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The Relation between Solutions of Higher Order Linear Differential Equations and Functions of Small Growth*

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Abstract: In this paper, the growth of solutions of higher order linear differential equation is investigates, $f^{(k)}+A_{k-1}(z)f^{(k-1)}+A_{k-2}(z)f^{(k-2)}+\cdots+A_2(z)f''+A_1(z)e^{az}f'+A_0(z)e^{bz}f=0$ in the $A_j(z)\not\equiv 0$ were entire functions, $\sigma(A_j)<1\,(j=0,1,2,\cdots,k-1)$, a,b are non-zero constant obtains their 1st, 2nd derivatives, differential polynomial of differential equations with function of small growth.

Keywords: Linear Differential Equations; Entire Function; Small Function; Exponent of Convergence

高阶线性微分方程解与其小函数的关系*

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摘　要：本文研究了高阶线性齐次微分方程
$f^{(k)}+A_{k-1}(z)f^{(k-1)}+A_{k-2}(z)f^{(k-2)}+\cdots+A_2(z)f''+A_1(z)e^{az}f'+A_0(z)e^{bz}f=0$ 解的增长性，其中 $A_j(z)\not\equiv 0$ 是整函数，$\sigma(A_j)<1\,(j=0,1,2,\cdots,k-1)$，$a,b$ 为非零复常数，得到了方程解的一阶导数，二阶导数，微分多项式与小函数之间的关系。

关键词：线性微分方程；整函数；小函数；收敛指数

1. 引言与主要结果

本文采用 Nevanlinna 值分布理论的标准记号[1-24]，用 $\sigma(f)$、$\lambda(f)$ 和 $\bar{\lambda}(f)$ 表示亚纯函数 $f(z)$ 的增长级、零点收敛指数和不同零点收敛指数，用 $\lambda(f-\varphi)$ 和 $\lambda(f-\varphi)$ 表示亚纯函数 $f(z)$ 取小函数的零点收敛指数和取小函数的不同零点收敛指数。设二阶线性微分方程

$$f''+A_1(z)e^{a_1z}f'+A_0(z)e^{a_0z}f=0, \tag{1.1}$$

其中 $A_j(z)(\neq 0)(j=0,1)$ 是整函数，且 $\sigma(A_j)<1$，$a_j\in C-\{0\}(j=0,1)$。陈宗煊[1]研究了微分方程(1.1)的解的增长性问题，大大推广和完善了 Frei M.[2]，Ozawa M.[3,4]，Gundersen G.[5]，Langley J. K.[6]关于二阶线性微分方程 $f''+e^{-z}f'+Q(z)f=0$ (其中 $Q(z)$ 为有限级整函数)解的增长性的结果。徐俊峰和仪洪勋在文[7]中进一步研究了

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二阶微分方程 $f''+A_1(z)e^{az}f'+A_0(z)e^{bz}f=0$ 的超越解与其小函数的关系。

本文在此基础上研究了高阶线性复微分方程

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+A_{k-2}(z)f^{(k-2)}+\cdots+A_2(z)f''+A_1(z)e^{az}f'+A_0(z)e^{bz}f=0 \tag{1.2}$$

的解 $f(z)$ 与其小函数 $\varphi(z)$ 的关系。得到了下述结论。

定理 设 $A_j(z)(\not\equiv 0)(j=0,1,2,\cdots,k-1)$ 是整函数，且 $\sigma(A_j)<1$，a,b 是复常数，且 $ab\neq 0$ 和 $a\neq b$ 和 $2b$。如果 $\varphi(z)$ 是不恒为零的整函数且 $\sigma(\varphi)<1$，则微分方程(1.2)的任意超越解 $f(z)$ 都满足

$$\overline{\lambda}(f-\varphi)=\overline{\lambda}(f'-\varphi)=\overline{\lambda}(f''-\varphi)=\infty .$$

2. 证明定理所需的引理

引理 1[8] 假设 $A_j=h_j(z)e^{a_jz}(j=0,1,2,\cdots,k-1)$，其中 $h_j(z)(\not\equiv 0)$ 为整函数且级小于 1，a_j 为互不相同的复常数。则

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+A_{k-2}(z)f^{(k-2)}+\cdots+A_1(z)f'+A_0(z)f=0$$

的所有超越解的级都为无穷。

引理 2[9-12] 设 $f(z)$ 是超越亚纯函数且 $\sigma(f)=\sigma<+\infty,H=\{(k_1,j_1)(k_2,j_2),\cdots,(k_q,j_q)\}$ 是不同的整数对的有限集合，满足 $k_i>j_i\geq 0(i=1,2,\cdots q)$。假设 $\varepsilon>0$ 是个给定常数，则存在一集合 $E\subset[0,2\pi)$，其线性测度为零，使得如果 $\varphi\in[0,2\pi)-E$，则存在常数 $R_0=R_0(\varphi)>1$，对满足 $\arg z=\varphi$ 及 $|z|\geq R_0$ 的所有 $(k,j)\in H$ 都有 $\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right|\leq|z|^{(k-j)(\sigma-1+\varepsilon)}$。

引理 3[13-15] 设 $f_1(z),f_2(z),\cdots,f_n(z)(n\geq 2)$ 为亚纯函数，$g_1(z),g_2(z),\cdots,g_n(z)$ 为整函数，满足下列条件：

1) $\sum_{j=1}^{n}f_j(z)e^{g_j(z)}\equiv 0$；

2) 当 $1\leq j<k\leq n$ 时，$g_j(z)-g_k(z)$ 是非常数；

3) 当 $0\leq j\leq n$，$1\leq h<k\leq n$ 时，$T(r,f_j)=o\{T(r,e^{g_h-g_k})\}(r\to\infty,r\in E)$。那么 $f_j(z)\equiv 0(j=1,2,\cdots,n)$。

在定理的证明中我们只需要如下形式：

引理 4 设 $f_1(z),f_2(z),\cdots,f_n(z),f_{n+1}(z)$ 为亚纯函数，$g_1(z),g_2(z),\cdots,g_n(z)$ 为整函数，满足下列条件：

1) $\sum_{j=1}^{n}f_j(z)e^{g_j(z)}\equiv f_{n+1}(z)$；

2) 当 $1\leq j\leq n+1$，$1\leq k\leq n$ 时，$f_j(z)$ 的级小于 $e^{g_k(z)}$ 的级。在 $n\geq 2$ 的情形，当 $0\leq j\leq n+1$，$1\leq h<k\leq n$ 时，$f_j(z)$ 的级也小于 $e^{g_h(z)-g_k(z)}$ 的级。那么 $f_j(z)\equiv 0(j=1,2,\cdots,n+1)$。

证明 因为条件 1)中的恒等式可以改写为 $\sum_{j=1}^{n}f_j(z)e^{g_j(z)}-f_{n+1}(z)e^{g_{n+1}(z)}\equiv 0$，$g_{n+1}(z)\equiv 0$，故由引理 3 即可得出 $f_j(z)\equiv 0(j=1,2,\cdots,n+1)$。

引理 5[16] 设 $A_0,A_1,\cdots,A_{k-1},F\not\equiv 0$ 都是有限级亚纯函数，如果 $f(z)$ 是方程

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=F(z)$$

的一个无穷级亚纯函数解，那么 $f(z)$ 满足 $\lambda(f)=\overline{\lambda}(f)=\sigma(f)=\infty$。

3. 定理的证明

证明 设 $f(z)$ 是方程(1.2)的任意超越解，则由已知和引理 1 可知 $\sigma(f)=\infty$。故 $\sigma(f-\varphi)=\infty$。

下面我们证明 $\overline{\lambda}(f'-\varphi)=\infty$ 和 $\overline{\lambda}(f''-\varphi)=\infty$。

1) 首先我们证明 $\overline{\lambda}(f'-\varphi)=\infty$。设 $g(z)=f'(z)-\varphi(z)$，则 $\sigma(g)=\sigma(f'-\varphi)=\sigma(f')=\sigma(f)=\infty$，$\overline{\lambda}(g)=\overline{\lambda}(f'-\varphi)$。

对微分方程(1.2)两边求导并整理得

$$f^{(k+1)}+A_{k-1}f^{(k)}+\sum_{j=1}^{k-3}\left(A'_{k-j}+A_{k-j-1}\right)f^{(k-j)}+\left(A_2+A_1e^{az}\right)f''+\left(\left(A_1e^{az}\right)'+A_0e^{bz}\right)f'+\left(A_0e^{bz}\right)'f=0, \quad (3.1)$$

由微分方程(1.2)得

$$f=-\frac{1}{A_0e^{bz}}\left(f^{(k)}+\sum_{j=1}^{k-3}A_{k-j}f^{(k-j)}+A_2f''+A_1e^{az}f'\right), \quad (3.2)$$

将(3.2)式代入(3.1)式并整理得

$$\begin{aligned}&f^{(k+1)}+\left(A_{k-1}-\frac{\left(A_0e^{bz}\right)'}{A_0e^{bz}}\right)f^{(k)}+\sum_{j=1}^{k-3}\left(A'_{k-j}+A_{k-j-1}-A_{k-j}\frac{\left(A_0e^{bz}\right)'}{A_0e^{bz}}\right)f^{(k-j)}\\&+\left(A'_2+A_1e^{az}-A_2\frac{\left(A_0e^{bz}\right)'}{A_0e^{bz}}\right)f''+\left(\left(A_1e^{az}\right)'+A_0e^{bz}-A_1e^{az}\frac{\left(A_0e^{bz}\right)'}{A_0e^{bz}}\right)f'=0\end{aligned}. \quad (3.3)$$

又由 $g(z)=f'(z)-\varphi(z)$ 可得

$$f'=g+\varphi,\quad f^{(k-j)}=g^{(k-j-1)}+\varphi^{(k-j-1)}\left(j=1,2,\cdots,k-2\right),\quad f^{(k)}=g^{(k-1)}+\varphi^{(k-1)},\quad f^{(k+1)}=g^{(k)}+\varphi^{(k)}.$$

将这 $k+1$ 个等式代入(3.3)并整理得

$$\begin{aligned}&g^{(k)}+\left(A_{k-1}-\frac{\left(A_0e^{bz}\right)'}{A_0e^{bz}}\right)g^{(k-1)}+\sum_{j=1}^{k-3}\left(A'_{k-j}+A_{k-j-1}-A_{k-j}\frac{\left(A_0e^{bz}\right)'}{A_0e^{bz}}\right)g^{(k-j-1)}\\&+\left(A'_2+A_1e^{az}-A_2\frac{\left(A_0e^{bz}\right)'}{A_0e^{bz}}\right)g'+\left(\left(A_1e^{az}\right)'+A_0e^{bz}-A_1e^{az}\frac{\left(A_0e^{bz}\right)'}{A_0e^{bz}}\right)g\\&=-\left\{\varphi^{(k)}+\left(A_{k-1}-\frac{\left(A_0e^{bz}\right)'}{A_0e^{bz}}\right)\varphi^{(k-1)}+\sum_{j=1}^{k-3}\left(A'_{k-j}+A_{k-j-1}-A_{k-j}\frac{\left(A_0e^{bz}\right)'}{A_0e^{bz}}\right)\varphi^{(k-j-1)}\right.\\&\left.+\left(A'_2+A_1e^{az}-A_2\frac{\left(A_0e^{bz}\right)'}{A_0e^{bz}}\right)\varphi'+\left(\left(A_1e^{az}\right)'+A_0e^{bz}-A_1e^{az}\cdot\frac{\left(A_0e^{bz}\right)'}{A_0e^{bz}}\right)\varphi\right\}\end{aligned}. \quad (3.4)$$

假设

$$h_k=A_{k-1}-\frac{\left(A_0e^{bz}\right)'}{A_0e^{bz}},\quad h_{k-j}(z)=A'_{k-j}+A_{k-j-1}-A_{k-j}\frac{\left(A_0e^{bz}\right)'}{A_0e^{bz}}\ \left(j=1,2,\cdots,k-3\right),$$

$$h_2(z)=A'_2+A_1e^{az}-A_2\frac{\left(A_0e^{bz}\right)'}{A_0e^{bz}},\quad h_1(z)=\left(A_1e^{az}\right)'+A_0e^{bz}-A_1e^{az}\cdot\frac{\left(A_0e^{bz}\right)'}{A_0e^{bz}}.$$

则(3.4)变为

$$g^{(k)}+h_k(z)g^{(k-1)}+\sum_{j=1}^{k-3}h_{k-j}(z)g^{(k-j-1)}+h_2(z)g'+h_1(z)g=h(z). \quad (3.5)$$

其中

$$h(z)=-\left\{\varphi^{(k)}+h_k(z)\varphi^{(k-1)}+\sum_{j=1}^{k-3}h_{k-j}(z)\varphi^{(k-j-1)}+h_2(z)\varphi'+h_1(z)\varphi\right\}.$$

若 $h(z)\equiv 0$，即

$$\varphi^{(k)}+h_k(z)\varphi^{(k-1)}+\sum_{j=1}^{k-3}h_{k-j}(z)\varphi^{(k-j-1)}+h_2(z)\varphi'+h_1(z)\varphi=0.$$

又 $\dfrac{\left(A_0e^{bz}\right)'}{A_0e^{bz}}=\dfrac{\left(A_0\right)'}{A_0}+b$，$\left(A_1e^{az}\right)'=A_1'e^{az}+aA_1e^{az}$ 故

$$\begin{aligned}&\frac{\varphi^{(k)}}{\varphi}+\left(A_{k-1}-\frac{A_0'}{A_0}-b\right)\frac{\varphi^{(k-1)}}{\varphi}+\sum_{j=1}^{k-3}\left(A_{k-j}'+A_{k-j-1}-A_{k-j}\frac{A_0'}{A_0}-bA_{k-j}\right)\frac{\varphi^{(k-j-1)}}{\varphi}+\left(A_2'-A_2\left(\frac{A_0'}{A_0}+b\right)\right)\frac{\varphi'}{\varphi}\\&+\left(A_1\frac{\varphi'}{\varphi}+aA_1+A_1'-A_1\frac{A_0'}{A_0}-bA_1\right)e^{az}+A_0e^{bz}=0\end{aligned}. \quad (3.6)$$

因为 $A_j(z)\not\equiv 0$ 是整函数，$\sigma\left(A_j\right)<1\left(j=0,1,2,\cdots,k-1\right)$，$a,b$ 是相互不同的复常数，$\varphi(z)$ 是不恒为零的整函数且 $\sigma(\varphi)<1$，所以由引理 2 知

$$\left(A_1\frac{\varphi'}{\varphi}+aA_1+A_1'-A_1\frac{A_0'}{A_0}-bA_1\right)e^{az},$$

和

$$\frac{\varphi^{(k)}}{\varphi}+\left(A_{k-1}-\frac{A_0'}{A_0}-b\right)\frac{\varphi^{(k-1)}}{\varphi}+\sum_{j=1}^{k-3}\left(A_{k-j}'+A_{k-j-1}-A_{k-j}\frac{A_0'}{A_0}-bA_{k-j}\right)\frac{\varphi^{(k-j-1)}}{\varphi}+\left(A_2'-A_2\left(\frac{A_0'}{A_0}+b\right)\right)\frac{\varphi'}{\varphi}.$$

的级都小于 1，由引理 4 和(3.6)式可知 $A_0\equiv 0$，这与定理的条件矛盾，因而 $h(z)\not\equiv 0$。

对于方程(3.5)来说，由于 $h(z)\not\equiv 0$ 及 $\sigma(g)=\infty$ 和引理 5 可知

$$\overline{\lambda}(g)=\overline{\lambda}(f'-\varphi)=\sigma(g)=\sigma(f)=\infty.$$

2) 其次我们证明 $\overline{\lambda}(f''-\varphi)=\infty$。对方程(3.1)两边求导并整理得

$$\begin{aligned}&f^{(k+2)}+A_{k-1}f^{(k+1)}+\left(2A_{k-1}'+A_{k-1}\right)f^{(k)}+\sum_{j=1}^{k-4}\left(A_{k-j}''+2A_{k-j-1}'+A_{k-j-2}\right)f^{(k-j)}+\left(A_3''+2A_2'+A_1e^{az}\right)f'''\\&+\left(A_2''+2\left(A_1e^{az}\right)'+A_0e^{bz}\right)f''+\left(\left(A_1e^{az}\right)''+2\left(A_0e^{bz}\right)'\right)f'+\left(A_0e^{az}\right)''f=0\end{aligned} \quad (3.7)$$

将(3.2)式代入(3.7)式并整理得

$$\begin{aligned}&f^{(k+2)}+A_{k-1}f^{(k+1)}+\left(2A_{k-1}'+A_{k-1}-\frac{\left(A_0e^{bz}\right)''}{A_0e^{bz}}\right)f^{(k)}+\sum_{j=1}^{k-4}\left(A_{k-j}''+2A_{k-j-1}'+A_{k-j-2}-A_{k-j}\frac{\left(A_0e^{bz}\right)''}{A_0e^{bz}}\right)f^{(k-j)}\\&+\left(A_3''+2A_2'+A_1e^{az}-A_3\frac{\left(A_0e^{bz}\right)''}{A_0e^{bz}}\right)f'''+\left(A_2''+2\left(A_1e^{az}\right)'+A_0e^{bz}-A_2\frac{\left(A_0e^{bz}\right)''}{A_0e^{bz}}\right)f''\\&+\left(\left(A_1e^{az}\right)''+2\left(A_0e^{bz}\right)'-A_1e^{az}\cdot\frac{\left(A_0e^{bz}\right)''}{A_0e^{bz}}\right)f'=0\end{aligned}. \quad (3.8)$$

令

$$\begin{aligned}\varphi_1(z)&=\left(A_1e^{az}\right)''+2\left(A_0e^{bz}\right)'-A_1e^{az}\frac{\left(A_0e^{bz}\right)''}{A_0e^{bz}}\\&=\left(A_1''+2aA_1'+a^2A_1\right)e^{az}+2\left(A_0'+bA_0\right)e^{bz}-A_1e^{az}\left(\frac{A_0''}{A_0}+2b\frac{A_0'}{A_0}+b^2\right)\\&=\left(A_1''+2aA_1'+a^2A_1-A_1\frac{A_0''}{A_0}-2bA_1\frac{A_0'}{A_0}+b^2A_1\right)e^{az}+2(A_0'+bA_0)e^{bz}\end{aligned}$$

$$\begin{aligned}\varphi_2(z)&=\left(A_1e^{az}\right)'+A_0e^{bz}-A_1e^{az}\frac{\left(A_0e^{bz}\right)'}{A_0e^{bz}}=A_1'e^{az}+aA_1e^{az}+A_0e^{bz}-A_1e^{az}\left(\frac{A_0'}{A_0}+b\right).\\&=\left(A_1'+aA_1-A_1\frac{A_0'}{A_0}-bA_1\right)e^{az}+A_0e^{bz}\end{aligned}$$

由引理 4 易知$\varphi_2(z)\not\equiv 0$，且$\varphi_1(z)$和$\varphi_2(z)$都是亚纯函数，$\sigma(\varphi_1)<1$，$\sigma(\varphi_2)<1$。由(3.3)可得

$$\begin{aligned}f'=-\frac{1}{\varphi_2}\Bigg\{&f^{(k+1)}+\left(A_{k-1}-\frac{\left(A_0e^{bz}\right)'}{A_0e^{bz}}\right)f^{(k)}+\sum_{j=1}^{k-4}\left(A_{k-j}'+A_{k-j-1}-A_{k-j}\frac{\left(A_0e^{bz}\right)'}{A_0e^{bz}}\right)f^{(k-j)}\\&+\left(A_3'+A_2-A_3\frac{\left(A_0e^{bz}\right)'}{A_0e^{bz}}\right)f'''+\left(A_2'+A_1e^{az}-A_2\frac{\left(A_0e^{bz}\right)'}{A_0e^{bz}}\right)\Bigg\}.\end{aligned}\tag{3.9}$$

将(3.9)式代入(3.8)式并整理得

$$\begin{aligned}&f^{(k+2)}+\left(A_{k-1}-\frac{\varphi_1}{\varphi_2}\right)f^{(k+1)}+\left(2A_{k-1}'+A_{k-1}-\frac{\left(A_0e^{bz}\right)''}{A_0e^{bz}}-\frac{\varphi_1}{\varphi_2}\left(A_{k-1}-\frac{\left(A_0e^{bz}\right)'}{A_0e^{bz}}\right)\right)f^{(k)}\\&+\sum_{j=1}^{k-4}\left(A_{k-j}''+2A_{k-j-1}'+A_{k-j-2}-A_{k-j}\frac{\left(A_0e^{bz}\right)''}{A_0e^{bz}}-\frac{\varphi_1}{\varphi_2}\left(A_{k-j}'+A_{k-j-1}-A_{k-j}\frac{\left(A_0e^{bz}\right)'}{A_0e^{bz}}\right)\right)f^{(k-j)}\\&+\left(A_3''+2A_2'+A_1e^{az}-A_3\frac{\left(A_0e^{bz}\right)''}{A_0e^{bz}}-\frac{\varphi_1}{\varphi_2}\left(A_3'+A_2-A_3\frac{\left(A_0e^{bz}\right)'}{A_0e^{bz}}\right)\right)f'''\\&+\left(A_2''+2\left(A_1e^{az}\right)'+A_0e^{bz}-A_2\cdot\frac{\left(A_0e^{bz}\right)''}{A_0e^{bz}}-\frac{\varphi_1}{\varphi_2}\left(A_2'+A_1e^{az}-A_2\cdot\frac{\left(A_0e^{bz}\right)'}{A_0e^{bz}}\right)\right)f''=0\end{aligned}.\tag{3.10}$$

令

$$H_{k-1}=A_{k-1}-\frac{\varphi_1}{\varphi_2},\tag{3.11}$$

$$H_{k-2}=2A_{k-1}'+A_{k-1}-\left(\frac{A_0''}{A_0}+2b\frac{A_0'}{A_0}+b^2\right)-\frac{\varphi_1}{\varphi_2}\left(A_{k-1}-\frac{A_0'}{A_0}-b\right),\tag{3.12}$$

$$H_{k-j-1}=\left(A''_{k-j}+2A'_{k-j-1}+A_{k-j-2}-A_{k-j}\left(\frac{A''_0}{A_0}+2b\frac{A'_0}{A_0}+b^2\right)\right)-\frac{\varphi_1}{\varphi_2}\left(A'_{k-j}+A_{k-j-1}-A_{k-j}\left(\frac{A'_0}{A_0}+b\right)\right)\left(j=1,2,\cdots,k-3\right), \tag{3.13}$$

$$H_1=A''_3+2A'_2+A_1e^{az}-A_3\left(\frac{A''_0}{A_0}+2b\frac{A'_0}{A_0}+b^2\right)-\frac{\varphi_1}{\varphi_2}\left(A'_3+A_2-A_3\left(\frac{A'_0}{A_0}+b\right)\right), \tag{3.14}$$

$$H_0=A''_2+2\left(A_1e^{az}\right)'+A_0e^{bz}-A_2\left(\frac{A''_0}{A_0}+2b\frac{A'_0}{A_0}+b^2\right)-\frac{\varphi_1}{\varphi_2}\left(A'_2+A_1e^{az}-A_2\left(\frac{A'_0}{A_0}+b\right)\right). \tag{3.15}$$

则 $H_0,H_1,H_{k-j-1}\left(j=1,2,\cdots,k-3\right),H_{k-1},H_k$ 都是亚函数，并且

$$\sigma\left(H_0\right)<1，\sigma\left(H_1\right)<1，\sigma\left(H_{k-j-1}\right)<1\left(j=1,2,\cdots,k-3\right)，\sigma\left(H_{k-1}\right)<1，\sigma\left(H_k\right)<1.$$

设 $h(z)=f''-\varphi$，则 $\sigma(h)=\sigma(f'')=\sigma(f)=\infty$ 和 $\overline{\lambda}(h)=\overline{\lambda}(f''-\varphi)$，且

$$f''=h+\varphi，f^{(k-j)}=h^{(k-j-2)}+\varphi^{(k-j-2)}\left(j=1,2,\cdots,k-3\right),$$

$$f^{(k)}=h^{(k-2)}+\varphi^{(k-2)}，f^{(k+1)}=h^{(k-1)}+\varphi^{(k-1)}，f^{(k+2)}=h^{(k)}+\varphi^{(k)}.$$

将这 $k+1$ 个等式代入(3.10)式并整理得

$$h^{(k)}+H_{k-1}h^{(k-1)}+H_{k-2}h^{(k-2)}+\sum_{j=1}^{k-4}H_{k-j-1}h^{(k-j-2)}+H_1h'+H_0h=-\left(\varphi^{(k)}+H_{k-1}\varphi^{(k-1)}+H_{k-2}\varphi^{(k-2)}+\sum_{j=1}^{k-4}H_{k-j-1}\varphi^{(k-j-2)}+H_1\varphi'+H_0\varphi\right). \tag{3.16}$$

令

$$B_1=A''_1+2aA'_1+a^2A_1-A_1\frac{A''_0}{A_0}-2bA_1\frac{A'_0}{A_0}+b^2A_1，B_2=A'_1+aA_1-A_1\frac{A'_0}{A_0}-bA_1.$$

则 B_1,B_2 都是亚纯函数，且 $\sigma\left(B_1\right)<1,\sigma\left(B_2\right)<1$。由 φ_1 和 φ_2 的表达式以及(11)~(15)式我们有

$$H_{k-1}=\frac{1}{\varphi_2}F_{k-1}，H_{k-2}=\frac{1}{\varphi_2}F_{k-2}，H_{k-j-1}=\frac{1}{\varphi_2}F_{k-j-1}\left(j=1,2,\cdots,k-3\right)，H_1=\frac{1}{\varphi_2}F_1，H_0=\frac{1}{\varphi_2}\left(F_0+A_0^2e^{2bz}\right).$$

其中

$$F_{k-1}=\left(B_1+A_1'A_{k-1}+aA_1A_{k-1}-A_1A_{k-1}\frac{A'_0}{A_0}-bA_1A_{k-1}\right)e^{az}+\left(A_0A_{k-1}-2A'_0-2bA_0\right)e^{bz},$$

$$F_{k-2}=\left\{\left(2A'_{k-1}+A_{k-1}-\frac{A''_0}{A_0}-2b\frac{A'_0}{A_0}-b^2\right)B_2-\left(A_{k-1}-\frac{A'_0}{A_0}-b\right)B_1\right\}e^{az}+\left\{A_0\left(2A'_{k-1}+A_{k-1}-\frac{A''_0}{A_0}-2b\frac{A'_0}{A_0}-b^2\right)-2\left(A_{k-1}-\frac{A'_0}{A_0}-b\right)\left(A'_0+bA_0\right)\right\}e^{bz}$$

$$F_{k-j-1}=\left\{\left(A''_{k-1}+2A'_{k-j-1}+A_{k-j-2}-A_{k-j}\frac{A''_0}{A_0}-2bA_{k-j}\frac{A'_0}{A_0}-b^2A_{k-j}\right)B_2-\left(A'_{k-j}+A_{k-j-1}-A_{k-j}\frac{A'_0}{A_0}-bA_{k-j}\right)B_1\right\}e^{az}+\left\{A_0\left(A''_{k-1}+2A'_{k-j-1}+A_{k-j-2}-A_{k-j}\frac{A''_0}{A_0}-2bA_{k-j}\frac{A'_0}{A_0}-b^2A_{k-j}\right)-2\left(A'_0+bA_0\right)\left(A'_{k-j}+A_{k-j-1}-A_{k-j}\frac{A'_0}{A_0}-bA_{k-j}\right)\right\}e^{bz},$$

$\left(j=1,2,\cdots,k-3\right)$

$$F_1=\left\{\left(A_3''+2A_2'-\frac{A_0''}{A_0}-2bA_3\frac{A_0'}{A_0}-b^2A_3\right)B_2+\left(A_3'+A_2-A_3\frac{A_0'}{A_0}-bA_3\right)B_1\right\}e^{az}+A_1\left(A_1'+aA_1-A_1\frac{A_0'}{A_0}-bA_1\right)e^{2az}$$
$$+A_0A_1e^{(a+b)z}+\left\{A_0\left(A_3''+2A_2'-A_3\frac{A_0''}{A_0}-2bA_3\frac{A_0'}{A_0}-b^2A_3\right)-2\left(A_0'+bA_0\right)\left(A_3'+A_2-A_3\frac{A_0'}{A_0}-bA_3\right)\right\}e^{bz},$$

$$F_0=\left\{\left(A_2''-A_2\frac{A_0''}{A_0}-2bA_2\frac{A_0'}{A_0}-b^2A_2\right)B_2+\left(A_3'+A_2-A_3\frac{A_0'}{A_0}-bA_3\right)B_1\right\}e^{az}+\left\{A_0B_2+2A_0\left(A_1'+2aA_1\right)\right\}e^{(a+b)z}$$
$$+\left\{\left(A_2''-A_2\frac{A_0''}{A_0}+2bA_2\frac{A_0'}{A_0}+b^2A_2\right)A_0-2\left(A_0'+bA_0\right)\left(A_3'+A_2-A_3\frac{A_0'}{A_0}-bA_3\right)\right\}e^{bz}+2\left(A_1'+bA_1\right)B_2e^{2az}+A_0^2e^{2bz}.$$

故$\sigma\left(F_j\right)<1\left(j=0,1,2,\cdots,k-1\right)$。令

$$H=-\left(\varphi^{(k)}+H_{k-1}\varphi^{(k-1)}+H_{k-2}\varphi^{(k-2)}+\sum_{j=1}^{k-3}H_{k-j-1}\varphi^{(k-j-2)}+H_1\varphi'+H_0\varphi\right),$$

由方程(3.16)可知

$$h^{(k)}+H_{k-1}h^{(k-1)}+H_{k-2}h^{(k-2)}+\sum_{j=1}^{k-4}H_{k-j-1}h^{(k-j-2)}+H_1h'+H_0h=H\ . \tag{3.17}$$

因此

$$-\left(\frac{\varphi^{(k)}}{\varphi}+H_{k-1}\frac{\varphi^{(k-1)}}{\varphi}+H_{k-2}\frac{\varphi^{(k-2)}}{\varphi}+\sum_{j=1}^{k-3}H_{k-j-1}\frac{\varphi^{(k-j-2)}}{\varphi}+H_1\frac{\varphi'}{\varphi}+H_0\right)$$
$$=-\frac{1}{\varphi_2}\left(\frac{\varphi^{(k)}}{\varphi}\varphi_2+F_{k-1}\frac{\varphi^{(k-1)}}{\varphi}+F_{k-2}\frac{\varphi^{(k-2)}}{\varphi}+\sum_{j=1}^{k-3}F_{k-j-1}\frac{\varphi^{(k-j-2)}}{\varphi}+F_1\frac{\varphi'}{\varphi}+F_0+A_0^2e^{2bz}\right).$$

由前面的分析，可将H改写为如下形式

$$H=f_1(z)e^{az}+f_2(z)e^{bz}+f_3(z)e^{(a+b)z}+f_4(z)e^{2az}+f_5(z)e^{2bz}.$$

类似前面的方法可证明$\sigma\left(f_j(z)\right)<1\left(1\le j\le 5\right)$满足引理 4 的条件，如果$H\equiv 0$，分两种情况：如果$2a\ne b$，再由定理的条件$a\ne b$或$a\ne 2b$有 $f_5(z)\equiv 0$。类似地，如果$2a=b$，同样也可以得到结论 $f_5(z)\equiv 0$。

另一方面，我们知道 $f_5(z)=A_0^2(z)\not\equiv 0$。从而得出矛盾。即证明了$H\not\equiv 0$从而也有

$$\varphi^{(k)}+H_{k-1}\varphi^{(k-1)}+H_{k-2}\varphi^{(k-2)}+\sum_{j=1}^{k-3}H_{k-j-1}\varphi^{(k-j-2)}+H_1\varphi'+H_0\varphi\not\equiv 0\ .$$

所以，对于方程(3.17)由$H\not\equiv 0$及$\sigma(h)=\infty$，由引理 5 可知

$$\bar{\lambda}(h)=\bar{\lambda}(f''-\varphi)=\bar{\lambda}(f'')=\sigma(h)=\sigma(f)=\infty\ .$$

由(I)和(II)可知，在定理条件下有

$$\bar{\lambda}(f-\varphi)=\bar{\lambda}(f'-\varphi)=\bar{\lambda}(f''-\varphi)=\infty\ .$$

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