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Weighted L^p Estimates for the Bi-Harmonic Parabolic Equation

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Abstract

Schauder estimates and L^p estimates are the basic regularity estimates in the partial differential equations. In this paper we mainly study a new class of regularity estimates, weighted L^p estimates for the bi-harmonic parabolic equation.

Keywords

Bi-Harmonic Parabolic Equation, Weighted, L^p Estimates, Regularity Estimates

双调和抛物方程的加权L^p估计

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摘　要

Schauder估计和 L^p 估计是偏微分方程基本的正则性估计。本文我们主要研究双调和抛物方程的一类新的正则性估计——加权 L^p 估计。

关键词

双调和抛物方程，加权，L^p 估计，正则性估计

1. 引言

微分方程解的存在唯一性及其正则性是偏微分方程中的经典问题，其中 Schauder 估计和 L^p 估计在正则性理论的研究中具有非常重要的作用。2003 年，Wang [1]在 Caffarelli 和 Peral [2]基础上利用紧方法、Vitali 覆盖引理、极大函数等技巧给出了 Poisson 方程和热传导方程的 L^p 内估计的几何化方法证明。之后，Byun 和 Wang 利用类似的技巧在[3]-[7]中得到了各类二阶散度型椭圆方程与抛物型方程在不同的区域(Reifenberg, Lipschitz)区域等中的全局 L^p 估计。近几年，Mengesha 和 Phuc 在[8] [9]中应用类似的技术方法研究了散度型椭圆问题的全局加权 $W^{1,p}$ 估计。Byun，Palagachev 和 Ryu 在文章[10]中也讨论了散度型抛物方程在不光滑区域上的加权 $W^{1,p}$ 估计，另外，Yao 在[11]中讨论了 $p(x)$ 调和椭圆方程的梯度加权 L^p 估计。我们的工作主要是把相应的结论推广到双调和抛物型方程以及更高阶的多调和抛物型方程。

考虑下述四阶的双调和抛物方程 S

$$u_t+\Delta^2 u=f,\ \ z\in Q_2\subset\mathbb{R}^{n+1} \tag{1.1}$$

这里 $Q_r=B_r\times\left(-r^4,r^4\right)$，而 $Q_r\left(z_0\right)=Q_r+z_0$，其中 $z_0=\left(x_0,t_0\right)$。另外，我们记

$$\left|D^4u\right|=\left|D_x^4u(x,t)\right|=\left(\sum_{i,j,k,l=1,\cdots,n}\left|D_{x_ix_jx_kx_l}u(x,t)\right|^2\right)^{\frac{1}{2}} \text{ 且 } \left\|D^4u\right\|_{L^p\left(\mathbb{R}^{n+1}\right)}=\sum_{i,j,k,l=1,\cdots,n}\left\|D_{x_ix_jx_kx_l}u(x,t)\right\|_{L^p\left(\mathbb{R}^{n+1}\right)},$$

这里 $p>1$ 且 $\left\|D_{x_ix_jx_kx_l}u(x,t)\right\|_{L^p\left(\mathbb{R}^{n+1}\right)}=\left(\int_{\mathbb{R}^{n+1}}\left|D_{x_ix_jx_kx_l}u(x,t)\right|^p\mathrm{d}x\mathrm{d}t\right)^{1/p}$。

本文主要目的是得到问题(1.1)解 u 的如下估计

$$\int_{Q_1}\left|D_tu\right|^p w(z)\mathrm{d}z+\int_{Q_1}\left|D^4u\right|^p w(z)\mathrm{d}z\le C\left(\int_{Q_2}\left|D^4u\right|^2 w(z)\mathrm{d}z\right)^{\frac{p}{2}}+C\int_{Q_2}|f|^p w(z)\mathrm{d}z \ \ \text{任意的} p>2, \tag{1.2}$$

这里 $C>0$ 是不依赖于 u 和 f 的常数。事实上，如果 $w\equiv 1$，那么上述估计即为经典的 L^p 内估计。现在我们首先介绍一下加权空间的一些概念与性质(见[8]-[13])。

定义 1.1. 假设 $q>1$，我们称权函数 $w\in A_q$，如果 $w\in L^1_{loc}\left(\mathbb{R}^{n+1}\right)$，几乎处处 $w>0$，且存在常数 $C>0$ 使得满足下述不等式

$$\left(\int_{Q_r}w(z)\mathrm{d}z\right)\left(\int_{Q_r}w(z)^{\frac{-1}{q-1}}\mathrm{d}z\right)^{q-1}\le C,$$

对任意 $\mathbb{R}^{n+1}$ 中的 Q_r 上成立。我们记

$$A_\infty=\bigcup_{1<q<\infty}A_q \text{ 且 } w(Q_r)=\int_{Q_r}w(z)\mathrm{d}z.$$

另外，相应的加权空间 $L_w^q\left(Q_r\right)$ 是由满足 $\|h\|_{L_w^q(Q_r)}=\left(\int_{Q_r}|h|^q w(z)\mathrm{d}z\right)^{1/q}<\infty$ 条件的全体函数 h 组成。

注 1.2. 1) 事实上，我们可得 $A_{q_1}\subset A_{q_2}$，这里 $1<q_1\le q_2<\infty$。

2) 如果 $g(z)\in L_w^q$，那么，

$$\int_{\mathbb{R}^{n+1}}|g|^q w(z)\mathrm{d}z = q\int_0^\infty \mu^{q-1}w\left(\left\{z\in\mathbb{R}^{n+1}:|g|>\mu\right\}\right)\mathrm{d}\mu$$

$$=(q-q_2)\int_0^\infty \mu^{q-q_2-1}\left\{\int_{\{z\in\mathbb{R}^{n+1}:|g|>\mu\}}|g|^{q_2}w(z)\mathrm{d}z\right\}\mathrm{d}\mu \quad 任意 q>q_2.$$

3) 如果 $w(z)\in A_p$，那么，$w\left(\left\{z\in\mathbb{R}^{n+1}:\mathcal{M}g(z)>\mu\right\}\right)\le\frac{C}{\mu^p}\int_{\mathbb{R}^{n+1}}|g(z)|^p w(z)\mathrm{d}z$。

4) 如果 $q>q_1>1$，那么 $L_w^q(Q_r)\subset L^{q_1}(Q_r)$。

我们进一步给出重要的反 Hölder 不等式和 A_q 权的一些其它性质。

引理 1.3. 如果 $w\in A_q\ (q>1)$，那么存在常数 $C>0$，使得对任意 $Q_r\subset\mathbb{R}^{n+1}$ 有

$$\left(\frac{1}{|Q_r|}\int_{Q_r}w(z)^{1+\epsilon}\mathrm{d}z\right)^{\frac{1}{1+\epsilon}}\le C\frac{1}{|Q_r|}\int_{Q_r}w(z)\mathrm{d}z.$$

引理 1.4. 如果 $w\in A_q\ (q>1)$ 且 $Q_r\subset Q_R\subset\mathbb{R}^{n+1}$，那么存在常数 $C_1>0$ 有

$$\frac{w(Q_R)}{w(Q_r)}\le C_1\left(\frac{|Q_R|}{|Q_r|}\right)^q=C_1\left(\frac{R}{r}\right)^{(n+4)q}.$$

引理 1.5. 如果 $w\in A_q\ (q>1)$ 且 $Q_r\subset Q_R\subset\mathbb{R}^{n+1}$，那么存在常数 $\sigma>0$ 及 $C>0$ 使得

$$\frac{w(Q_r)}{w(Q_R)}\le C\left(\frac{|Q_r|}{|Q_R|}\right)^\sigma=C\left(\frac{r}{R}\right)^{(n+4)\sigma}.$$

现在我们给出本文的主要结论。

定理 1.6. 假设 $w\in A_p\ (p>2)$ 且 $f\in L_w^p(Q_2)$。如果 u 满足双调和抛物方程且 $|D^4u|\in L_w^2$，那么我们可得加权 L^p 估计(1.2)式。

注 1.7. 上述定理的结论可以推广到更高阶的多调和抛物型方程，即方程

$$u_t+(-\Delta)^m u=f,\ z\in Q_2\subset\mathbb{R}^{n+1} \tag{1.3}$$

有类似于定理 1.6 的加权 L^p 估计

$$\int_{Q_1}|D_tu|^p w(z)\mathrm{d}z+\int_{Q_1}|D^{2m}u|^p w(z)\mathrm{d}z\le C\left(\int_{Q_2}|D^{2m}u|^2 w(z)\mathrm{d}z\right)^{\frac{p}{2}}+C\int_{Q_2}|f|^p w(z)\mathrm{d}z \quad 任意 p>2. \tag{1.4}$$

2. 主要结论的证明

本节我们将完成主要结论定理 1.6 的证明。现在我们记

$$\lambda_0^2=\frac{1}{w(Q_2)}\int_{Q_2}|D^4u|^2 w(z)\mathrm{d}z+\frac{1}{\delta w(Q_2)}\int_{Q_2}|f|^2 w(z)\mathrm{d}z, \tag{2.1}$$

这里 $\delta\in(0,1)$ 是一个充分小的待定正常数。令

$$f_\lambda=f/\lambda,\ u_\lambda=u/\lambda \text{ 且 } J_\lambda[Q]=\frac{1}{w(Q)}\int_Q|D^4u_\lambda|^2 w(z)\mathrm{d}z+\frac{1}{\delta w(Q)}\int_Q|f_\lambda|^2 w(z)\mathrm{d}z \tag{2.2}$$

对任意的 $\lambda>0$ 和 $Q\subset\mathbb{R}^{n+1}$。另外，我们把水平集 $E_\lambda(1)$ 定义为 $E_\lambda(1)=\left\{z\in Q_1:|D^4u_\lambda|>1\right\}$.

下面，我们将分解水平集 $E_\lambda(1)$ 并给出相应的性质。

引理 2.1. 任意给定 $\lambda>\left(C_1 20^{(n+4)p}\right)^{1/2}\lambda_0$。如果 $w(z)\in A_p$，那么存在一族互不相交的 $\{Q_{\rho_i}(z_i)\}_{i\ge1}$，使

得

$$J_\lambda\left[Q_{\rho_i}(z_i)\right]=1,\ J_\lambda\left[Q_\rho(z_i)\right]<1\quad \text{任意的}\rho>\rho_i; \tag{2.3}$$

且

$$E_\lambda(1)\subset\bigcup_{i\ge1}Q_{5\rho_i}(z_i)\cup\text{零测集}, \tag{2.4}$$

这里 $z_i\in E_\lambda(1)$ 且 $\rho_i=\rho(z_i)\in\left(0,\frac{1}{10}\right)$。进一步，我们可得

$$w\left(Q_{\rho_i}(z_i)\right)\le2\int_{\left\{z\in Q_{\rho_i}(z_i):\left|D^4u_\lambda\right|^2>1/4\right\}}\left|D^4u_\lambda\right|^2w(z)\mathrm{d}z+\frac{2}{\delta}\int_{\left\{z\in Q_{\rho_i}(z_i):\left|f_\lambda\right|^2>\delta/4\right\}}\left|f_\lambda\right|^2w(z)\mathrm{d}z. \tag{2.5}$$

证明：1) 首先，对几乎处处的 $z\in E_\lambda(1)$，由基本的测度理论我们可得

$$\begin{aligned}\lim_{\rho\to0}J_\lambda\left[Q_\rho(z)\right]&=\lim_{\rho\to0}\left(\frac{1}{w\left(Q_{\rho(z)}\right)}\int_{Q_{\rho(z)}}\left|D^4u_\lambda\right|^2w(z)\mathrm{d}z+\frac{1}{\delta w\left(Q_{\rho(z)}\right)}\int_{Q_{\rho(z)}}\left|f_\lambda\right|^2w(z)\mathrm{d}z\right)\\&>\lim_{\rho\to0}\frac{1}{w\left(Q_{\rho(z)}\right)}\int_{Q_{\rho(z)}}w(z)\mathrm{d}z>1.\end{aligned}$$

故存在某个 $0<\rho<1/10$ 满足：

$$J_\lambda\left[Q_\rho(z)\right]>1. \tag{2.6}$$

另外，对某一固定的 $z\in Q_1$，且任意的 $\rho\in\left[\frac{1}{10},1\right]$ 由(2.1)、(2.2)和引理 1.4，可得

$$\begin{aligned}J_\lambda\left[Q_\rho(z)\right]&\le\frac{w(Q_2)}{w\left(Q_\rho(z)\right)}\left(\frac{1}{w(Q_2)}\int_{Q_2}\left|D^4u_\lambda\right|^2w(z)\mathrm{d}z+\frac{1}{\delta w(Q_2)}\int_{Q_2}\left|f_\lambda\right|^2w(z)\mathrm{d}z\right)\\&\le C_1\left(\frac{|Q_2|}{\left|Q_\rho(z)\right|}\right)^p\frac{\lambda_0^2}{\lambda^2}\le C_1 20^{(n+4)p}\frac{\lambda_0^2}{\lambda^2}.\end{aligned}$$

当 $\lambda>\left(C_1 20^{(n+4)p}\right)^{1/2}\lambda_0$ 时，进一步可得：$J_\lambda\left[Q_\rho(z)\right]\le1$。

所以，我们有

$$\sup_{z\in Q_1}\sup_{\rho\in\left[\frac{1}{10},1\right]}J_\lambda\left[Q_\rho(z)\right]\le1\text{。}$$

因此，由上式和(2.6)式可知，存在 $\rho_z\in\left(0,\frac{1}{10}\right)$ 使得：

$$J_\lambda\left[Q_{\rho_z}(z)\right]=1,\quad J_\lambda\left[Q_\rho(z)\right]<1\quad\text{任意的 }\rho>\rho_z\text{。}$$

综上所述，可得对几乎处处的 $z\in E_\lambda(1)$ 都存在满足如上条件的区域 $Q_{\rho_z}(z)$。然后，利用 Vitali 覆盖引理，我们找到一族互不相交的 $\left\{Q_{\rho_i}(z_i)\right\}_{i\ge1}$ 并且满足(2.3)和(2.4)。从而第一部分结论得证！

2) 由(2.2)和(2.3)，可得：

$$w\left(Q_{\rho_i}(z_i)\right)=\int_{Q_{\rho_i}(z_i)}\left|D^4u_\lambda\right|^2w(z)\mathrm{d}z+\frac{1}{\delta}\int_{Q_{\rho_i}(z_i)}\left|f_\lambda\right|^2w(z)\mathrm{d}z\text{。}$$

再将上式中右边的二个积分都拆为两项，可得

$$w\left(Q_{\rho_i}(z_i)\right) \le \int_{\left\{z\in Q_{\rho_i}(z_i):\left|D^4u_\lambda\right|^2>1/4\right\}}\left|D^4u_\lambda\right|^2 w(z)\mathrm{d}z + w\left(Q_{\rho_i}(z_i)\right)/4$$
$$+\frac{1}{\delta}\int_{\left\{z\in Q_{\rho_i}(z_i):|f_\lambda|^2>\delta/4\right\}}|f_\lambda|^2 w(z)\mathrm{d}z + w\left(Q_{\rho_i}(z_i)\right)/4.$$

整理后即可得第二部分的结论!

事实上，由上面的引理可以得到如下两个不等式：

$$\frac{1}{w\left(Q_\rho(z_i)\right)}\int_{Q_\rho(z_i)}\left|D^4u_\lambda\right|^2 w(z)\mathrm{d}z < 1 \quad 任意的\rho>\rho_i, \tag{2.7}$$

$$\frac{1}{w\left(Q_\rho(z_i)\right)}\int_{Q_\rho(z_i)}|f_\lambda|^2 w(z)\mathrm{d}z < \delta \quad 任意的\rho>\rho_i. \tag{2.8}$$

进一步，由上面的不等式我们还可以得如下引理。

引理 2.2. 假设$1<p_1<2$且$\lambda>\left(C_1 20^{(n+4)p}\right)^{1/2}\lambda_0$。那么存在常数$C>0$使得

$$\frac{1}{\left|Q_\rho(z_i)\right|}\int_{Q_\rho(z_i)}\left|D^4u_\lambda\right|^{p_1}\mathrm{d}z \le C \text{ 且 } \frac{1}{\left|Q_\rho(z_i)\right|}\int_{Q_\rho(z_i)}|f_\lambda|^{p_1}\mathrm{d}z \le C\delta^{\frac{p_1}{2}} \quad 任意的\rho>\rho_i.$$

证明：这里我们只证明第一式，第二式同理可证。对任意的$\rho>\rho_i$，由 Hölder 不等式可得

$$\frac{1}{\left|Q_\rho(z_i)\right|}\int_{Q_\rho(z_i)}\left|D^4u_\lambda\right|^{p_1}\mathrm{d}z = \frac{1}{\left|Q_\rho(z_i)\right|}\int_{Q_\rho(z_i)}\left|D^4u_\lambda\right|^{p_1} w(z)^{\frac{p_1}{2}} w(z)^{-\frac{p_1}{2}}\mathrm{d}z$$
$$\le\left(\frac{1}{\left|Q_\rho(z_i)\right|}\int_{Q_\rho(z_i)}\left|D^4u_\lambda\right|^2 w(z)\mathrm{d}z\right)^{\frac{p_1}{2}}\left(\frac{1}{\left|Q_\rho(z_i)\right|}\int_{Q_\rho(z_i)} w(z)^{-\frac{1}{\frac{2}{p_1}-1}}\mathrm{d}z\right)^{1-\frac{p_1}{2}}.$$
$$=\left(\frac{1}{w\left(Q_\rho(z_i)\right)}\int_{Q_\rho(z_i)}\left|D^4u_\lambda\right|^2 w(z)\mathrm{d}z\right)^{\frac{p_1}{2}}\left(\frac{w\left(Q_\rho(z_i)\right)}{\left|Q_\rho(z_i)\right|}\right)^{\frac{p_1}{2}}\left(\frac{1}{\left|Q_\rho(z_i)\right|}\int_{Q_\rho(z_i)} w(z)^{-\frac{1}{\frac{2}{p_1}-1}}\mathrm{d}z\right)^{1-\frac{p_1}{2}}.$$

由定义 1.1 和(2.7)式，并且$A_{2/p_1}\subset A_2\subset A_p$，可得

$$\frac{1}{\left|Q_\rho(z_i)\right|}\int_{Q_\rho(z_i)}\left|D^4u_\lambda\right|^{p_1}\mathrm{d}z < \left[\left(\frac{1}{\left|Q_\rho(z_i)\right|}\int_{Q_\rho(z_i)} w(z)\mathrm{d}z\right)\left(\frac{1}{\left|Q_\rho(z_i)\right|}\int_{Q_\rho(z_i)} w(z)^{-\frac{1}{\frac{2}{p_1}-1}}\mathrm{d}z\right)^{\frac{2}{p_1}-1}\right]^{\frac{p_1}{2}} \le C.$$

最后，我们将完成主要结论：定理 1.6 的证明。通过基本的逼近理论(见[14]-[17])，我们不妨假设$\left|D^4u\right|\in L_w^p(Q_2)$。

证明：对于固定的$i\ge 1$，我们首先对f_λ作从$Q_{10\rho_i}(z_i)$到$\mathbb{R}^n\times(0,T]$上的零延拓，并记为$\overline{f_\lambda}$。则$\overline{f_\lambda}\in L^2\left(\mathbb{R}^n\times(0,T]\right)$，现在我们引入下述参照方程

$$\begin{cases} w_t+\Delta^2 w=\overline{f_\lambda}, & z\in\mathbb{R}^n\times(0,T], \\ w=0, & z\in\mathbb{R}^n\times\{t=0\}. \end{cases}$$

由 L^p 估计(见[18])，我们可得：$\left\|D^4u\right\|_{L^{p_1}\left(\mathbb{R}^n\times(0,T]\right)}\le C\left\|\overline{f_\lambda}\right\|_{L^{p_1}\left(\mathbb{R}^n\times(0,T]\right)}$.

从而可得

$$\int_{Q_{10\rho_i}(z_i)}\left|D^4w\right|^{p_1}\mathrm{d}z\le\int_{\mathbb{R}^n\times(0,T]}\left|D^4w\right|^{p_1}\mathrm{d}z\le C\int_{\mathbb{R}^n\times(0,T]}\left|\overline{f_\lambda}\right|^{p_1}\mathrm{d}z=C\int_{Q_{10\rho_i}(z_i)}\left|f_\lambda\right|^{p_1}\mathrm{d}z.$$

由上式及引理 2.2 得

$$\frac{1}{\left|Q_{10\rho_i}(z_i)\right|}\int_{Q_{10\rho}(z_i)}\left|D^4w\right|^{p_1}\mathrm{d}z\le C\frac{1}{\left|Q_{10\rho_i}(z_i)\right|}\int_{Q_{10\rho}(z_i)}\left|f_\lambda\right|^{p_1}\mathrm{d}z\le C\delta^{\frac{p_1}{2}}\le C \tag{2.9}$$

令 $v=u_\lambda-w$，可知 $v_t+\Delta^2v=0, z\in Q_{10\rho_i}(z_i)$。
由引理 2.2 和(2.9)可得

$$\frac{1}{\left|Q_{10\rho_i}(z_i)\right|}\int_{Q_{10\rho}(z_i)}\left|D^4v\right|^{p_1}\mathrm{d}z\le 2^{p_1-1}\left\{\frac{1}{\left|Q_{10\rho_i}(z_i)\right|}\int_{Q_{10\rho}(z_i)}\left|D^4w\right|^{p_1}\mathrm{d}z+\frac{1}{\left|Q_{10\rho_i}(z_i)\right|}\int_{Q_{10\rho}(z_i)}\left|D^4u_\lambda\right|^{p_1}\mathrm{d}z\right\}\le C.$$

故由基本的 $W_{loc}^{4,\infty}$ 内部正则性理论(见[19])，可得：$\sup\limits_{Q_{5\rho_i}(z_i)}\left|D^4v\right|\le N_1$，这里取 $N_1>1$。利用(2.9)和上式我们可得

$$\begin{aligned}
&\left|\left\{z\in Q_{5\rho_i}(z_i):\left|D^4u\right|>2N_1\lambda\right\}\right|=\left|\left\{z\in Q_{5\rho_i}(z_i):\left|D^4u_\lambda\right|>2N_1\right\}\right|\\
&\le\left|\left\{z\in Q_{5\rho_i}(z_i):\left|D^4w\right|>N_1\right\}\right|+\left|\left\{z\in Q_{5\rho_i}(z_i):\left|D^4v\right|>N_1\right\}\right|\\
&=\left|\left\{z\in Q_{5\rho_i}(z_i):\left|D^4w\right|>N_1\right\}\right|\le\frac{1}{N_1^{p_1}}\int_{Q_{5\rho_i}(z_i)}\left|D^4w\right|^{p_1}\mathrm{d}z\\
&\le\int_{Q_{10\rho_i}(z_i)}\left|D^4w\right|^{p_1}\mathrm{d}z\le C\int_{Q_{10\rho_i}(z_i)}\left|f_\lambda\right|^{p_1}\mathrm{d}z\\
&\le C\delta^{\frac{p_1}{2}}\left|Q_{10\rho_i}(z_i)\right|=C\delta^{\frac{p_1}{2}}\left|Q_{5\rho_i}(z_i)\right|.
\end{aligned}$$

那么由引理 1.4 和引理 1.5 及上式，可得

$$\frac{w\left(\left\{z\in Q_{5\rho_i}(z_i):\left|D^4u\right|>2N_1\lambda\right\}\right)}{w\left(Q_{\rho_i}(z_i)\right)}=\frac{w\left(\left\{z\in Q_{5\rho_i}(z_i):\left|D^4u\right|>2N_1\lambda\right\}\right)}{w\left(Q_{5\rho_i}(z_i)\right)}\frac{w\left(Q_{5\rho_i}(z_i)\right)}{w\left(Q_{\rho_i}(z_i)\right)}\le C\delta^{\frac{p_1\sigma}{2}}.$$

即：

$$w\left(\left\{z\in Q_{5\rho_i}(z_i):\left|D^4u\right|>2N_1\lambda\right\}\right)\le C\delta^{\frac{p_1\sigma}{2}}w\left(Q_{\rho_i}(z_i)\right)。$$

现在记 $\lambda^*=\left(C_1 20^{(n+4)p}\right)^{1/2}\lambda_0$，则对任意的 $\lambda>\lambda^*$，利用引理 2.1 中的(2.5)并将(2.2)式代入上式，可得

$$\begin{aligned}
&w\left(\left\{z\in Q_{5\rho_i}(z_i):\left|D^4u\right|>2N_1\lambda\right\}\right)\le C\delta^{\frac{p_1\sigma}{2}}w\left(Q_{\rho_i}(z_i)\right)\\
&\le\frac{C\delta^{\frac{p_1\sigma}{2}}}{\lambda^2}\left(\int_{\left\{z\in Q_{\rho_i}(z_i):\left|D^4u\right|^2>\lambda^2/4\right\}}\left|D^4u\right|^2w(z)\mathrm{d}z+\frac{1}{\delta}\int_{\left\{z\in Q_{\rho_i}(z_i):\left|f\right|^2>\lambda^2\delta/4\right\}}\left|f\right|^2w(z)\mathrm{d}z\right).
\end{aligned}$$

又因为 $\left\{Q_{\rho_i}(z_i)\right\}_{i\ge1}$ 的互不相交性和 $E_\lambda(1)=\left\{z\in Q_1:\left|D^4u_\lambda\right|>1\right\}\subset\bigcup\limits_{i\ge1}Q_{5\rho_i}(z_i)\cup$零测集，

我们可得：

$$
\begin{aligned}
& w\left(\left\{z\in Q_1:\left|D^4u\right|>2N_1\lambda\right\}\right)\le\sum_{i=1}^{\infty}w\left(\left\{z\in Q_{5\rho_i}\left(z_i\right):\left|D^4u\right|>2N_1\lambda\right\}\right) \\
& \le\frac{C\delta^{\frac{p_1\sigma}{2}}}{\lambda^2}\left(\int_{\left\{z\in Q_2:\left|D^4u\right|^2>\lambda^2/4\right\}}\left|D^4u\right|^2 w(z)\mathrm{d}z+\frac{1}{\delta}\int_{\left\{z\in Q_2:|f|^2>\lambda^2\delta/4\right\}}|f|^2 w(z)\mathrm{d}z\right).
\end{aligned}
\tag{2.10}
$$

利用注 1.2 计算可得

$$
\begin{aligned}
\int_{Q_1}\left|D^4u\right|^p w(z)\mathrm{d}z &= p\int_0^{\infty}\mu^{p-1}w\left(\left\{z\in Q_1:\left|D^4u\right|>\mu\right\}\right)\mathrm{d}\mu \\
&= p\int_0^{\infty}(2N_1\lambda)^{p-1}w\left(\left\{z\in Q_1:\left|D^4u\right|>2N_1\lambda\right\}\right)\mathrm{d}\left[2N_1\lambda\right] \\
&= (2N_1)^p\, p\int_0^{\infty}\lambda^{p-1}w\left(\left\{z\in Q_1:\left|D^4u\right|>2N_1\lambda\right\}\right)\mathrm{d}\lambda \\
&= (2N_1)^p\, p\left\{\int_0^{\lambda^*}+\int_{\lambda^*}^{\infty}\right\}\lambda^{p-1}w\left(\left\{z\in Q_1:\left|D^4u\right|>2N_1\lambda\right\}\right)\mathrm{d}\lambda \\
&= I_1+I_2.
\end{aligned}
$$

I_1 的估计。事实上，首先可得下面不等式

$$
\left(\int_{Q_2}|f|^2 w(z)\mathrm{d}z\right)^{\frac{p}{2}}=\left(\int_{Q_2}|f|^2 w(z)^{\frac{2}{p}}w(z)^{1-\frac{2}{p}}\mathrm{d}z\right)^{\frac{p}{2}}\le C\left(\int_{Q_2}|f|^p w(z)\mathrm{d}z\right)\cdot\left(\int_{Q_2}w(z)\mathrm{d}z\right)^{\frac{p-2}{2}}\le C\int_{Q_2}|f|^p w(z)\mathrm{d}z.
$$

再结合 λ_0 的定义，可得

$$
\begin{aligned}
I_1 &= (2N_1)^p\, p\int_0^{\lambda^*}\lambda^{p-1}w\left(\left\{z\in Q_1:\left|D^4u\right|>2N_1\lambda\right\}\right)\mathrm{d}\lambda \\
&\le Cw(Q_1)\lambda_0^p\le C\left(\int_{Q_2}\left|D^4u\right|^2 w(z)\mathrm{d}z+\frac{1}{\delta}\int_{Q_2}|f|^2 w(z)\mathrm{d}z\right)^{\frac{p}{2}} \\
&\le C\left(\int_{Q_2}\left|D^4u\right|^2 w(z)\mathrm{d}z\right)^{\frac{p}{2}}+C(\delta)\left(\int_{Q_2}|f|^2 w(z)\mathrm{d}z\right)^{\frac{p}{2}} \\
&\le C\left(\int_{Q_2}\left|D^4u\right|^2 w(z)\mathrm{d}z\right)^{\frac{p}{2}}+C(\delta)\int_{Q_2}|f|^p w(z)\mathrm{d}z,
\end{aligned}
$$

这里 $C(\delta)$ 是指仅依赖于 δ 的常数。

I_2 的估计，由于 $\lambda>\lambda^*$，利用(2.10)及注 1.2 中的(2)，可得

$$
\begin{aligned}
I_2 &= (2N_1)^p\, p\int_{\lambda^*}^{\infty}\lambda^{p-1}w\left(\left\{z\in Q_1:\left|D^4u\right|>2N_1\lambda\right\}\right)\mathrm{d}\lambda \\
&\le C\delta^{\frac{p_1\sigma}{2}}p\int_0^{\infty}\lambda^{p-3}\left\{\int_{\left\{z\in Q_2:\left|D^4u\right|^2>\lambda^2/4\right\}}\left|D^4u\right|^2\mathrm{d}z\right\}\mathrm{d}\lambda \\
&\quad +C\delta^{\frac{p_1\sigma}{2}}p\int_0^{\infty}\lambda^{p-3}\left\{\frac{1}{\delta}\int_{\left\{z\in Q_2:|f|^2>\lambda^2\delta/4\right\}}|f|^2\mathrm{d}z\right\}\mathrm{d}\lambda \\
&\le C\delta^{\frac{p_1\sigma}{2}}\int_{Q_2}\left|D^4u\right|^p w(z)\mathrm{d}z+C(\delta)\int_{Q_2}|f|^p w(z)\mathrm{d}z.
\end{aligned}
$$

结合 I_1 与 I_2 的估计，我们可得

$$
\int_{Q_1}\left|D^4u\right|^p w(z)\mathrm{d}z\le C\delta^{\frac{p_1\sigma}{2}}\int_{Q_2}\left|D^4u\right|^p w(z)\mathrm{d}z+C\left(\int_{Q_2}\left|D^4u\right|^2 w(z)\mathrm{d}z\right)^{\frac{p}{2}}+C(\delta)\int_{Q_2}|f|^p w(z)\mathrm{d}z.
$$

选取合适的 $\delta>0$ 满足 $C\delta^{\frac{p_1\sigma}{2}}<1/2$，利用基本的迭代引理(见[19] [20])吸收掉上式右端的第一个积分，从而可得

$$\int_{Q_1}\left|D^4u\right|^p w(z)\mathrm{d}z \le C\left(\int_{Q_2}\left|D^4u\right|^2 w(z)\mathrm{d}z\right)^{\frac{p}{2}}+C\int_{Q_2}|f|^p w(z)\mathrm{d}z \quad \text{任意的} p>2.$$

再由方程(1.1)即可得定理 1.6 的结论。

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