tilde{G}_{29}(x) = c - ax_1 - bx_2 + \theta\lambda^2$$

由于$\tilde{G}(x) > \theta\lambda^2$, 故无极小值.

30^0 $x_1 > \lambda\theta$,$x_2 > \lambda\theta$, $ax_1 + bx_2 = c$, 故

$$\tilde{G}(x) = \theta\lambda^2$$

由于$\tilde{G}(x)$是常量, 极小值依然为: $\theta\lambda^2$.

综合28^0-30^0, 在区域$S_7 = \{x_1 > \lambda\theta, x_2 > \lambda\theta\}$上存在最小值为$\theta\lambda^2$, 最优解为$x \in \{ax_1 + bx_2 = c, x_1 > \lambda\theta, x_2 > \lambda\theta\}$.

31^0 $0 \leq x_1 \leq \lambda\theta, x_2 > \lambda\theta$且当$\lambda\theta \geq \frac{c}{b}$时, $ax_1 + bx_2 \geq c$, 故

$$\tilde{G}_{31}(x_1, x_2) = ax_1 + bx_2 - c + \lambda x_1 - \frac{x_1^2}{2\theta} + \frac{1}{2}\theta\lambda^2,$$

由于

(i) $\tilde{G}(0, \lambda\theta) = b\lambda\theta - c + \frac{1}{2}\theta\lambda^2 > \frac{1}{2}\theta\lambda^2 = \tilde{G}(0, \frac{c}{b})$.

(ii) $\tilde{G}(\lambda\theta, \lambda\theta) = |a\lambda\theta + b\lambda\theta - c| + \theta\lambda^2 > \theta\lambda^2 > \tilde{G}(0, \frac{c}{b})$.

且$(0, \frac{c}{b})$不在该区域, 故无极小值

32^0 $0 \leq x_1 \leq \lambda\theta, x_2 > \lambda\theta$且当$\lambda\theta < \frac{c}{b}$时, $ax_1 + bx_2 < c$, 故

$$\tilde{G}_{32}(x) = c - ax_1 - bx_2 + \lambda x_1 - \frac{x_1^2}{2\theta} + \frac{1}{2}\theta\lambda^2.$$

由于

(i) $\tilde{G}(0,\lambda\theta)=c-b\lambda\theta+\frac{1}{2}\theta\lambda^2\geq\frac{1}{2}\theta\lambda^2=\tilde{G}(0,\frac{c}{b})$.

(ii) $\tilde{G}(\lambda\theta,\lambda\theta)=|a\lambda\theta+b\lambda\theta-c|+\theta\lambda^2>\theta\lambda^2>\tilde{G}(0,\frac{c}{b})$.

故该区域存在极小值点$(0,\frac{c}{b})$, 极小值为: $\frac{1}{2}\theta\lambda^2$.

综合31^0-32^0, 在区域$S_8=\{0\leq x_1\leq\lambda\theta,x_2>\lambda\theta\}$上存在最小值为$\frac{1}{2}\theta\lambda^2$, 最优解为: $(0,\frac{c}{b})$.

33^0 $x_1>\lambda\theta,0\leq x_2\leq\lambda\theta$, 且当$\lambda\theta\geq\frac{c}{a}$时, $ax_1+bx_2\geq c$, 故

$$\tilde{G}_{33}(x_1,x_2)=ax_1+bx_2-c+\lambda x_2-\frac{x_2^2}{2\theta}+\frac{1}{2}\theta\lambda^2,$$

由于

(i) $\tilde{G}(\lambda\theta,0)=a\lambda\theta-c+\frac{1}{2}\theta\lambda^2>\frac{1}{2}\theta\lambda^2=\tilde{G}(\frac{c}{a},0)$.

(ii) $\tilde{G}(\lambda\theta,\lambda\theta)=|a\lambda\theta+b\lambda\theta-c|+\theta\lambda^2>\theta\lambda^2>\tilde{G}(\frac{c}{a},0)$.

且$(\frac{c}{a},0)$不在该区域, 故不是该区域的最优解.

34^0 $x_1>\lambda\theta,0\leq x_2\leq\lambda\theta$, 且当$\lambda\theta<\frac{c}{a}$时, $ax_1+bx_2<c$, 故

$$\tilde{G}_{34}(x)=c-ax_1-bx_2+\lambda x_2-\frac{x_2^2}{2\theta}+\frac{1}{2}\theta\lambda^2$$

由于

(i) $\tilde{G}(\lambda\theta,0)=c-a\lambda\theta+\frac{1}{2}\theta\lambda^2\geq\frac{1}{2}\theta\lambda^2=\tilde{G}(\frac{c}{a},0)$.

(ii) $\tilde{G}(\lambda\theta,\lambda\theta)=|a\lambda\theta+b\lambda\theta-c|+\theta\lambda^2>\theta\lambda^2>\tilde{G}(\frac{c}{a},0)$.

故存在点$(\frac{c}{a},0)$处的极小值, 值为: $\frac{1}{2}\theta\lambda^2$.

综合33^0-34^0, 在区域$S_9=\{x_1>\lambda\theta,0\leq x_2\leq\lambda\theta\}$上存在最小值为$\frac{1}{2}\theta\lambda^2$, 最优解为: $(\frac{c}{a},0)$.

35^0 $0\leq x_1\leq\lambda\theta,0\leq x_2\leq\lambda\theta$且当$\lambda\theta<\frac{c}{a+b}$, $ax_1+bx_2<c$, 故

$$\tilde{G}_{35}(x_1,x_2)=c-ax_1-bx_2+\lambda x_1-\frac{x_1^2}{2\theta}+\lambda x_2-\frac{x_2^2}{2\theta},$$

由于

(i) $\tilde{G}(0,0)=c$.

(ii) $\tilde{G}(\lambda\theta,0)=c-a\lambda\theta+\frac{1}{2}\theta\lambda^2\geq\frac{1}{2}\theta\lambda^2=\tilde{G}(\frac{c}{a},0)$.

(iii) $\tilde{G}(0,\lambda\theta)=c-b\lambda\theta+\frac{1}{2}\theta\lambda^2\geq\frac{1}{2}\theta\lambda^2=\tilde{G}(0,\frac{c}{b})$.

(iv) $\tilde{G}(\lambda\theta,\lambda\theta)=c-a\lambda\theta-b\lambda\theta+\theta\lambda^2\geq\theta\lambda^2>\frac{1}{2}\theta\lambda^2$.

(v) $\tilde{G}(x)'=0$时, 得到驻点$((-a+\lambda)\theta,(-b+\lambda)\theta)$.

所以

A. 当$\frac{1}{2}\theta\lambda^2 > c$时, 点: $(0,0)$为此区域极小值点, 极小值为c.

B. 当$\frac{1}{2}\theta\lambda^2 = c$时, 点: $(0,0),(0,\frac{c}{b}),(\frac{c}{a},0)$为此区域极小值点, 极小值为$c$.

C. 当$\frac{1}{2}\theta\lambda^2 < c$时, 点: $(0,\frac{c}{b}),(\frac{c}{a},0)$为此区域极小值点, 极小值为$\frac{1}{2}\theta\lambda^2$.

D. 根据引理2.1, $A=C=-\frac{1}{\theta}<0, B=0, H=\frac{1}{\theta^2}>0$, 从而$\tilde{G}_{35}((-a+\lambda)\theta,(-b+\lambda)\theta)$, 是该区域的极大值点, 故不是该区间极小值点.

36^0 $0\le x_1\le\lambda\theta, 0\le x_2\le\lambda\theta$且当$\lambda\theta>\frac{c}{a+b}$时, $ax_1+bx_2>c$, 故

$$\tilde{G}_{36}(x_1,x_2)=ax_1+bx_2-c+\lambda x_1-\frac{x_1^2}{2\theta}+\lambda x_2-\frac{x_2^2}{2\theta},$$

由于

(i) $\tilde{G}(0,0)=-c$.

(ii) $\tilde{G}(\lambda\theta,0)=a\lambda\theta-c+\frac{1}{2}\theta\lambda^2\ge\frac{1}{2}\theta\lambda^2=\tilde{G}(\frac{c}{a},0)$.

(iii) $\tilde{G}(0,\lambda\theta)=b\lambda\theta-c+\frac{1}{2}\theta\lambda^2\ge\frac{1}{2}\theta\lambda^2=\tilde{G}(0,\frac{c}{b})$.

(iv) $\tilde{G}(\lambda\theta,\lambda\theta)=a\lambda\theta+b\lambda\theta-c+\theta\lambda^2\ge\theta\lambda^2>\frac{1}{2}\theta\lambda^2$.

(v) $\tilde{G}(x)'=0$时, 得到驻点$((a+\lambda)\theta,(b+\lambda)\theta)$.

所以

A. 当$\frac{1}{2}\theta\lambda^2 > -c$时, 点: $(0,0)$为此区域极小值点, 极小值为$-c$.

B. 当$\frac{1}{2}\theta\lambda^2 = -c$时, 点: $(0,0),(0,\frac{c}{b}),(\frac{c}{a},0)$为此区域极小值点, 极小值为$-c$.

C. 当$\frac{1}{2}\theta\lambda^2 < -c$时, 点: $(0,\frac{c}{b}),(\frac{c}{a},0)$为此区域极小值点, 极小值为$\frac{1}{2}\theta\lambda^2$.

D. 根据引理2.1, $A=C=-\frac{1}{\theta}<0, B=0, H=\frac{1}{\theta^2}>0$, 从而$\tilde{G}_{36}((a+\lambda)\theta,(b+\lambda)\theta)$, 是该区域的极大值点, 故不是该区间极小值点.

综上35^0-36^0,在区域$S_5=\{0\le x_1\le\lambda\theta, 0\le x_2\le\lambda\theta\}$上存在最优解:

$$x=\begin{cases}(0,0), & \frac{1}{2}\theta\lambda^2>|c|,\\ (0,0),(\frac{c}{a},0)\text{或}(0,\frac{c}{b}), & \frac{1}{2}\theta\lambda^2=|c|,\\ (\frac{c}{a},0)\text{或}(0,\frac{c}{b}), & \frac{1}{2}\theta\lambda^2<|c|,\end{cases}$$

综上1^0-36^0, 且由于$\theta\lambda=2$, 故全局最优解为:

$$x^*=\begin{cases}(0,0), & \lambda>|c|,\\ (0,0),(\frac{c}{a},0)\text{或}(0,\frac{c}{b}), & \lambda=|c|,\\ (\frac{c}{a},0)\text{或}(0,\frac{c}{b}), & \lambda<|c|.\end{cases}$$

全局最优值为:

$$\min \tilde{G}(x) = \begin{cases} |c|, & x_1 = 0, x_2 = 0 \\ \lambda, & x_1 = \frac{c}{a}, x_2 = 0 \text{或} x_1 = 0, x_2 = \frac{c}{b}, \end{cases}$$

即证明了问题(2)和(3)具有相同的全局最优解和最优值. □

3. 总结

本文在二维情况下探讨了由最小一乘问题的ℓ_0惩罚和MCP惩罚得到的两个含参绝对值优化问题全局最优解的关系. 在一定条件下, 证明了这两个问题具有相同的全局最优解和最优值. 这为进一步研究相应的高维问题提供了启发和借鉴.

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