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广义变系数K(m,n)方程的精确解

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摘 要

本文的目的是利用约化技巧化简高阶非线性方程的思想来研究广义变系数K(m,n)方程的精确解。通过符号计算获得了该方程的新的精确解。

关键词

精确解，广义K(m,n)方程，符号计算

Exact Solutions for the Generalized K(m,n) Equation with Variable Coefficients

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Abstract

The objective of this paper is to investigate exact solutions for the generalized K(m,n)

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with variable coefficients. An extended approach is proposed for reducing the order of the equations with higher order nonlinearity. New exact solutions are found by symbolic computation.

Keywords

Exact Solutions, Generalized K(m,n) Equation, Symbolic Computation

1. 引言

众所周知，Korteweg-de Vries (KdV)方程

$$u_t + auu_x + bu_{xxx} = 0, \tag{1.1}$$

是一个用来描述小浅水波在非粘性流体表面的运动 [1], 其中a 和b 是任意非零常数。

近年来，由于$K(n,n)$方程的广泛应用，一些研究学者将目光转向变系数的KdV型方程 [2–6]。通过下面的变换

$$u(x,t) = v^{\frac{1}{n-1}}(x,t), \tag{1.2}$$

与辅助方程技巧，Lv等人 [7]获得了如下$K(n,n)$方程的许多显式的具有不同结构的行波解

$$u_t + a(t)u_x + b(t)(u^n)_x + k(t)(u^n)_{xxx} = 0, \quad n \neq 0, 1. \tag{1.3}$$

这些解包括三角函数周期波解和双曲函数解。

这篇文章，受文章 [7]的启发，我们通过如下变换研究广义的KdV方程：

$$u_t + a(t)u_x + b(t)(u^m)_x + k(t)(u^n)_{xxx} = 0, \quad m, n \neq 0, 1, \tag{1.4}$$

其中$a(t)b(t)k(t) \neq 0$.

这里，我们只对方程(1.4)的compacton解感兴趣。compacton解(就是具有紧支集的孤立波解)。Compacton解已经在文献 [8–12]被广泛研究。但是对于当$m \neq n$时方程(1.4)的compacton解还没有被研究。

2. 方程(1.4)的约化

为了获得compacton解，我们将下面方程

$$u(x,t) = c_1 + c_2 z(\xi), \quad \xi = p(t)x + q(t), \tag{2.1}$$

代入到(1.4)中，其中$z' = \epsilon\sqrt{a_1 + a_2 z + a_3 z^2}$，$\epsilon = \pm 1$，$a_1$，$a_2$，$a_3$，$c_1$ 与c_2 是实常数，$p(t)$ 与$q(t)$ 是未知函数。让$\Delta = a_2^2 - 4a_1 a_3$，就会有下面的事实 [13]。当$a_3 < 0$ 与$\Delta > 0$，我们有$z(\xi) = \frac{\epsilon\sqrt{\Delta}}{2a_3}\sin(\sqrt{-a_3}\xi) - \frac{a_2}{2a_3}$ 与$z(\xi) = \frac{\epsilon\sqrt{\Delta}}{2a_3}\cos(\sqrt{-a_3}\xi) - \frac{a_2}{2a_3}$.

性质1. 当$m = n$，利用变换

$$u = v^p, \tag{3.1}$$

其中$p = \frac{\alpha}{n-1}$，$\alpha \in \mathbb{Z}$ 与$\alpha \neq 0$，方程(1.4)能够变成下面两种情形。

(i) 如果$\alpha \geq 2$，方程(1.4)能变换成

$$v_t + a(t)v_x + b(t)nv^\alpha v_x + k(t)n(np-1)(np-2)v^{\alpha-2}v_x^3 + 3k(t)n(np-1)v^{\alpha-1}v_x v_{xx} + k(t)nv^\alpha v_{xxx} = 0. \tag{3.2}$$

(ii) 如果$\alpha < 2$，方程(1.4)能变换成

$$v^{2-\alpha}v_t + a(t)v^{2-\alpha}v_x + b(t)nv^2 v_x + k(t)n(np-1)(np-2)v_x^3 + 3k(t)n(np-1)vv_x v_{xx} + k(t)nv^2 v_{xxx} = 0. \tag{3.3}$$

性质2. 当$m \neq n$ 和$\frac{n-1}{m-1} = \frac{\beta+2}{\alpha} \neq 1$，利用变换

$$u = v^p, \tag{3.4}$$

其中α，$\beta \in \mathbb{Z}$，$\alpha \neq 0$，$\beta \neq -2$ and $p = \frac{\alpha}{m-1} = \frac{\beta+2}{n-1}$，方程(1.4)能化简成

$$v_t + a(t)v_x + b(t)mv^\alpha v_x + k(t)n(np-1)(np-2)v^\beta v_x^3 + 3k(t)n(np-1)v^{\beta+1}v_x v_{xx} + k(t)nv^{\beta+2}v_{xxx} = 0. \tag{3.5}$$

性质1和2的证明 利用变换(3.1)，我们有

$$\begin{cases} u_t = pv^{p-1}v_t, \\ u_x = pv^{p-1}v_x, \\ (u^m)_x = mpv^{mp-1}v_x, \\ (u^n)_{xxx} = np(np-1)(np-2)v^{np-3}v_x^3 + 3np(np-1)v^{np-2}v_x v_{xx} + npv^{np-1}v_{xxx}. \end{cases} \tag{3.6}$$

将(3.6)代入到方程(1.4)，我们得到

$$\begin{aligned} &pv^{p-1}v_t + a(t)pv^{p-1}v_x + b(t)mpv^{mp-1}v_x + k(t)np(np-1)(np-2)v^{np-3}v_x^3 \\ &+3k(t)np(np-1)v^{np-2}v_x v_{xx} + k(t)npv^{np-1}v_{xxx} = 0. \end{aligned} \tag{3.7}$$

上式两端乘以v^s, 我们有

$$v^s[pv^{p-1}v_t + a(t)pv^{p-1}v_x + b(t)mpv^{mp-1}v_x + k(t)np(np-1)(np-2)v^{np-3}v_x^3 \\ +3k(t)np(np-1)v^{np-2}v_xv_{xx} + k(t)npv^{np-1}v_{xxx}] = 0, \tag{3.8}$$

或者

$$pv^{s+p-1}v_t + a(t)pv^{s+p-1}v_x + b(t)mpv^{s+mp-1}v_x + k(t)np(np-1)(np-2)v^{s+np-3}v_x^3 \\ +3k(t)np(np-1)v^{s+np-2}v_xv_{xx} + k(t)npv^{s+np-1}v_{xxx} = 0, \tag{3.9}$$

其中s是一个任意常数。让

$$\begin{cases} s+p-1 = A, \\ s+mp-1 = B, \\ s+np-3 = C, \end{cases} \tag{3.10}$$

其中A, B 和C 是任意常数。从(3.10)我们可以得到

$$p = \frac{B-A}{m-1} = \frac{C-A+2}{n-1}. \tag{3.11}$$

下面，我们将分$m = n$ 和$m \neq n$来分别证明性质1和2。

Case (1) 当$m = n$, i.e. $B = C + 2$, 通过利用(3.10) 和(3.11), 方程(3.9)可以化为

$$v^Av_t + a(t)v^Av_x + b(t)mv^Bv_x + k(t)n(np-1)(np-2)v^{B-2}v_x^3 \\ +3k(t)n(np-1)v^{B-1}v_xv_{xx} + k(t)nv^Bv_{xxx} = 0. \tag{3.12}$$

当$B-2 \geq A$, 上式两端乘以v^{-A}, 我们得到

$$v_t + a(t)v_x + b(t)mv^{B-A}v_x + k(t)n(np-1)(np-2)v^{B-A-2}v_x^3 \\ +3k(t)n(np-1)v^{B-A-1}v_xv_{xx} + k(t)nv^{B-A}v_{xxx} = 0. \tag{3.13}$$

为了简单起见，我们让$B-A = \alpha$. 方程(3.13)能花成方程(3.2)。当$B-2 < A$, 上式两端乘以v^{-B+2}, 我们得到

$$v^{A-B+2}v_t + a(t)v^{A-B+2}v_x + b(t)mv^2v_x + k(t)n(np-1)(np-2)v_x^3 \\ +3k(t)n(np-1)vv_xv_{xx} + k(t)nv^2v_{xxx} = 0. \tag{3.14}$$

进一步的方程(3.14)就能化成方程(3.3). 于是我们就完成了性质1的证明。

Case (2) 当$m \neq n$, 利用(3.10)与(3.11), 方程(3.9)能够写成

$$v_t + a(t)v_x + b(t)mv^{B-A}v_x + k(t)n(np-1)(np-2)v^{C-A}v_x^3 \\ +3k(t)n(np-1)v^{C-A+1}v_xv_{xx} + k(t)nv^{C-A+2}v_{xxx} = 0. \tag{3.15}$$

让$B-A = \alpha$和$C-A = \beta$, 我们就完成了性质2的证明。

3. 方程(1.4)的Compacton解

3.1 $m = n$

让$\alpha = 2$, 方程(3.2)变成

$$v_t + a(t)v_x + nb(t)v^2v_x + \frac{2n(n+1)}{(n-1)^2}k(t)v_x^3 + \frac{3n(n+1)}{n-1}k(t)vv_xv_{xx} + nk(t)v^2v_{xxx} = 0. \tag{4.1}$$

将(2.1)代入(4.1)让系数$x^s z^i(\xi)\sqrt{a_1 + a_2 z + a_3 z^2}(s = 0, 1, i = 0, 1, 2)$等于0, 我们就可以得到关于$c_2$, $p(t)$ 与$q(t)$的代数方程。通过数学软件Mathematica求解这些方程，我们得到

$$\begin{cases} c_2 = \frac{2a_3c_1}{a_2}, \\ p(t) = \pm\frac{n-1}{2n}\sqrt{\frac{-b(t)}{a_3k(t)}}, \\ p'(t) = 0, \\ q'(t) = \mp\frac{n-1}{4n^2a_2^2}\sqrt{\frac{-b(t)}{a_3k(t)}}\left(2na(t)a_2^2 + (n+1)b(t)(a_2^2 - 4a_1a_3)c_1^2\right), \end{cases} \tag{4.2}$$

其中$a_2a_3c_1 \neq 0$, a_1, a_2, a_3 与c_1是任意常数。在(4.2), 我们要求$\frac{b(t)}{k(t)}$必须是一个常数。

(i) 如果$a_3 < 0$, $\Delta > 0$ 且$\frac{b(t)}{k(t)} > 0$, 我们得到

$$u_1(x,t) = \left[\frac{c_1\sqrt{\Delta}}{a_2}\sin\left(\frac{n-1}{2n}\sqrt{\frac{b(t)}{k(t)}}x + \sqrt{-a_3}q(t)\right)\right]^{\frac{2}{n-1}},$$

和

$$u_2(x,t) = \left[\frac{c_1\sqrt{\Delta}}{a_2}\cos\left(\frac{n-1}{2n}\sqrt{\frac{b(t)}{k(t)}}x + \sqrt{-a_3}q(t)\right)\right]^{\frac{2}{n-1}}.$$

3.2 $m \neq n$

当$m \neq n$, 方程(3.5)可以写成

$$v^3v_t + a(t)v^3v_x + mb(t)v^4v_x + k(t)(2-m)\left(\frac{(3-2m)(4-3m)}{(m-1)^2}v_x^3 + \frac{9-6m}{m-1}vv_xv_{xx} + v^2v_{xxx}\right) = 0. \tag{4.7}$$

将(2.1)代入到(4.7)并让系数$x^s z^i(\xi)\sqrt{a_1 + a_2 z + a_3 z^2}(s = 0, 1, i = 0, 1, 2, 3)$等于0 ，我们可以获得关于$a_3$, c_2, $p(t)$ 和$q(t)$的代数方程。通过软件Mathematica，我们可以得到

$$\begin{cases} q'(t) = \frac{-16M_2^3M_5a(t)a_1^3 - mM_3^3b(t)a_2^3c_1^4}{4M_2^3M_6a_1^3}, \\ c_2 = \frac{M_3a_2c_1}{4M_2a_1}, \\ p(t) = \pm\frac{4M_5}{M_6}, \\ p'(t) = 0, \\ a_3 = \frac{3M_4a_2^2}{8(3m-4)M_1^2a_1}, \end{cases} \tag{4.8}$$

其中$M_1=14m^2-41m+29$, $M_2=-116+251m-179m^2+42m^3$, $M_3=-164+374m-281m^2+69m^3$, $M_4=-2460+8726m-12305m^2+8618m^3-2997m^4+414m^5$, $M_5=\sqrt{(m-1)^2m(3m-4)M_1^2b(t)a_1c_1^2}$, $M_6=\sqrt{(m-2)^3M_4k(t)a_2^2}$, $a_1a_2a_3c_1\neq 0$. 在(4.8)中，我们要求因子$\frac{b(t)}{k(t)}$必须是一个常数。进一步的，我们有：

(i) 当$a_3<0$与$\Delta>0$, 我们获得了方程(1.4)的解

$$u_6(x,t)=\left[c_1+c_2\left(\frac{\epsilon\sqrt{\Delta}}{2a_3}\sin(\sqrt{-a_3}(p(t)x+q(t)))-\frac{a_2}{2a_3}\right)\right]^{\frac{1}{m-1}},$$

与

$$u_7(x,t)=\left[c_1+c_2\left(\frac{\epsilon\sqrt{\Delta}}{2a_3}\cos(\sqrt{-a_3}(p(t)x+q(t)))-\frac{a_2}{2a_3}\right)\right]^{\frac{1}{m-1}}.$$

4. 结论

这篇文章中我们通过一个简单的技巧获得了广义K(m,n)方程的精确解。事实上，当我们通过下面的变换

$$u(x,t)=\sum_{i=0}^{n}c_iz^i(\xi),\quad \xi=p(t)x+q(t),$$

其中$c_i(i=0,1,2,3,\cdots)$ 是实常数。同时我们引入辅助方程

$$z'=\epsilon\sqrt{a_1+a_2z+a_3z^2+a_4z^3+a_5z^4},$$

其中$a_i(i=1,2,3,4,5)$ 是实常数。我们可以获得广义K(m,n)方程的其余的精确解。

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