Pure Mathematics 理论数学, 2020, 10(9), 906-913
Published Online September 2020 in Hans. http://www.hanspub.org/journal/pm
https://doi.org/10.12677/pm.2020.109105

Hans汉斯

曲率流的拼挤估计

刘晋鹏，阿迪拉·阿布都热依木，米雅薇

新疆师范大学，新疆 乌鲁木齐
Email: 1239343981@qq.com

收稿日期：2020年8月30日；录用日期：2020年9月20日；发布日期：2020年9月27日

摘 要

我们通过对平均曲率流的拼挤估计，得到对一般数量曲率流的拼挤估计。对于 $\frac{\partial}{\partial t}X(x,t)=\sigma_k^{1/k}n$，当 $k=1$ 时，$\frac{\partial}{\partial t}X(x,t)=Hn$，此时$H$是平均曲率流；当 $k=2$ 时，$\frac{\partial}{\partial t}X(x,t)=R^{\frac{1}{2}}n$，$R^{\frac{1}{2}}$ 是常数量曲率流，本文得到 $k=2$ 时的拼挤估计。

关键词

平均曲率流，拼挤估计，常数量

Squeezing Estimation of Curvature Flow

Jinpeng Liu, Adila·Abudureyimu, Yawei Mi

Xinjiang Normal University, Urumqi Xinjiang
Email: 1239343981@qq.com

Received: Aug. 30th, 2020; accepted: Sep. 20th, 2020; published: Sep. 27th, 2020

Abstract

In this paper, we focus on the estimation of the numerical curvature flow and some related problems. By means of the mean curvature flow squeezing estimation, we get the squeezing estimation of the general number of curvature flow squeezing. For $\frac{\partial}{\partial t}X(x,t)=\sigma_k^{1/k}n$, when $k=1$, $\frac{\partial}{\partial t}X(x,t)=Hn$, where H is the average curvature flow; When $k=2$, $\frac{\partial}{\partial t}X(x,t)=R^{\frac{1}{2}}n$, $R^{\frac{1}{2}}$ is a constant number of curvature flows. In this paper, the squeezing estimate is obtained when $k=2$.

文章引用: 刘晋鹏, 阿迪拉·阿布都热依木, 米雅薇. 曲率流的拼挤估计[J]. 理论数学, 2020, 10(9): 906-913.
DOI: 10.12677/pm.2020.109105

Keywords

Mean Curvature Flows, Crowding Estimation, Often the Number

1. 引言

在现代微分几何中偏微分方程是一种非常有力的工具。特别的，通过抛物极大值原理作为主要工具，抛物发展方程(几何热流)已经成功的应用于研究流行的几何量。一些学者已经对平均曲率流的拼挤估计[1] [2]进行了研究[3]，得出的先验条件[4] [5]，本文对满足先验条件的数量曲率流进行了拼挤估计。

2. 预备知识

设 M^n 是一个 n 维光滑流形， $X(\cdot,t):M^n\to R^{n+1}$ 是一个 R^{n+1} 中的光滑浸入超曲面[6]。对于 $\frac{\partial}{\partial t}X(x,t)=\sigma_k^{1/k}\boldsymbol{n}$，如果 $k=2$，则 $\frac{\partial}{\partial t}X(x,t)=R^{\frac{1}{2}}\boldsymbol{n}$ ($x\in M^n$， $t>0$)。其中 $R^{\frac{1}{2}}$ 和 $\boldsymbol{n}$ 分别是常数量曲率流和 $X(\cdot,t)$ 的单位法向量。在局部坐标系 $\{x_i\}$ 下， $X(\cdot,t)$ 的度量和第二基本形式为：

$g_{ij}(x,t)=\left(\frac{\partial X(x,t)}{\partial x^i},\frac{\partial X(x,t)}{\partial x^j}\right)$； $h_{ij}(x,t)=\left(\boldsymbol{n}(x,t),\frac{\partial^2 X(x,t)}{\partial x^i\partial x^j}\right)$。在 $X(\cdot,t)$ 上的联络系数为 $\Gamma_{ij}^k=\frac{1}{2}g^{kl}\left(\frac{\partial}{\partial x^i}g_{jl}+\frac{\partial}{\partial x^j}g_{jl}-\frac{\partial}{\partial x^l}g_{ij}\right)$；向量 v 在 $X(\cdot,t)$ 上的协变导数为 $\nabla_j v^i=\frac{\partial}{\partial x^j}v^i+\Gamma_{jk}^i v^k$。黎曼集合张量，里奇张量和标量曲率通过高斯方程给定： $R_{ijkl}=h_{ik}h_{jl}-h_{il}h_{jk}$； $R_{ik}=Hh_{ik}-h_{il}g^{lj}h_{jk}$； $R=H^2-|A|^2$，其中 $|A|^2=g^{ij}g^{kl}h_{ik}h_{jl}$。

回忆 Guass-Weingarten 关系式为 $\frac{\partial^2 X}{\partial x^i\partial x^j}=\Gamma_{ij}^k\frac{\partial X}{\partial x^k}+h_{ij}\boldsymbol{n}$； $\frac{\partial \boldsymbol{n}}{\partial x^j}=-h_{jl}g^{lm}\frac{\partial X}{\partial x^m}$。

引理 1 (Hamiltion [4] [7])：极大值原理是研究抛物方程的有用的工具。现在我们提出一个关于张量的极大值原理。设对称张量 M_{ij}，如果对所有向量 v^i 有 $M_{ij}v^iv^j\geq 0$，则 $M_{ij}\geq 0$。让 $N_{ij}=P(M_{ij},g_{ij})$ 是 M_{ij} 中的多项式，是 M_{ij} 和它自己的度量的乘积形成的。

假设 $0\leq t\leq T$，

$$\frac{\partial}{\partial t}M_{ij}=\Delta M_{ij}+u^k\nabla_k M_{ij}+N_{ij}$$

其中 $N_{ij}=P(M_{ij},g_{ij})$ 满足

$N_{ij}v^iv^j\geq 0$，无论何时 $M_{ij}v^j=0$

如果在 $t=0$ 时 $M_{ij}\geq 0$，则在 $0\leq t\leq T$ 时 $M_{ij}\geq 0$ 仍然成立。

3. 主要结论及其证明

根据 $\frac{\partial}{\partial t}X(x,t)=R^{\frac{1}{2}}\boldsymbol{n}$，为了得到关于超曲面 $X(\cdot,t)$ 的几何量发展方程，需要下面几个引理：

引理 2： $\frac{1}{2}\nabla_i\nabla_j R = \Box h_{ij} + \nabla_i H\nabla_j H - \nabla_i h_{kl}\nabla_j h_{mn} + \left(H|A|^2 - C\right)h_{ij} - Rh_{im}h_j^m$

证明：

$$
\begin{aligned}
\frac{1}{2}\nabla_i\nabla_j R &= H\nabla_i\nabla_j H + \nabla_i H\nabla_j H - \frac{1}{2}\nabla_i\nabla_j\left(g^{km}g^{ln}h_{kl}h_{mn}\right) \\
&= Hg^{kl}\nabla_i\nabla_j h_{kl} - g^{km}g^{ln}h_{mn}\nabla_i\nabla_j h_{kl} + \nabla_i H\nabla_j H - g^{km}g^{ln}\nabla_i h_{kl}\nabla_j h_{mn} \\
&= \left(Hg^{kl} - h^{kl}\right)\nabla_i\nabla_j h_{kl} + \nabla_i H\nabla_j H - g^{km}g^{ln}\nabla_i h_{kl}\nabla_j h_{mn} \\
&= \left(Hg^{kl} - h^{kl}\right)\nabla_k\nabla_l h_{ij} + \left(Hg^{kl} - h^{kl}\right)\left(R_{ikjm}h_l^m + R_{iklm}h_j^m\right) + \nabla_i H\nabla_j H - g^{km}g^{ln}\nabla_i h_{kl}\nabla_j h_{mn} \\
&= \Box h_{ij} + \nabla_i\nabla_j H - g^{km}g^{ln}\nabla_i h_{kl}\nabla_j h_{mn} + *
\end{aligned}
$$

其中：

$$
\begin{aligned}
* &= \left(Hg^{kl} - h^{kl}\right)\left[\left(h_{ij}h_{km} - h_{im}h_{kj}\right)h_l^m + \left(h_{il}h_{km} - h_{im}h_{kl}\right)h_j^m\right] \\
&= Hg^{kl}h_{km}h_l^m h_{ij} - h^{kl}h_{ij}h_{km}h_l^m - Hg^{kl}h_{im}h_{kj}h_l^m + h^{kl}h_{im}h_{kj}h_l^m + Hg^{kl}h_{il}h_{km}h_j^m - h^{kl}h_{il}h_{km}h_j^m - Hg^{kl}h_{im}h_{kl}h_j^m \\
&= H|A|^2 h_{ij} - h^{kl}h_l^m h_{mk}h_{ij} - H^2 h_{im}h_j^m + |A|^2 h_{im}h_j^m \\
&= \left(H|A|^2 - h^{kl}h_l^m h_{mk}\right)h_{ij} - Rh_{im}h_j^m \\
&= \left(H|A|^2 - C\right)h_{ij} - Rh_{im}h_j^m
\end{aligned}
$$

引理 3： $-H^2\nabla\left(\frac{|A|^2}{H^2}\right) + \frac{2R}{H}\nabla H = \nabla R$

$$
\begin{aligned}
&-H^2\nabla\left(\frac{|A|^2}{H^2}\right) + \frac{2R}{H}\nabla H \\
&= -H^2\left(\frac{\nabla|A|^2}{H^2} - \frac{2|A|^2\nabla H}{H^3}\right) + \frac{2H^2\nabla H - 2|A|^2\nabla H}{H}
\end{aligned}
$$

证明：

$$
\begin{aligned}
&= -\nabla|A|^2 + 2H\nabla H \\
&= \nabla\left(H^2 - |A|^2\right) \\
&= \nabla R
\end{aligned}
$$

引理 4：

$$
-\frac{1}{H^2}\left(H\nabla_i h_{kl} - \nabla_i H h_{kl}\right)\left(H\nabla_j h_{kl} - \nabla_j H h_{kl}\right) = -\nabla_i h_{kl}\nabla_j h_{kl} - \frac{|A|^2}{H^2}\nabla_i H\nabla_j H + 2\frac{\nabla_j H}{H}h_{kl}\nabla_i h_{kl}
$$

$$
-\frac{1}{H^2}\left(H\nabla_i h_{kl} - \nabla_i H h_{kl}\right)\left(H\nabla_j h_{kl} - \nabla_j H h_{kl}\right)
$$

证明：

$$
\begin{aligned}
&= -\frac{1}{H^2}\left(H^2\nabla_i h_{kl}\nabla_j h_{kl} - 2H\nabla_j H h_{kl}\nabla_i h_{kl} + \nabla_i H\nabla_j H|A|^2\right) \\
&= -\nabla_i h_{kl}\nabla_j h_{kl} - \frac{|A|^2}{H^2}\nabla_i H\nabla_j H + 2\frac{\nabla_j H}{H}h_{kl}\nabla_i h_{kl}
\end{aligned}
$$

引理 5： $\frac{\partial}{\partial t}H^2 = R^{-\frac{1}{2}}\left(\Box H^2 - 2|\nabla H|^2_{Hg-h} - \frac{2}{H}\left|H\nabla h_{kl} - \nabla H h_{kl}\right|^2 - \frac{H^5}{2R}\left|\nabla\frac{|A|^2}{H^2}\right|^2 + 2\left(H|A|^2 - C\right)H^2\right)$

$$\frac{\partial}{\partial t}H=\frac{\partial}{\partial t}g^{ij}h_{ij}+g^{ij}\frac{\partial}{\partial t}h_{ij}$$

证明：
$$=2R^{\frac{1}{2}}|A|^2+R^{-\frac{1}{2}}\left(\Box H-\frac{1}{H^2}\left|H\nabla h_{kl}-\nabla Hh_{kl}\right|^2-\frac{H^4}{4R}\left|\nabla\frac{|A|^2}{H^2}\right|^2+\left(H|A|^2-C\right)H-2R|A|^2\right)$$

$$=R^{-\frac{1}{2}}\left(\Box H-\frac{1}{H^2}\left|H\nabla h_{kl}-\nabla Hh_{kl}\right|^2-\frac{H^4}{4R}\left|\nabla\frac{|A|^2}{H^2}\right|^2+\left(H|A|^2-C\right)H\right)$$

因为，

$$\begin{aligned}\Box H^2&=\left(Hg_{kl}-h_{kl}\right)\nabla_k\nabla_l H^2\\&=\left(Hg_{kl}-h_{kl}\right)\nabla_k\left(2H\nabla_l H\right)\\&=2\left(Hg_{kl}-h_{kl}\right)H\nabla_k\nabla_l H+2\left(Hg_{kl}-h_{kl}\right)\nabla_k H\nabla_l H\\&=2H\Box H+2\left|\nabla H\right|_{Hg-h}\end{aligned}$$

所以，

$$\frac{\partial}{\partial t}H^2=2H\frac{\partial}{\partial t}H=R^{-\frac{1}{2}}\left(\Box H^2-2\left|\nabla H\right|_{Hg-h}^2-\frac{2}{H}\left|H\nabla h_{kl}-\nabla Hh_{kl}\right|^2-\frac{H^5}{2R}\left|\nabla\frac{|A|^2}{H^2}\right|^2+2\left(H|A|^2-C\right)H^2\right)$$

定理 1：根据数量曲率流，我们得到下面的发展方程：

1) $\frac{\partial}{\partial t}g_{ij}=-2R^{\frac{1}{2}}h_{ij}$

2) $\frac{\partial \boldsymbol{n}}{\partial t}=-\nabla^j R^{\frac{1}{2}}\frac{\partial X}{\partial x^j}$

3)
$$\frac{\partial h_{ij}}{\partial t}=R^{-\frac{1}{2}}\left(\Box h_{ij}-\frac{1}{H^2}\left(H\nabla_i h_{kl}-\nabla_i Hh_{kl}\right)\left(H\nabla_j h_{kl}-\nabla_j Hh_{kl}\right)-\frac{H^4}{4R}\nabla_i\left(\frac{|A|^2}{H^2}\right)\nabla_j\left(\frac{|A|^2}{H^2}\right)\right.$$
$$\left.+\left(H|A|^2-C\right)h_{ij}-2Rh_{ik}h_{kj}\right)$$

4) $\frac{\partial}{\partial t}R^{\frac{1}{2}}=R^{\frac{1}{2}}\Box R^{\frac{1}{2}}+R\left(H|A|^2-C\right)$

证明：

根据度量的定义、超曲面 $X(\cdot,t)$ 第二基本形式和 Guass-Weingarten 关系式，我们得到：

1)
$$\begin{aligned}\frac{\partial}{\partial t}g_{ij}&=\frac{\partial}{\partial t}\left(\frac{\partial X}{\partial x^i},\frac{\partial X}{\partial x^j}\right)\\&=2R^{\frac{1}{2}}\left(\frac{\partial \boldsymbol{n}}{\partial x^i},\frac{\partial X}{\partial x^j}\right)\\&=-2R^{\frac{1}{2}}\left(\boldsymbol{n},\frac{\partial X}{\partial x^i\partial x^j}\right)\\&=-2R^{\frac{1}{2}}h_{ij}\end{aligned}$$

2)
$$\begin{aligned}\frac{\partial \boldsymbol{n}}{\partial t}&=\left(\frac{\partial \boldsymbol{n}}{\partial t},\frac{\partial X}{\partial x^i}\right)\frac{\partial X}{\partial x^j}g^{ij}\\&=-\left(\boldsymbol{n},\frac{\partial}{\partial t}\frac{\partial X}{\partial x^i}\right)\frac{\partial X}{\partial x^j}g^{ij}\\&=-\frac{\partial}{\partial x^i}R^{\frac{1}{2}}\frac{\partial X}{\partial x^j}g^{ij}\\&=-\nabla^j R^{\frac{1}{2}}\frac{\partial X}{\partial x^j}\end{aligned}$$

3)
$$\begin{aligned}\frac{\partial h_{ij}}{\partial t}&=\frac{\partial}{\partial t}\left(\frac{\partial^2 X}{\partial x_i\partial x_j},\boldsymbol{n}\right)\\&=\left\langle\frac{\partial^2}{\partial x_i\partial x_j}\left(R^{\frac{1}{2}}\boldsymbol{n}\right),\boldsymbol{n}\right\rangle+\left\langle\frac{\partial^2 X}{\partial x_i\partial x_j},-\frac{\partial R^{\frac{1}{2}}}{\partial x^i}\frac{\partial X}{\partial x^j}g^{ij}\right\rangle\\&=\left\langle\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}R^{\frac{1}{2}}\boldsymbol{n}-R^{\frac{1}{2}}h_{jk}g^{kl}\frac{\partial X}{\partial x_l}\right),\boldsymbol{n}\right\rangle+\left\langle\Gamma_{ij}^k\frac{\partial X}{\partial x_k}+h_{ij}\boldsymbol{n},-\frac{\partial R^{\frac{1}{2}}}{\partial x^i}\frac{\partial X}{\partial x^j}g^{ij}\right\rangle\\&=\frac{\partial^2 R^{\frac{1}{2}}}{\partial x^i\partial x^j}-\left(h_{jk}g^{kl}\frac{\partial X}{\partial x^l}\frac{\partial}{\partial x_i},\boldsymbol{n}\right)-\left(\Gamma_{ij}^k\frac{\partial X}{\partial x_k},\frac{\partial R^{\frac{1}{2}}}{\partial x^i}\frac{\partial X}{\partial x^j}g^{ij}\right)-\left(h_{ij}\boldsymbol{n},\frac{\partial R^{\frac{1}{2}}}{\partial x^i}\frac{\partial X}{\partial x^j}g^{ij}\right)\\&=\frac{\partial^2 R^{\frac{1}{2}}}{\partial x^i\partial x^j}-R^{\frac{1}{2}}h_{jk}g^{kl}h_{il}-\Gamma_{ij}^k\frac{\partial R^{\frac{1}{2}}}{\partial x^k}\\&=\nabla_i\nabla_j R^{\frac{1}{2}}-R^{\frac{1}{2}}h_{jk}g^{kl}h_{il}\\&=-\frac{1}{4}R^{-\frac{3}{2}}\nabla_i R\nabla_j R+\frac{1}{2}R^{-\frac{1}{2}}\nabla_i\nabla_j R-R^{\frac{1}{2}}h_{jk}h_{il}\\&=R^{-\frac{1}{2}}\left(\frac{1}{2}\nabla_i R\nabla_j R-\frac{1}{4R}\nabla_i R\nabla_j R-Rh_{jk}h_{il}\right)\end{aligned}\tag{3.1}$$

根据引理 2 (3.1)括号里的式子可以表示为：

$$\Box h_{ij}+\nabla_i H\nabla_j H-\nabla_i h_{kl}\nabla_j h_{kl}+\left(H|A|^2-C\right)h_{ij}-\frac{1}{4R}\nabla_i R\nabla_j R-2Rh_{ik}h_{kj}\tag{3.2}$$

(3.2)式括号中的 $-\frac{1}{4R}\nabla_i R\nabla_j R$ 根据引理 3 可以计算为：

$$\begin{aligned}&-\frac{1}{4R}\nabla_i R\nabla_j R\\&=-\frac{1}{4R}\left(-H^2\nabla_i\left(\frac{|A|^2}{H^2}\right)+\frac{2R}{H}\nabla_i H\right)\left(-H^2\nabla_j\left(\frac{|A|^2}{H^2}\right)+\frac{2R}{H}\nabla_j H\right)\\&=-\frac{1}{4R}\left(H^4\nabla_i\left(\frac{|A|^2}{H^2}\right)\nabla_j\left(\frac{|A|^2}{H^2}\right)+\frac{4R^2}{H^2}\nabla_i H\nabla_j H-4HR\nabla_i\left(\frac{|A|^2}{H^2}\right)\nabla_j H\right)\end{aligned}$$

再根据引理 4，我们可以计算出：

$$
\begin{aligned}
&\nabla_i H\nabla_j H-\nabla_i h_{kl}\nabla_j h_{kl}-\frac{1}{4R}\nabla_i R\nabla_j R\\
&=\nabla_i H\nabla_j H-\nabla_i h_{kl}\nabla_j h_{kl}-\frac{H^4}{4R}\nabla_i\left(\frac{|A|^2}{H^2}\right)\nabla_j\left(\frac{|A|^2}{H^2}\right)-\frac{R}{H^2}\nabla_i H\nabla_j H+H\nabla_i\left(\frac{|A|^2}{H^2}\right)\nabla_j H\\
&=\frac{|A|^2}{H^2}\nabla_i H\nabla_j H-\nabla_i h_{kl}\nabla_j h_{kl}-\frac{H^4}{4R}\nabla_i\left(\frac{|A|^2}{H^2}\right)\nabla_j\left(\frac{|A|^2}{H^2}\right)+\frac{2H\nabla_j H h_{kl}\nabla_i h_{kl}}{H^2}-\frac{2H|A|^2\nabla_j H\nabla_i H}{H^3}\\
&=-\nabla_i h_{kl}\nabla_j h_{kl}-\frac{|A|^2}{H^2}\nabla_i H\nabla_j H+\frac{2\nabla_j H}{H}h_{kl}\nabla_i h_{kl}-\frac{H^4}{4R}\nabla_i\left(\frac{|A|^2}{H^2}\right)\nabla_j\left(\frac{|A|^2}{H^2}\right)\\
&=-\frac{1}{H^2}\left(H\nabla_i h_{kl}-\nabla_i H h_{kl}\right)\left(H\nabla_j h_{kl}-\nabla_j H h_{kl}\right)-\frac{H^4}{4R}\nabla_i\left(\frac{|A|^2}{H^2}\right)\nabla_j\left(\frac{|A|^2}{H^2}\right)
\end{aligned}
$$

所以我们得到：

$$
\begin{aligned}
\frac{\partial}{\partial t}h_{ij}&=R^{-\frac{1}{2}}\left(h_{ij}+\nabla_i H\nabla_j H-\nabla_i h_{kl}\nabla_j h_{kl}-\frac{1}{4R}\nabla_i R\nabla_j R+\left(H|A|^2-C\right)h_{ij}-2Rh_{ik}h_{kj}\right)\\
&=R^{-\frac{1}{2}}\Bigg(\Box h_{ij}-\frac{1}{H^2}\left(H\nabla_i h_{kl}-\nabla_i H h_{kl}\right)\left(H\nabla_j h_{kl}-\nabla_j H h_{kl}\right)-\frac{H^4}{4R}\nabla_i\left(\frac{|A|^2}{H^2}\right)\nabla_j\left(\frac{|A|^2}{H^2}\right)\\
&\quad+\left(H|A|^2-C\right)h_{ij}-2Rh_{ik}h_{kj}\Bigg)
\end{aligned}
$$

4) 根据引理 4 可得

$$
\begin{aligned}
\frac{\partial}{\partial t}R^{\frac{1}{2}}&=\frac{1}{2}R^{-\frac{1}{2}}\frac{\partial}{\partial t}R\\
&=\frac{1}{2}R^{-\frac{1}{2}}\frac{\partial}{\partial t}\left(H^2-|A|^2\right)\\
&=\frac{1}{2}\left[\Box R-2|\nabla H|^2_{Hg-h}+2|\nabla A|^2_{Hg-h}-\frac{2}{H^2}\left|H\nabla h_{kl}-\nabla H h_{kl}\right|^2_{Hg-h}-\frac{H^4}{2R}\left|\nabla\frac{|A|^2}{H^2}\right|^2_{Hg-h}+2R\left(H|A|^2-C\right)\right]\\
&=\frac{1}{2}\Box R-|\nabla H|^2_{Hg-h}+|\nabla A|^2_{Hg-h}-\frac{1}{H^2}\left|H\nabla h_{kl}-\nabla H h_{kl}\right|^2_{Hg-h}-\frac{H^4}{4R}\left|\nabla\frac{|A|^2}{H^2}\right|^2_{Hg-h}+R\left(H|A|^2-C\right)
\end{aligned}
\tag{4.1}
$$

$$
\because \nabla_i R=-H^2\nabla_i\left(\frac{|A|^2}{H^2}\right)+\frac{2R}{H}\nabla_i H
$$

$$
\therefore H^2\nabla_i\left(\frac{|A|^2}{H^2}\right)=\nabla_i R-\frac{2R}{H}\nabla_i H
$$

则

$$-\frac{H^4}{4R}\left|\nabla\frac{|A|^2}{H^2}\right|^2_{Hg-h}=-\frac{1}{4R}\left\langle\nabla R-\frac{2R}{H}\nabla H,\nabla R-\frac{2R}{H}\nabla H\right\rangle_{Hg-h}$$
$$=-\frac{1}{4R}|\nabla R|^2_{Hg-h}+\frac{2}{4R}\left\langle\nabla R,\frac{2R}{H}\nabla H\right\rangle_{Hg-h}-\frac{1}{4R}\frac{4R^2}{H^2}|\nabla H|^2_{Hg-h} \quad (4.2)$$
$$=-\frac{1}{4R}|\nabla R|^2_{Hg-h}+\frac{1}{H}\langle\nabla R,\nabla H\rangle_{Hg-h}-\frac{R}{H^2}|\nabla H|^2_{Hg-h}$$

$$\begin{aligned}\Box R^{\frac{1}{2}}&=\left(Hg_{ij}-h_{ij}\right)\nabla_i\nabla_j R^{\frac{1}{2}}\\&=\left(Hg_{ij}-h_{ij}\right)\nabla_i\left(\frac{1}{2}R^{-\frac{1}{2}}\nabla_j R\right)\\&=\frac{1}{2}\left(Hg_{ij}-h_{ij}\right)\left(-\frac{1}{2}R^{-\frac{3}{2}}\nabla_i R\nabla_j R+R^{-\frac{1}{2}}\nabla_i\nabla_j R\right)\\&=-\frac{1}{4}R^{-\frac{3}{2}}|\nabla R|^2_{Hg-h}+\frac{1}{2}R^{-\frac{1}{2}}\Box R\end{aligned} \quad (4.3)$$

$$\begin{aligned}|\nabla A|^2_{Hg-h}&=\frac{1}{H^2}\left(H^2|\nabla A|^2_{Hg-h}-H\left\langle\nabla|A|^2,\nabla H\right\rangle_{Hg-h}+A^2|\nabla H|^2_{Hg-h}\right)+\frac{1}{H}\left\langle\nabla|A|^2,\nabla H\right\rangle_{Hg-h}-\frac{A^2}{H^2}|\nabla H|^2_{Hg-h}\\&=\frac{1}{H^2}|H\nabla h_{kl}-\nabla Hh_{kl}|^2_{Hg-h}+\frac{1}{H}\left\langle\nabla|A|^2,\nabla H\right\rangle_{Hg-h}-\frac{A^2}{H^2}|\nabla H|^2_{Hg-h}\end{aligned} \quad (4.4)$$

把(4.2)，(4.3)，(4.4)代入(4.1)得到

$$\begin{aligned}\frac{\partial}{\partial t}R^{\frac{1}{2}}&=R^{\frac{1}{2}}\Box R^{\frac{1}{2}}+\frac{1}{4R}|\nabla R|^2_{Hg-h}+\frac{1}{H}\left\langle\nabla|A|^2,\nabla H\right\rangle_{Hg-h}-\frac{|A|^2}{H^2}|\nabla H|^2_{Hg-h}-|\nabla H|^2_{Hg-h}-\frac{1}{4R}|\nabla R|^2_{Hg-h}\\&\quad+\frac{1}{H}\langle\nabla R,\nabla H\rangle_{Hg-h}-\frac{R}{H^2}|\nabla H|^2_{Hg-h}+R\left(H|A|^2-C\right)\\&=R^{\frac{1}{2}}\Box R^{\frac{1}{2}}+R\left(H|A|^2-C\right)\end{aligned}$$

定理 2 (拼挤估计)对于函数$\frac{\partial}{\partial t}X(x,t)=R^{\frac{1}{2}}\boldsymbol{n}$ ($x\in M^n$， $t>0$)，如果$t=0$时$R^{\frac{1}{2}}\geq 0$，则$t>0$时$R^{\frac{1}{2}}\geq 0$仍然成立。

证明：

根据命题 1，$R^{\frac{1}{2}}$的发展方程是

$$\begin{aligned}\frac{\partial}{\partial t}R^{\frac{1}{2}}&=R^{\frac{1}{2}}\Box R^{\frac{1}{2}}+\frac{1}{4R}|\nabla R|^2_{Hg-h}+\frac{1}{H}\left\langle\nabla|A|^2,\nabla H\right\rangle_{Hg-h}-\frac{|A|^2}{H^2}|\nabla H|^2_{Hg-h}-|\nabla H|^2_{Hg-h}-\frac{1}{4R}|\nabla R|^2_{Hg-h}\\&\quad+\frac{1}{H}\langle\nabla R,\nabla H\rangle_{Hg-h}-\frac{R}{H^2}|\nabla H|^2_{Hg-h}+R\left(H|A|^2-C\right)\end{aligned}$$

$$\because\frac{1}{H}\left\langle\nabla|A|^2,\nabla H\right\rangle_{Hg-h}-\frac{|A|^2}{H^2}|\nabla H|^2_{Hg-h}-|\nabla H|^2_{Hg-h}+\frac{1}{H}\left\langle 2H\nabla H-\nabla A^2,\nabla H\right\rangle_{Hg-h}-\frac{H^2-|A|^2}{H^2}|\nabla H|^2_{Hg-h}=0$$

$\therefore$存在向量v^i满足$R^{\frac{1}{2}}v^j=0$

则$\left(H|A|^2-C\right)Rv^j=0$

由引理 3 我们可以得出结论。

基金项目

新疆师范大学重点实验室(XJNUSYS082018A02)。

参考文献

[1] Hamilton, R.S. (1995) Isoperimtric Estimates for the Curve Shrinking Flow in the Plan. *Annals of Mathematics Studies*, **137**, 201-222. https://doi.org/10.1515/9781400882571-014

[2] Hamilton, R.S. (1995) Harnack Estimates for the Mean Curvature Flow. *Journal of Differential Geometry*, **41**, 215-226. https://doi.org/10.4310/jdg/1214456010

[3] 陈旭忠. 关于曲率流的某些问题[D]: [博士学位论文]. 杭州: 浙江大学, 2006.

[4] Hamilton, R.S. (1982) Three-Manifolds with Positive Ricci Curvature. *Journal of Differential Geometry*, **17**, 255-306. https://doi.org/10.4310/jdg/1214436922

[5] Hamilton, R.S. (1986) Four-Manifolds with Positive Curvature Operator. *Journal of Differential Geometry*, **24**, 153-179. https://doi.org/10.4310/jdg/1214440433

[6] Hamilton, R.S. (1994) Convex Hypersurfaces with Pinched Second Fundamental Form. *Communications in Analysis and Geometry*, **2**, 167-172. https://doi.org/10.4310/CAG.1994.v2.n1.a10

[7] Hamilton, R.S. (1995) A Compactness Property for Solution of the Ricci Flow. *American Journal of Mathematics*, **117**, 545-572. https://doi.org/10.2307/2375080