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基于毕达哥拉斯三角模糊语言Sugeno-Weber集成算子的COPRAS多属性决策方法

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摘　要

针对属性权重未知且属性值为毕达哥拉斯三角模糊语言数的多属性决策问题，研究毕达哥拉斯三角模糊语言集的集成算子，考虑Sugeno-Weber范数在信息集成中的优势，提出一种基于毕达哥拉斯三角模糊语言Sugeno-Weber集成算子的COPRAS (COmplex PRoportional ASsessment)决策方法。首先，定义毕达哥拉斯三角模糊语言集的Sugeno-Weber运算法则，提出几种新型集成算子并讨论其性质。其次，建立基于离差最大化的权重模型确定属性权重。最后，基于所提算子构建改进的COPRAS方法并通过案例分析验证其有效性、实用性和可行性。所提出的毕达哥拉斯三角模糊语言Sugeno-Weber集成算子丰富了其集成算子理论。

关键词

多属性群决策，毕达哥拉斯三角模糊语言集，Sugeno-Weber，COPRAS方法

A COPRAS Approach for Multi-Attribute Decision-Making Based on Pythagorean Triangular Fuzzy Linguistic Sugeno-Weber Aggregation Operators

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Abstract

This study addresses multi-attribute decision-making problems where attribute weights are unknown and attribute values are expressed as Pythagorean Triangular fuzzy linguistic numbers. Focusing on aggregation operators for Pythagorean Triangular fuzzy linguistic sets, and leveraging the advantages of the Sugeno-Weber norms in information aggregation, we propose an improved COPRAS (COmplex PRoportional ASsessment) decision-making method based on novel Pythagorean Triangular fuzzy linguistic Sugeno-Weber aggregation operators. Firstly, the Sugeno-Weber operational laws for Pythagorean Triangular fuzzy linguistic sets are defined, and several new aggregation operators are proposed, along with a discussion of their properties. Secondly, a weight determination model based on maximum deviation is established to ascertain attribute weights. Finally, an improved COPRAS method is constructed utilizing the proposed operators, and its effectiveness, practicability, and feasibility are validated through a case study. The proposed Pythagorean Triangular fuzzy linguistic set Sugeno-Weber aggregation operators enrich the theory of aggregation operators.

Keywords

Multi-Attribute Decision-Making, Pythagorean Triangular Fuzzy Linguistic Sets, Sugeno-Weber, COPRAS Approach

1. 引言

鉴于客观事物的复杂性，模糊性和决策者认知能力的不确定性，以实数为背景的评价信息难以处理现实具有模糊特征的决策问题。基于此，Zadeh [1]于 1965 年创始性地提出了模糊集理论并广泛应用于处理不确定多属性决策问题。Atanassov [2]引入非隶属度和犹豫度的概念提出直觉模糊集理论且其隶属度和非隶属度和小于或等于 1。此后，基于直觉模糊集理论的扩展形式，如区间值直觉模糊集[3]，语言直觉模糊集[4]和区间直觉不确定语言集[5]等不确定信息表示模型被提出以丰富不确定信息表示模型理论。

实际决策环境的复杂性使得专家提供的评价信息往往不能满足直觉模糊集的限制条件，出现隶属度和非隶属度和大于 1 的情形。基于此，Yager [6]提出了毕达哥拉斯模糊集理论，其限制条件扩充为隶属度和非隶属度的平方和小于或等于 1。作为直觉模糊集的扩展形式，毕达哥拉斯模糊集为决策者表征模糊信息提供了更加广阔的表达空间。自其被提出以来，毕达哥拉斯模糊集在诸多领域，如决策分析[7] [8]、模糊集理论扩展[9] [10]、集成算子理论[11] [12]及实际问题分析[13] [14]。此后，Du 等[15]提出了毕达哥拉斯三角模糊语言集的概念并提出了基于 Hamacher 范数的集成算子并构建决策方法。Jing 等[16]提出了毕达哥拉斯三角模糊语言 Bonferroni Mean 算子并构建决策模型考虑属性间相互关系的决策问题。

Sugeno-Weber 模[17]是一种新型的三角模并被诸多学者拓展到模糊环境构建其基本运算[18]-[20]。Petchimuthu 等[21]提出了复值广义正交图模糊环境下的 Sugeno-Weber 算子并构建温室气体减排战略评

价模型。Rani 等[22]提出了 Fermatean 模糊 Sugeno-Weber 算子和距离测度并构建太阳能电池板选择框架。Wang 等[23]提出了基于广义正交模糊 Sugeno-Weber 幂算子的太阳能电池板选型方法。Ashraf 等[24]提出基于球形模糊 Z 数 Sugeno-Weber 算子的温室效应气候变化多准则评价模型。目前，尚未有学者研究基于 Sugeno-Weber 范数的毕达哥拉斯三角模糊语言集成算子。

基于效用理论的 COPRAS 方法是由 Zavadskas 和 Kaklauskas 提出的一种常用的多属性决策方法，其核心思想是同时通过效益型属性和成本型属性的综合评估值来评估备选方案的相对效用和优先级排序。COPRAS 方法的优势是同时考虑效益属性和成本型属性，无需对属性进行归一化处理且计算过程简单、易于操作。目前已被学者们广泛应用于选址、冷链物流服务质量评价、数字化转型等决策问题中[25] [26]。Liu 等[27]提出基于区间值犹豫费马模糊集 COPRAS 方法的海水淡化技术选择模型。Gao 等[28]提出基于球形模糊 COPRAS 方法的数字供应链合作伙伴选择框架。目前尚未发现 COPRAS 方法在毕达哥拉斯三角模糊语言集中的研究。

鉴于毕达哥拉斯三角模糊语言集和 Sugeno-Weber 算子在不确定信息表示和集成算子研究中的优势。本文将建立基于 Sugeno-Weber 范数的毕达哥拉斯三角模糊语言数运算法则并提出毕达哥拉斯三角模糊语言 Sugeno-Weber 加权平均算子、有序加权平均算子、加权几何算子和有序加权几何算子，并讨论了上述算子的幂等性、单调性和有界性等性质。其次，基于毕达哥拉斯三角模糊语言集的距离测度，提出了改进的离差最大法确定属性权重信息。最后，提出基于所提算子的 COPRAS 多属性决策方法。

2. 预备知识

定义 1 [15] 设 Y 为一个非空经典集合，则毕达哥拉斯三角模糊语言集 Q 表示为：

$$Q=\left\{\left\langle y,\left[s_{\alpha(y)},s_{\beta(y)},s_{\chi(y)}\right],\left(\delta(y),\varepsilon(y)\right)\right\rangle \middle| y\in Y\right\}, \tag{1}$$

其中 $S=\{s_0,s_1,\cdots,s_l\}$ 为有限的语言术语集。$\delta(y)\in[0,1]$，$\varepsilon(y)\in[0,1]$，$\delta(y)$ 和 $\varepsilon(y)$ 分别表示 y 属于三角模糊语言 $\left[s_{\alpha(y)},s_{\beta(y)},s_{\chi(y)}\right]$ 的隶属度与非隶属度且满足 $0\le(\delta(y))^2+(\varepsilon(y))^2\le 1$。犹豫度 $\pi(y)$ 定义为 $\pi(y)=\sqrt{1-(\delta(y))^2-(\varepsilon(y))^2}$。为简便起见，毕达哥拉斯三角模糊语言数(PyTrFLN)可记作 $q=\left\langle\left[s_{\alpha_q},s_{\beta_q},s_{\chi_q}\right],\left(\delta_q,\varepsilon_q\right)\right\rangle$。

定义 2 [15] 设 $q_j=\left\langle\left[s_{\alpha_{q_j}},s_{\beta_{q_j}},s_{\chi_{q_j}}\right],\left(\delta_{q_j},\varepsilon_{q_j}\right)\right\rangle (j=1,2)$ 为两个 PyTrFLNs。则

(1) $q_1\oplus q_2=\left\langle\left[s_{\alpha_{q1}+\alpha_{q2}},s_{\beta_{q1}+\beta_{q2}},s_{\chi_{q1}+\chi_{q2}}\right],\left(\sqrt{(\delta_{q_1})^2+(\delta_{q_2})^2-(\delta_{q_1})^2(\delta_{q_2})^2},\varepsilon_{q_1}\varepsilon_{q_2}\right)\right\rangle$；

(2) $q_1\otimes q_2=\left\langle\left[s_{\alpha_{q1}\times\alpha_{q2}},s_{\beta_{q1}\times\beta_{q2}},s_{\chi_{q1}\times\chi_{q2}}\right],\left(\delta_{q_1}\delta_{q_2},\sqrt{(\varepsilon_{q_1})^2+(\varepsilon_{q_2})^2-(\varepsilon_{q_1})^2(\varepsilon_{q_2})^2}\right)\right\rangle$；

(3) $\lambda q_j=\left\langle\left[s_{\lambda\times\alpha_{q_j}},s_{\lambda\times\beta_{q_j}},s_{\lambda\times\chi_{q_j}}\right],\left(\sqrt{1-\left(1-(\delta_{q_j})^2\right)^\lambda},(\varepsilon_{q_j})^\lambda\right)\right\rangle,\lambda>0;$

(4) $(q_j)^\lambda=\left\langle\left[s_{(\alpha_{q_j})^\lambda},s_{(\beta_{q_j})^\lambda},s_{(\chi_{q_j})^\lambda}\right],\left((\delta_{q_j})^\lambda,\sqrt{1-\left(1-(\varepsilon_{q_j})^2\right)^\lambda},\right)\right\rangle,\lambda>0.$

定义 3 [15] 设 $q_j=\left\langle\left[s_{\alpha_{q_j}},s_{\beta_{q_j}},s_{\chi_{q_j}}\right],\left(\delta_{q_j},\varepsilon_{q_j}\right)\right\rangle (j=1,2)$ 为两个 PyTrFLNs。若 $SF(q_1)>SF(q_2)$，则 $q_1\succ q_2$；若 $SF(q_1)=SF(q_2)$，若 $AF(q_1)>AF(q_2)$，则 $q_1\succ q_2$；若 $SF(q_1)=SF(q_2)$，若 $AF(q_1)<AF(q_2)$，则 $q_1\prec q_2$；

若 $SF(q_1)=SF(q_2)$，若 $AF(q_1)=AF(q_2)$，则 $q_1 \sim q_2$。其中 $SF(q_j)=\left(\left(\left(\delta_{q_j}\right)^2-\left(\varepsilon_{q_j}\right)^2\right)\left(\alpha_{q_j}+\beta_{q_j}+\chi_{q_j}\right)\right)\Big/3$，$AF(q_j)=\left(\left(\left(\delta_{q_j}\right)^2+\left(\varepsilon_{q_j}\right)^2\right)\left(\alpha_{q_j}+\beta_{q_j}+\chi_{q_j}\right)\right)\Big/3$ 分别表示 q_j 的得分函数和精确函数。

定义 4 [15] 设 $q_j=\left\langle\left[s_{\alpha_{q_j}},s_{\beta_{q_j}},s_{\chi_{q_j}}\right],\left(\delta_{q_j},\varepsilon_{q_j}\right)\right\rangle(j=1,2)$ 为两个 PyTrFLNs，$S=\{s_0,s_1,\cdots,s_l\}$ 为有限的语言术语集。则这两个 PTrFLNs 之间的海明距离可定义为：

$$d(q_1,q_2)=\frac{1}{2(l-1)}\left(\left|\left(1+\left(\delta_{q_1}\right)^2-\left(\varepsilon_{q_1}\right)^2\right)\times\frac{\alpha_{q_1}+\beta_{q_1}+\chi_{q_1}}{3}-\left(1+\left(\delta_{q_2}\right)^2-\left(\varepsilon_{q_2}\right)^2\right)\times\frac{\alpha_{q_2}+\beta_{q_2}+\chi_{q_2}}{3}\right|\right) \tag{2}$$

定义 5 [17]. Sugeno-Weber T-范数 $\left(T_{sw}^{\aleph}\right)_{\aleph\in[0,\infty)}$ 和 S-范数 $\left(S_{sw}^{\aleph}\right)_{\aleph\in[0,\infty)}$ 定义如下：

$$T_{sw}^{\aleph}(a,b)=\begin{cases}T_D(a,b),\text{if } \aleph=-1\\ \max\left(0,\dfrac{a+b-1+\aleph ab}{1+\aleph}\right),\aleph\in(-1,+\infty),\\ T_P(a,b),\text{if } \aleph=+\infty\end{cases}\qquad S_{sw}^{\aleph}(a,b)=\begin{cases}S_D(a,b),\text{if } \aleph=-1\\ \min\left(1,a+b-\dfrac{\aleph}{1+\aleph}ab\right),\aleph\in(-1,+\infty)\\ S_P(a,b),\text{if } \aleph=+\infty\end{cases} \tag{3}$$

其中 $T_D(a,b)$ 和 $S_D(a,b)$ 表示 Drastic T-范数和 S-范数，$T_P(a,b)$ 和 $S_P(a,b)$ 表示和 Product T-范数和 S-范数。

3. 毕达哥拉斯三角模糊语言环境下的 Sugeno-Weber 算子

本节定义基于 Sugeno-Weber 的 PTrFLN 运算法则并提出相应的加权平均和几何算子。

3.1. 基于 Sugeno-Weber 模的毕达哥拉斯三角模糊语言数运算法则

定义 6 毕达哥拉斯模糊环境下的 Sugeno-Weber T-范数和 S-范数定义为：

$$T_{sw}^{\aleph}(a,b)=a\otimes_{sw}b=\sqrt{\frac{a^2+b^2-1+\aleph a^2b^2}{1+\aleph}},-1<\aleph<+\infty; \tag{4}$$

$$S_{sw}^{\aleph}(a,b)=a\oplus_{sw}b=\sqrt{a^2+b^2-\frac{\aleph}{1+\aleph}a^2b^2},-1<\aleph<+\infty \tag{5}$$

基于毕达哥拉斯模糊环境下的 Sugeno-Weber T-范数和 S-范数和毕达哥拉斯三角模糊语言的定义，则基于 Sugeno-Weber 模的毕达哥拉斯三角模糊语言数运算法则定义如下。

定义 7 设 $q_j=\left\langle\left[s_{\alpha_{q_j}},s_{\beta_{q_j}},s_{\chi_{q_j}}\right],\left(\delta_{q_j},\varepsilon_{q_j}\right)\right\rangle(j=1,2)$ 为两个 PyTrFLNs。则：

(1)

$$q_1\oplus_{sw}q_2=\left\langle\left[s_{\alpha_{q_1}+\alpha_{q_2}},s_{\beta_{q_1}+\beta_{q_2}},s_{\chi_{q_1}+\chi_{q_2}}\right],\left(\sqrt{\left(\delta_{q_1}\right)^2+\left(\delta_{q_2}\right)^2-\frac{\aleph}{1+\aleph}\left(\delta_{q_1}\right)^2\left(\delta_{q_2}\right)^2},\sqrt{\frac{\left(\varepsilon_{q_1}\right)^2+\left(\varepsilon_{q_2}\right)^2-1+\aleph\left(\varepsilon_{q_1}\right)^2\left(\varepsilon_{q_2}\right)^2}{1+\aleph}}\right)\right\rangle;$$

(2)

$$q_1\otimes_{sw}q_2=\left\langle\left[s_{\alpha_{q_1}\times\alpha_{q_2}},s_{\beta_{q_1}\times\beta_{q_2}},s_{\chi_{q_1}\times\chi_{q_2}}\right],\left(\sqrt{\frac{\left(\delta_{q_1}\right)^2+\left(\delta_{q_2}\right)^2-1+\aleph\left(\delta_{q_1}\right)^2\left(\delta_{q_2}\right)^2}{1+\aleph}},\sqrt{\left(\varepsilon_{q_1}\right)^2+\left(\varepsilon_{q_2}\right)^2-\frac{\aleph}{1+\aleph}\left(\varepsilon_{q_1}\right)^2\left(\varepsilon_{q_2}\right)^2}\right)\right\rangle;$$

(3)

$$\lambda q_j=\left\langle\left[s_{\lambda\times\alpha_{q_j}},s_{\lambda\times\beta_{q_j}},s_{\lambda\times\chi_{q_j}}\right],\left(\sqrt{\frac{1+\aleph}{\aleph}\left(1-\left(1-\left(\delta_{q_j}\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\lambda}\right)},\sqrt{\frac{1}{\aleph}\left((1+\aleph)\left(\frac{\aleph\left(\varepsilon_{q_j}\right)^2+1}{1+\aleph}\right)^{\lambda}-1\right)}\right)\right\rangle,\lambda>0;$$

(4)

$$\left(q_j\right)^{\lambda}=\left\langle\left[s_{\left(\alpha_{q_j}\right)^{\lambda}},s_{\left(\beta_{q_j}\right)^{\lambda}},s_{\left(\chi_{q_j}\right)^{\lambda}}\right],\left(\sqrt{\frac{1}{\aleph}\left((1+\aleph)\left(\frac{\aleph\left(\delta_{q_j}\right)^2+1}{1+\aleph}\right)^{\lambda}-1\right)},\sqrt{\frac{1+\aleph}{\aleph}\left(1-\left(1-\left(\varepsilon_{q_j}\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\lambda}\right)}\right)\right\rangle,\lambda>0.$$

基于毕达哥拉斯三角模糊语言 Sugeno-Weber 运算法则，下面定义毕达哥拉斯三角模糊语言环境下的 Sugeno-Weber 集成算子。

3.2. 毕达哥拉斯三角模糊语言环境下的 Sugeno-Weber 加权平均算子

定义 8 设 $q_j=\left\langle\left[s_{\alpha_{q_j}},s_{\beta_{q_j}},s_{\chi_{q_j}}\right],\left(\delta_{q_j},\varepsilon_{q_j}\right)\right\rangle(j=1,2,\cdots,n)$ 为一组 PyTrFLNs 集合。ϖ_j 是毕达哥拉斯三角模糊语言数 q_j 的权重且满足 $\sum_{j=1}^{n}\varpi_j=1$, $\varpi_j\in[0,1]$。则毕达哥拉斯三角模糊语言 Sugeno-Weber 加权平均(PyTrFLSWWA)算子 $\text{PyTrFLSWWA}:\Omega^n\to\Omega$ 定义如下：

$$\text{PyTrFLSWWA}\left(q_1,q_2,\cdots,q_n\right)=\bigoplus_{sw\,j=1}^{n}\varpi_j q_j \tag{6}$$

定理 1 设 $q_j=\left\langle\left[s_{\alpha_{q_j}},s_{\beta_{q_j}},s_{\chi_{q_j}}\right],\left(\delta_{q_j},\varepsilon_{q_j}\right)\right\rangle(j=1,2,\cdots,n)$ 为一组 PyTrFLNs 集合。则利用 PyTrFLSWWA 算子集成后的结果仍是 PyTrFLN 且

$$PyTrFLSWWA\left(q_1,q_2,\cdots,q_n\right)=\bigoplus_{sw\,j=1}^{n}\varpi_j q_j$$

$$=\left\langle\left[s_{\bigoplus_{sw\,j=1}^{n}\varpi_j\alpha_{q_j}},s_{\bigoplus_{sw\,j=1}^{n}\varpi_j\beta_{q_j}},s_{\bigoplus_{sw\,j=1}^{n}\varpi_j\chi_{q_j}}\right],\left(\sqrt{\frac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{n}\left(1-\left(\delta_{q_j}\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\varpi_j}\right)},\sqrt{\frac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{n}\left(\frac{\aleph\left(\varepsilon_{q_j}\right)^2+1}{1+\aleph}\right)^{\varpi_j}-1\right)}\right)\right\rangle. \tag{7}$$

证明： 定理 1 可通过数学归纳法证明。当 $n=2$ 时，

$$\text{PyTrFLSWWA}\left(q_1,q_2\right)=\bigoplus_{sw\,j=1}^{2}\varpi_j q_j=\varpi_1 q_1\oplus_{sw}\varpi_2 q_2$$

$$=\left\langle\left[s_{\bigoplus_{sw\,j=1}^{2}\varpi_j\alpha_{q_j}},s_{\bigoplus_{sw\,j=1}^{2}\varpi_j\beta_{q_j}},s_{\bigoplus_{sw\,j=1}^{2}\varpi_j\chi_{q_j}}\right],\left(\sqrt{\frac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{2}\left(1-\left(\delta_{q_j}\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\varpi_j}\right)},\sqrt{\frac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{2}\left(\frac{\aleph\left(\varepsilon_{q_j}\right)^2+1}{1+\aleph}\right)^{\varpi_j}-1\right)}\right)\right\rangle.$$

则当 $n=2$ 时，公式(7)成立。假设当 $n=\ell$，公式(7)成立。

$$\text{PyTrFLSWWA}(q_1,q_2,\cdots,q_\ell)=\oplus_{sw\,j=1}^{\ell}\varpi_j q_j$$

$$=\left\langle\left[s_{\oplus_{sw\,j=1}^{\ell}\varpi_j\alpha_{q_j}},s_{\oplus_{sw\,j=1}^{\ell}\varpi_j\beta_{q_j}},s_{\oplus_{sw\,j=1}^{\ell}\varpi_j\chi_{q_j}}\right],\left(\sqrt{\frac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{\ell}\left(1-\left(\delta_{q_j}\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\varpi_j}\right)},\sqrt{\frac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{\ell}\left(\frac{\aleph\left(\varepsilon_{q_j}\right)^2+1}{1+\aleph}\right)^{\varpi_j}-1\right)}\right)\right\rangle.$$

则当 $n=\ell+1$ 时，

$$\text{PyTrFLSWWA}(q_1,q_2,\cdots,q_\ell,q_{\ell+1})=\text{PyTrFLSWWA}(q_1,q_2,\cdots,q_\ell)\oplus_{sw}\varpi_{\ell+1}q_{\ell+1}$$

$$=\left\langle\left[s_{\oplus_{sw\,j=1}^{\ell}\varpi_j\alpha_{q_j}},s_{\oplus_{sw\,j=1}^{\ell}\varpi_j\beta_{q_j}},s_{\oplus_{sw\,j=1}^{\ell}\varpi_j\chi_{q_j}}\right],\left(\sqrt{\frac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{\ell}\left(1-\left(\delta_{q_j}\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\varpi_j}\right)},\sqrt{\frac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{\ell}\left(\frac{\aleph\left(\varepsilon_{q_j}\right)^2+1}{1+\aleph}\right)^{\varpi_j}-1\right)}\right)\right\rangle$$

$$\oplus_{sw}\left\langle\left[s_{\varpi_{\ell+1}\times\alpha_{q_{\ell+1}}},s_{\varpi_{\ell+1}\times\beta_{q_{\ell+1}}},s_{\varpi_{\ell+1}\times\chi_{q_{\ell+1}}}\right],\left(\sqrt{\frac{1+\aleph}{\aleph}\left(1-\left(1-\left(\delta_{q_{\ell+1}}\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\varpi_{\ell+1}}\right)},\sqrt{\frac{1}{\aleph}\left((1+\aleph)\left(\frac{\aleph\left(\varepsilon_{q_{\ell+1}}\right)^2+1}{1+\aleph}\right)^{\varpi_{\ell+1}}-1\right)}\right)\right\rangle$$

$$=\left\langle\left[s_{\oplus_{sw\,j=1}^{\ell+1}\varpi_j\alpha_{q_j}},s_{\oplus_{sw\,j=1}^{\ell+1}\varpi_j\beta_{q_j}},s_{\oplus_{sw\,j=1}^{\ell+1}\varpi_j\chi_{q_j}}\right],\left(\sqrt{\frac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{\ell+1}\left(1-\left(\delta_{q_j}\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\varpi_j}\right)},\sqrt{\frac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{\ell+1}\left(\frac{\aleph\left(\varepsilon_{q_j}\right)^2+1}{1+\aleph}\right)^{\varpi_j}-1\right)}\right)\right\rangle$$

因此，当 $n=\ell+1$ 时，公式(7)成立。

接下来，我们将探讨 PyTrFLSWWA 算子的性质。

性质 1 (幂等性) 设 $q_j=\left\langle\left[s_{\alpha_{q_j}},s_{\beta_{q_j}},s_{\chi_{q_j}}\right],\left(\delta_{q_j},\varepsilon_{q_j}\right)\right\rangle(j=1,2,\cdots,n)$ 为一组 PyTrFLNs 集合。若 $q_j=q_0=\left\langle\left[s_{\alpha_{q_0}},s_{\beta_{q_0}},s_{\chi_{q_0}}\right],\left(\delta_{q_0},\varepsilon_{q_0}\right)\right\rangle$，则 $\text{PyTrFLSWWA}(q_1,q_2,\cdots,q_n)=q_0$。

证明： 因为 $q_j=q_0=\left\langle\left[s_{\alpha_{q_0}},s_{\beta_{q_0}},s_{\chi_{q_0}}\right],\left(\delta_{q_0},\varepsilon_{q_0}\right)\right\rangle$

$$\text{PyTrFLSWWA}(q_1,q_2,\cdots,q_n)$$

$$=\left\langle\left[s_{\oplus_{sw\,j=1}^{n}\varpi_j\alpha_{q_j}},s_{\oplus_{sw\,j=1}^{n}\varpi_j\beta_{q_j}},s_{\oplus_{sw\,j=1}^{n}\varpi_j\chi_{q_j}}\right],\left(\sqrt{\frac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{n}\left(1-\left(\delta_{q_j}\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\varpi_j}\right)},\sqrt{\frac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{n}\left(\frac{\aleph\left(\varepsilon_{q_j}\right)^2+1}{1+\aleph}\right)^{\varpi_j}-1\right)}\right)\right\rangle$$

$$=\left\langle\left[s_{\alpha_{q_0}},s_{\beta_{q_0}},s_{\chi_{q_0}}\right],\left(\sqrt{\frac{1+\aleph}{\aleph}\left(1-\left(1-\left(\delta_{q_0}\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\sum_{j=1}^{n}\varpi_j}\right)},\sqrt{\frac{1}{\aleph}\left((1+\aleph)\left(\frac{\aleph\left(\varepsilon_{q_0}\right)^2+1}{1+\aleph}\right)^{\sum_{j=1}^{n}\varpi_j}-1\right)}\right)\right\rangle$$

$$=\left\langle\left[s_{\alpha_{q_0}},s_{\beta_{q_0}},s_{\chi_{q_0}}\right],\left(\sqrt{\frac{1+\aleph}{\aleph}\left(\left(\delta_{q_0}\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)},\sqrt{\frac{1}{\aleph}\left(\aleph\left(\varepsilon_{q_0}\right)^2\right)}\right)\right\rangle=\left\langle\left[s_{\alpha_{q_0}},s_{\beta_{q_0}},s_{\chi_{q_0}}\right],\left(\delta_{q_0},\varepsilon_{q_0}\right)\right\rangle=q_0$$

性质 2 (单调性) 设 q_j 和 $\tilde{q}_j$ 是两组 PyTrFLNs。若 $q_j\le\tilde{q}_j$，则

$$\text{PyTrFLSWWA}(q_1,q_2,\cdots,q_n)\le\text{PyTrFLSWWA}(\tilde{q}_1,\tilde{q}_2,\cdots,\tilde{q}_n)$$

性质 3 (有界性)设 $q_j=\left\langle\left[s_{\alpha_{q_j}},s_{\beta_{q_j}},s_{\chi_{q_j}}\right],\left(\delta_{q_j},\varepsilon_{q_j}\right)\right\rangle(j=1,2,\cdots,n)$ 为 PyTrFLNs 集合。若 $s_{\alpha^-}=\min_{1\le j\le n}\left\{s_{\alpha_{q_j}}\right\}$，$s_{\alpha^+}=\max_{1\le j\le n}\left\{s_{\alpha_{q_j}}\right\}$，$s_{\beta^-}=\min_{1\le j\le n}\left\{s_{\beta_{q_j}}\right\}$，$s_{\beta^+}=\max_{1\le j\le n}\left\{s_{\beta_{q_j}}\right\}$，$s_{\chi^-}=\min_{1\le j\le n}\left\{s_{\chi_{q_j}}\right\}$，$s_{\alpha^+}=\max_{1\le j\le n}\left\{s_{\alpha_{q_j}}\right\}$，$\delta_{\min}=\min_{1\le j\le n}\left\{\delta_{q_j}\right\}$，$\delta_{\max}=\max_{1\le j\le n}\left\{\delta_{q_j}\right\}$，$\varepsilon_{\min}=\min_{1\le j\le n}\left\{\varepsilon_{q_j}\right\}$，$\varepsilon_{\max}=\max_{1\le j\le n}\left\{\varepsilon_{q_j}\right\}$。则

$$\left\langle\left[s_{\alpha^-},s_{\beta^-},s_{\chi^-}\right],\left(\delta_{\min},\varepsilon_{\max}\right)\right\rangle\le\text{PyTrFLSWWA}(q_1,q_2,\cdots,q_n)\le\left\langle\left[s_{\alpha^+},s_{\beta^+},s_{\chi^+}\right],\left(\delta_{\max},\varepsilon_{\min}\right)\right\rangle$$

定义 9 设 $q_j=\left\langle\left[s_{\alpha_{q_j}},s_{\beta_{q_j}},s_{\chi_{q_j}}\right],\left(\delta_{q_j},\varepsilon_{q_j}\right)\right\rangle(j=1,2,\cdots,n)$ 为一组 PyTrFLNs 集合。ϖ_j 是毕达哥拉斯三角模糊语言数 q_j 的权重且满足 $\sum_{j=1}^{n}\varpi_j=1,\ \varpi_j\in[0,1]$。则毕达哥拉斯三角模糊语言 Sugeno-Weber 有序加权平均(PyTrFLSWOWA)算子 $\text{PyTrFLSWOWA}:\Omega^n\to\Omega$ 定义如下：

$$\text{PyTrFLSWOWA}(q_1,q_2,\cdots,q_n)=\oplus_{sw\,j=1}^{\;n}\varpi_j q_{\tau(j)} \tag{8}$$

其中 Ω 是全体毕 PTrFLN 集合，$(\tau(1),\tau(2),\cdots,\tau(n))$ 是 $(1,2,\cdots,n)$ 的置换使得 $q_{\tau(n-1)}\ge q_{\tau(n)}$，$\forall j=2,3,\cdots,n$。

定理 2 设 $q_j=\left\langle\left[s_{\alpha_{q_j}},s_{\beta_{q_j}},s_{\chi_{q_j}}\right],\left(\delta_{q_j},\varepsilon_{q_j}\right)\right\rangle(j=1,2,\cdots,n)$ 为一组 PyTrFLNs 集合。则利用 PyTrFLSWOWA 算子集成后的结果仍是 PyTrFLNs 且集成结果表示为

$$PyTrFLSWOWA(q_1,q_2,\cdots,q_n)=\oplus_{sw\,j=1}^{\;n}\varpi_j q_{\tau(j)}$$

$$=\left\langle\left[s_{\oplus_{sw\,j=1}^{\;n}\varpi_j\alpha_{q_{\tau(j)}}},s_{\oplus_{sw\,j=1}^{\;n}\varpi_j\beta_{q_{\tau(j)}}},s_{\oplus_{sw\,j=1}^{\;n}\varpi_j\chi_{q_{\tau(j)}}}\right],\left(\sqrt{\frac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{n}\left(1-\left(\delta_{q_{\tau(j)}}\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\varpi_j}\right)},\sqrt{\frac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{n}\left(\frac{\aleph\left(\varepsilon_{q_{\tau(j)}}\right)^2+1}{1+\aleph}\right)^{\varpi_j}-1\right)}\right)\right\rangle. \tag{9}$$

证明：与定理 1 相似。

3.3. 毕达哥拉斯三角模糊语言数 Sugeno-Weber 加权几何算子

定义 10 设 $q_j=\left\langle\left[s_{\alpha_{q_j}},s_{\beta_{q_j}},s_{\chi_{q_j}}\right],\left(\delta_{q_j},\varepsilon_{q_j}\right)\right\rangle(j=1,2,\cdots,n)$ 为一组 PyTrFLNs 集合。ϖ_j 是 PyTrFLNs q_j 的权重且满足 $\sum_{j=1}^{n}\varpi_j=1,\ \varpi_j\in[0,1]$，$\Omega$ 是全体 PyTrFLNs 集合。则毕达哥拉斯三角模糊语言 Sugeno-Weber 加权几何(PyTrFLSWWG)算子 $\text{PyTrFLSWWG}:\Omega^n\to\Omega$ 定义如下：

$$\text{PyTrFLSWWG}(q_1,q_2,\cdots,q_n)=\otimes_{sw\,j=1}^{\;n}\left(q_j\right)^{\varpi_j} \tag{10}$$

定理 3 设 $q_j=\left\langle\left[s_{\alpha_{q_j}},s_{\beta_{q_j}},s_{\chi_{q_j}}\right],\left(\delta_{q_j},\varepsilon_{q_j}\right)\right\rangle(j=1,2,\cdots,n)$ 为一组 PyTrFLNs 集合。则利用 PyTrFLSWWG 算子集成后的结果仍是 PyTrFLNs 且集成结果表示为

$$\text{PyTrFLSWWG}(q_1,q_2,\cdots,q_n)=\otimes_{sw\,j=1}^{\;n}\left(q_j\right)^{\varpi_j}$$

$$=\left\langle\left[s_{\prod_{j=1}^{n}\left(\alpha_{q_j}\right)^{\varpi_j}},s_{\prod_{j=1}^{n}\left(\beta_{q_j}\right)^{\varpi_j}},s_{\prod_{j=1}^{n}\left(\chi_{q_j}\right)^{\varpi_j}}\right],\left(\sqrt{\frac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{n}\left(\frac{\aleph\left(\delta_{q_j}\right)^2+1}{1+\aleph}\right)^{\varpi_j}-1\right)},\sqrt{\frac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{n}\left(1-\left(\varepsilon_{q_j}\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\varpi_j}\right)}\right)\right\rangle. \tag{11}$$

证明： 与定理 1 相似。

接下来，我们将探讨 PyTrFLSWWG 算子的特殊性质。

性质 4 (幂等性) 设 $q_j=\left\langle\left[s_{\alpha_{q_j}},s_{\beta_{q_j}},s_{\chi_{q_j}}\right],\left(\delta_{q_j},\varepsilon_{q_j}\right)\right\rangle(j=1,2,\cdots,n)$ 为一组 PyTrFLNs 集合。若 $q_j=q_0=\left\langle\left[s_{\alpha_{q_0}},s_{\beta_{q_0}},s_{\chi_{q_0}}\right],\left(\delta_{q_0},\varepsilon_{q_0}\right)\right\rangle$，则 $\text{PyTrFLSWWG}(q_1,q_2,\cdots,q_n)=q_0$。

性质 5 (单调性) 设 q_j 和 $\tilde{q}_j$ 是两组 PyTrFLNs。若 $q_j\le\tilde{q}_j$，则 $\text{PyTrFLSWWG}(q_1,q_2,\cdots,q_n)\le\text{PyTrFLSWWG}(\tilde{q}_1,\tilde{q}_2,\cdots,\tilde{q}_n)$。

性质 6 (有界性) 设 $q_j=\left\langle\left[s_{\alpha_{q_j}},s_{\beta_{q_j}},s_{\chi_{q_j}}\right],\left(\delta_{q_j},\varepsilon_{q_j}\right)\right\rangle(j=1,2,\cdots,n)$ 为一组 PyTrFLNs 集合。若 $s_{\alpha^-}=\min\limits_{1\le j\le n}\left\{s_{\alpha_{q_j}}\right\}$，$s_{\alpha^+}=\max\limits_{1\le j\le n}\left\{s_{\alpha_{q_j}}\right\}$，$s_{\beta^-}=\min\limits_{1\le j\le n}\left\{s_{\beta_{q_j}}\right\}$，$s_{\beta^+}=\max\limits_{1\le j\le n}\left\{s_{\beta_{q_j}}\right\}$，$s_{\chi^-}=\min\limits_{1\le j\le n}\left\{s_{\chi_{q_j}}\right\}$，$s_{\alpha^+}=\max\limits_{1\le j\le n}\left\{s_{\alpha_{q_j}}\right\}$，$\delta_{\min}=\min\limits_{1\le j\le n}\left\{\delta_{q_j}\right\}$，$\delta_{\max}=\max\limits_{1\le j\le n}\left\{\delta_{q_j}\right\}$，$\varepsilon_{\min}=\min\limits_{1\le j\le n}\left\{\varepsilon_{q_j}\right\}$，$\varepsilon_{\max}=\max\limits_{1\le j\le n}\left\{\varepsilon_{q_j}\right\}$。则 $\left\langle\left[s_{\alpha^-},s_{\beta^-},s_{\chi^-}\right],\left(\delta_{\min},\varepsilon_{\max}\right)\right\rangle\le\text{PyTrFLSWWG}(q_1,q_2,\cdots,q_n)\le\left\langle\left[s_{\alpha^+},s_{\beta^+},s_{\chi^+}\right],\left(\delta_{\max},\varepsilon_{\min}\right)\right\rangle$。

定义 11 设 $q_j=\left\langle\left[s_{\alpha_{q_j}},s_{\beta_{q_j}},s_{\chi_{q_j}}\right],\left(\delta_{q_j},\varepsilon_{q_j}\right)\right\rangle(j=1,2,\cdots,n)$ 为一组 PyTrFLNs 集合。ϖ_j 是毕达哥拉斯三角模糊语言数 q_j 的权重且满足 $\sum_{j=1}^{n}\varpi_j=1,\ \varpi_j\in[0,1]$。则毕达哥拉斯三角模糊语言 Sugeno-Weber 有序加权

几何(PyTrFLSWOWG)算子 $\text{PyTrFLSWOWG}:\Omega^n\to\Omega$ 定义如下：

$$\text{PyTrFLSWOWG}(q_1,q_2,\cdots,q_n)=\bigotimes_{sw\,j=1}^{n}\left(q_{\tau(j)}\right)^{\varpi_j}\tag{12}$$

其中 $(\tau(1),\tau(2),\cdots,\tau(n))$ 是 $(1,2,\cdots,n)$ 的置换使得 $q_{\tau(n-1)}\ge q_{\tau(n)}$，$\forall j=2,3,\cdots,n$。

定理 4 设 $q_j=\left\langle\left[s_{\alpha_{q_j}},s_{\beta_{q_j}},s_{\chi_{q_j}}\right],\left(\delta_{q_j},\varepsilon_{q_j}\right)\right\rangle(j=1,2,\cdots,n)$ 为 PyTrFLNs 集合。则利用 PyTrFLSWOWG 算子集成后的结果仍是 PyTrFLNs 且集成结果表示为

$$\begin{aligned}&\text{PyTrFLSWOWG}(q_1,q_2,\cdots,q_n)=\bigotimes_{sw\,j=1}^{n}\left(q_{\tau(j)}\right)^{\varpi_j}\\&=\left\langle\left[s_{\prod_{j=1}^{n}\left(\alpha_{q_{\tau(j)}}\right)^{\varpi_j}},s_{\prod_{j=1}^{n}\left(\beta_{q_{\tau(j)}}\right)^{\varpi_j}},s_{\prod_{j=1}^{n}\left(\chi_{q_{\tau(j)}}\right)^{\varpi_j}}\right],\left(\sqrt{\frac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{n}\left(\frac{\aleph\left(\delta_{q_{\tau(j)}}\right)^2+1}{1+\aleph}\right)^{\varpi_j}-1\right)},\sqrt{\frac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{n}\left(1-\left(\varepsilon_{q_{\tau(j)}}\right)^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\varpi_j}\right)}\right)\right\rangle.\end{aligned}\tag{13}$$

证明： 与定理 1 相似。

4. 基于毕达哥拉斯三角模糊语言 Sugeno-Weber 算子的 COPRAS 多属性决策方法

本章基于毕达哥拉斯三角模糊语言集理论、所提的 Sugeno-Weber 算子和离差最大法，提出属性权重信息完全未知的 COPRAS 多属性决策方法。在所提方法中，决策者通过毕达哥拉斯三角模糊语言数表示其评价信息以表征决策者提供信息时的不确定性和模糊性。为确定权重信息，提出基于毕达哥拉斯三角模糊语言海明距离测度的离差最大法。此外，提出基于毕达哥拉斯三角模糊语言 Sugeno-Weber 算子的 COPRAS 多属性决策方法确定备选方案的排序。

毕达哥拉斯三角模糊语言多属性决策问题涉及的概念和符号定义如下。$H=\left\{H_i\,|\,i=1(1)m\right\}$ 为一组备

选方案，$C=\left\{C_j \middle| j=1(1)n\right\}$ 为属性集合且其权重向量为 $\varpi=\left\{\varpi_j \middle| j=1(1)n\right\}$ 且满足 $\varpi_j\in[0,1],\sum_{j=1}^{n}\varpi_j=1$。专家对备选方案 $H_i\ (i=1,2,\cdots,m)$ 在属性 $C_j\ (j=1,2,\cdots,n)$ 下的评价值用毕达哥拉斯三角模糊语言数表示并构成决策矩阵 $\tilde{F}=\left(\tilde{f}_{ij}\right)_{m\times n}$，$\tilde{f}_{ij}=\left\langle\left[s_{\tilde{\alpha}_{ij}},s_{\tilde{\beta}_{ij}},s_{\tilde{\chi}_{ij}}\right],\left(\tilde{\delta}_{ij},\tilde{\varepsilon}_{ij}\right)\right\rangle\ (i=1,2,\cdots,m;\ j=1,2,\cdots,n)$。基于上述定义，所提基于毕达哥拉斯三角模糊语言 Sugeno-Weber 算子的 COPRAS 多属性决策方法步骤如下：

步骤 1：确定属性权重。

属性权重是确定备选方案的关键步骤，本文基于毕达哥拉斯三角模糊语言海明距离，提出离差最大法确定属性的客观权重，该方法是通过考虑所有备选方案在属性 $C_j\ (j=1,2,\cdots,n)$ 下的差异大小进而确定属性对方案的排序影响程度来确定权重。基于毕达哥拉斯三角模糊语言数得分函数的离差最大法步骤描述如下：

步骤 1.1：计算方案 H_i 与方案 H_k 在属性 $C_j\ (j=1,2,\cdots,n)$ 之间的距离 $D_{ij}(\varpi)$：

$$\begin{aligned}D_{ij}(\varpi)&=\sum_{k=1}^{m}d\left(f_{ij},f_{kj}\right)\varpi_j\\&=\frac{1}{2(l-1)}\sum_{k=1}^{m}\varpi_j\left(\left|\left(1+\left(\delta_{ij}\right)^2-\left(\varepsilon_{ij}\right)^2\right)\times\frac{\alpha_{ij}+\beta_{ij}+\chi_{ij}}{3}-\left(1+\left(\delta_{kj}\right)^2-\left(\varepsilon_{kj}\right)^2\right)\times\frac{\alpha_{kj}+\beta_{kj}+\chi_{kj}}{3}\right|\right)\end{aligned}\tag{14}$$

步骤 1.2：计算所有方案与其他方案在所有属性 $C_j\ (j=1,2,\cdots,n)$ 下的总离差 $D_j(\varpi)$：

$$\begin{aligned}D_j(\varpi)&=\sum_{i=1}^{m}D_{ij}(\varpi)\\&=\frac{1}{2(l-1)}\sum_{i=1}^{m}\sum_{k=1}^{m}\varpi_j\left(\left|\left(1+\left(\delta_{ij}\right)^2-\left(\varepsilon_{ij}\right)^2\right)\times\frac{\alpha_{ij}+\beta_{ij}+\chi_{ij}}{3}-\left(1+\left(\delta_{kj}\right)^2-\left(\varepsilon_{kj}\right)^2\right)\times\frac{\alpha_{kj}+\beta_{kj}+\chi_{kj}}{3}\right|\right)\end{aligned}\tag{15}$$

步骤 1.3：根据上述分析，为使得所有属性对所有方案的总离差最大进而构建目标函数：

$$\max D(\varpi)=\frac{1}{2(l-1)}\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m}\varpi_j\left(\left|\left(1+\left(\delta_{ij}\right)^2-\left(\varepsilon_{ij}\right)^2\right)\times\frac{\alpha_{ij}+\beta_{ij}+\chi_{ij}}{3}-\left(1+\left(\delta_{kj}\right)^2-\left(\varepsilon_{kj}\right)^2\right)\times\frac{\alpha_{kj}+\beta_{kj}+\chi_{kj}}{3}\right|\right)\tag{16}$$

步骤 1.4：构建如下的最优化模型。

$$\begin{cases}\max D(\varpi)=\dfrac{1}{2(l-1)}\displaystyle\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m}\varpi_j\left(\left|\left(1+\left(\delta_{ij}\right)^2-\left(\varepsilon_{ij}\right)^2\right)\times\frac{\alpha_{ij}+\beta_{ij}+\chi_{ij}}{3}-\left(1+\left(\delta_{kj}\right)^2-\left(\varepsilon_{kj}\right)^2\right)\times\frac{\alpha_{kj}+\beta_{kj}+\chi_{kj}}{3}\right|\right)\\ \text{s.t.}\displaystyle\sum_{j=1}^{n}\left(\varpi_j\right)^2=1,\varpi_j\geq 0.\end{cases}\tag{17}$$

步骤 1.5：通过构造拉格朗日函数得到初始权重为

$$\varpi_j^{*}=\frac{\displaystyle\sum_{i=1}^{m}\sum_{k=1}^{m}\left(\left|\left(1+\left(\delta_{ij}\right)^2-\left(\varepsilon_{ij}\right)^2\right)\times\frac{\alpha_{ij}+\beta_{ij}+\chi_{ij}}{3}-\left(1+\left(\delta_{kj}\right)^2-\left(\varepsilon_{kj}\right)^2\right)\times\frac{\alpha_{kj}+\beta_{kj}+\chi_{kj}}{3}\right|\right)}{\sqrt{\displaystyle\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{k=1}^{m}\left(\left|\left(1+\left(\delta_{ij}\right)^2-\left(\varepsilon_{ij}\right)^2\right)\times\frac{\alpha_{ij}+\beta_{ij}+\chi_{ij}}{3}-\left(1+\left(\delta_{kj}\right)^2-\left(\varepsilon_{kj}\right)^2\right)\times\frac{\alpha_{kj}+\beta_{kj}+\chi_{kj}}{3}\right|\right)\right]^2}}\tag{18}$$

对初始权重进行归一化处理，可得最终权重为：

$$\varpi_j=\frac{\displaystyle\sum_{i=1}^{m}\sum_{k=1}^{m}\left(\left|\left(1+\left(\delta_{ij}\right)^2-\left(\varepsilon_{ij}\right)^2\right)\times\frac{\alpha_{ij}+\beta_{ij}+\chi_{ij}}{3}-\left(1+\left(\delta_{kj}\right)^2-\left(\varepsilon_{kj}\right)^2\right)\times\frac{\alpha_{kj}+\beta_{kj}+\chi_{kj}}{3}\right|\right)}{\displaystyle\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m}\left(\left|\left(1+\left(\delta_{ij}\right)^2-\left(\varepsilon_{ij}\right)^2\right)\times\frac{\alpha_{ij}+\beta_{ij}+\chi_{ij}}{3}-\left(1+\left(\delta_{kj}\right)^2-\left(\varepsilon_{kj}\right)^2\right)\times\frac{\alpha_{kj}+\beta_{kj}+\chi_{kj}}{3}\right|\right)}\tag{19}$$

步骤 2： 根据属性的类型，通过公式(20)-(21)确定备选方案在属性 C_j 下的综合评估值：

$$Q_i^+ = \text{PyTrFLSWWA}(q_1, q_2, \cdots, q_n) = \oplus_{sw\, j=1}^{t} \varpi_j f_{ij}$$

$$= \left\langle \left[s_{\oplus_{sw\, j=1}^{t} \varpi_j \alpha_{ij}}, s_{\oplus_{sw\, j=1}^{t} \varpi_j \beta_{ij}}, s_{\oplus_{sw\, j=1}^{t} \varpi_j \chi_{ij}} \right], \left(\sqrt{\frac{1+\aleph}{\aleph}\left(1-\prod_{j=1}^{n}\left(1-(\delta_{ij})^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\varpi_j}\right)}, \sqrt{\frac{1}{\aleph}\left((1+\aleph)\prod_{j=1}^{n}\left(\frac{\aleph(\varepsilon_{ij})^2+1}{1+\aleph}\right)^{\varpi_j}-1\right)} \right) \right\rangle. \tag{20}$$

$$Q_i^- = \text{PyTrFLSWWA}(q_1, q_2, \cdots, q_n) = \oplus_{sw\, j=t+1}^{t} \varpi_j f_{ij}$$

$$= \left\langle \left[s_{\oplus_{sw\, j=1}^{t} \varpi_j \alpha_{ij}}, s_{\oplus_{sw\, j=1}^{t} \varpi_j \beta_{ij}}, s_{\oplus_{sw\, j=1}^{t} \varpi_j \chi_{ij}} \right], \left(\sqrt{\frac{1+\aleph}{\aleph}\left(1-\prod_{j=t+1}^{n}\left(1-(\delta_{ij})^2\left(\frac{\aleph}{1+\aleph}\right)\right)^{\varpi_j}\right)}, \sqrt{\frac{1}{\aleph}\left((1+\aleph)\prod_{j=t+1}^{n}\left(\frac{\aleph(\varepsilon_{ij})^2+1}{1+\aleph}\right)^{\varpi_j}-1\right)} \right) \right\rangle. \tag{21}$$

其中 t 表示效益型属性的数量， $n-t$ 表示成本型属性的数量。 Q_i^+ 和 Q_i^- 分别表示备选方案在效益型和成本型属性 C_j $(j=1,2,\cdots,n)$ 下的综合评估值。

步骤 3： 通过公式(22)计算备选方案的相对评估值：

$$R_i = SF(Q_i^+) + \left(\sum_{i=1}^{m} SF(Q_i^-)\right) \Bigg/ \left(SF(Q_i^-)\sum_{i=1}^{m}\frac{1}{SF(Q_i^-)}\right). \tag{22}$$

步骤 4： 根据 R_i 值的对备选方案进行降序排列确定备选方案的优先级。

5. 实例分析

本章通过案例分析、参数讨论比较分析讨论所提基于毕达哥拉斯三角模糊语言数 WASPAS 多属性决策方法的实用性，稳定性和有效性。

5.1. 决策实施过程

本章考虑某地区在大力开展招商引资时，由一家投资公司在本地区遴选一些小型企业作为备选。现有五家小型企业($H = \{H_i | i = 1(1)5\}$)作为备选方案。经过商定确定四个指标作为遴选时的标准，企业的风险规避能力(C_1)，企业的环境下(C_2)，企业的规模(C_3)和企业的成长能力(C_4)。为遴选最优的企业进行投资，五家供应商在这四个属性下的评价值由决策专家以毕达哥拉斯三角模糊语言数的形式给出，评价矩阵见表 1。

Table 1. Pythagorean triangular fuzzy linguistic number evaluation matrix
表 1. 毕达哥拉斯三角模糊语言数评价矩阵

	C_1	C_2	C_3	C_4
H_1	$\langle [s_2,s_3,s_4],(0.7,0.5)\rangle$	$\langle [s_1,s_3,s_4],(0.6,0.5)\rangle$	$\langle [s_2,s_3,s_5],(0.5,0.5)\rangle$	$\langle [s_1,s_3,s_5],(0.8,0.4)\rangle$
H_2	$\langle [s_2,s_3,s_5],(0.65,0.5)\rangle$	$\langle [s_2,s_3,s_4],(0.8,0.3)\rangle$	$\langle [s_2,s_3,s_6],(0.7,0.5)\rangle$	$\langle [s_1,s_3,s_4],(0.7,0.5)\rangle$
H_3	$\langle [s_2,s_3,s_6],(0.6,0.5)\rangle$	$\langle [s_2,s_3,s_5],(0.8,0.35)\rangle$	$\langle [s_2,s_4,s_6],(0.75,0.5)\rangle$	$\langle [s_2,s_3,s_4],(0.75,0.5)\rangle$
H_4	$\langle [s_3,s_4,s_6],(0.8,0.5)\rangle$	$\langle [s_1,s_3,s_5],(0.85,0.3)\rangle$	$\langle [s_1,s_4,s_6],(0.75,0.5)\rangle$	$\langle [s_1,s_3,s_5],(0.75,0.4)\rangle$
H_5	$\langle [s_3,s_4,s_5],(0.75,0.5)\rangle$	$\langle [s_1,s_3,s_6],(0.8,0.3)\rangle$	$\langle [s_2,s_4,s_6],(0.8,0.5)\rangle$	$\langle [s_1,s_2,s_4],(0.75,0.5)\rangle$

步骤 1： 确定属性权重。通过公式(18)~(19)确定属性的权重为 $\varpi_1 = 0.3038$，$\varpi_2 = 0.2462$，$\varpi_3 = 0.2633$，$\varpi_4 = 0.1867$。

步骤 2： 根据属性的类型，通过公式(20)~(21)确定备选方案在属性 $C_j\ (j=1,2,\cdots,n)$ 下的综合评估值，因为所有属性均为效益型属性，所以只需计算备选方案的 Q_i^+ 值。

$Q_1^+ = \langle [1.5671, 3.0000, 4.4500], (0.6572, 0.4820) \rangle$，$Q_2^+ = \langle [1.8133, 3.0000, 4.8304], (0.7141, 0.4549) \rangle$，
$Q_3^+ = \langle [2.0000, 3.2633, 5.3804], (0.7253, 0.4652) \rangle$，$Q_4^+ = \langle [1.6075, 3.5671, 5.5671], (0.7924, 0.4362) \rangle$，
$Q_5^+ = \langle [1.8708, 3.3804, 5.3229], (0.7764, 0.4549) \rangle$。

步骤 3： 通过公式(22)计算备选方案的相对评估值：

$R_1 = 0.5999$，$R_2 = 0.9739$，$R_3 = 1.0987$，$R_4 = 1.5668$，$R_5 = 1.3953$。

步骤 4： 根据 R_i 值可确定备选方案的排序为 $H_4 \succ H_5 \succ H_3 \succ H_2 \succ H_1$，即 H_4 为最优投资企业。

5.2. 灵敏度分析

本节将对所提基于毕达哥拉斯三角模糊语言数 COPRAS 决策方法中涉及的参数进行讨论进而分析所提方法的鲁棒性和稳定性。针对 PyTrFLSWWA 算子中的参数 $\aleph$，取不同值获得备选方案的相对评估值和排序结果如表 2 和图 1 所示。由表 2 和图 1 可知，随着参数 $\aleph$ 的逐渐增大，不同备选方案的相对评估值随之增大，但是备选方案的排序没有发生任何变化，因此所提方法对该参数敏感性较低且具有极高的稳定性。

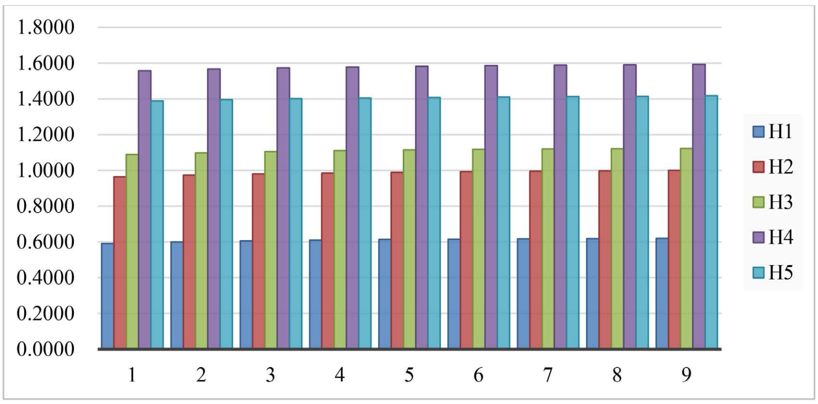

Figure 1. Relative evaluation values of alternatives based on different parameter $\aleph$ values
图 1. 基于不同参数 $\aleph$ 值的备选方案的相对评估值

Table 2. Relative evaluation values and ranking results of alternatives based on different parameter $\aleph$ values
表 2. 基于不同参数 $\aleph$ 值的备选方案的相对评估值和排序结果

$\aleph$	H_1	H_2	H_3	H_4	H_5	排序
1	0.5899	0.9649	1.0883	1.5582	1.3887	$H_4 \succ H_5 \succ H_3 \succ H_2 \succ H_1$
2	0.5999	0.9739	1.0987	1.5668	1.3953	$H_4 \succ H_5 \succ H_3 \succ H_2 \succ H_1$
3	0.6060	0.9803	1.1055	1.5731	1.4003	$H_4 \succ H_5 \succ H_3 \succ H_2 \succ H_1$
4	0.6101	0.9852	1.1103	1.5780	1.4044	$H_4 \succ H_5 \succ H_3 \succ H_2 \succ H_1$
5	0.6132	0.9892	1.1140	1.5820	1.4077	$H_4 \succ H_5 \succ H_3 \succ H_2 \succ H_1$
6	0.6155	0.9924	1.1169	1.5852	1.4105	$H_4 \succ H_5 \succ H_3 \succ H_2 \succ H_1$

续表

7	0.6173	0.9950	1.1193	1.5880	1.4129	$H_4 \succ H_5 \succ H_3 \succ H_2 \succ H_1$
8	0.6188	0.9973	1.1213	1.5903	1.4149	$H_4 \succ H_5 \succ H_3 \succ H_2 \succ H_1$
9	0.6200	0.9993	1.1229	1.5923	1.4167	$H_4 \succ H_5 \succ H_3 \succ H_2 \succ H_1$

5.3. 比较分析

为验证本文所提方法的有效性，本节基于决策矩阵和属性权重，利用文献[15]中基于毕达哥拉斯三角模糊语言 Hamacher 加权平均算子求解本文的决策问题，获得的备选方案排序与本文一致，备选方案的排序为 $H_4 \succ H_5 \succ H_3 \succ H_2 \succ H_1$，因此本文所提基于 Sugeno-Weber 集成算子的改进 COPRAS 决策方法是有效的。

6. 结论

本文针对属性权重未知、评估信息为毕达哥拉斯三角模糊语言数的多属性决策问题，提出了一种基于 Sugeno-Weber 集成算子的改进 COPRAS 方法。主要工作与结论如下：首次定义了适用于毕达哥拉斯三角模糊语言数的 Sugeno-Weber 运算法则并提出新型集成算子：基于新法则，提出了毕达哥拉斯三角模糊语言 Sugeno-Weber 加权平均和几何等新型集成算子。其次，提出基于离差最大化的权重确定模型，通过最大化各属性下方案评价值的总离差来求取客观权重。再次，将所提算子与离差最大化权重模型结合，构建了改进的 COPRAS 决策框架。综上，本文提出的毕达哥拉斯三角模糊语言集成算子、离差最大化权重模型及改进 COPRAS 方法，为解决属性权重未知且评估信息为毕达哥拉斯三角模糊语言数的复杂决策问题提供了一种新颖、有效的实用方案。

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