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一类具有多个激波层结构粘性守恒律方程的粘性消失极限问题

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摘 要

本文研究的是带有两个无相互作用的激波层的一维拟线性粘性方程的柯西问题解的渐近极限。目的是理解无相互作用的粘性激波层的进化与构造以及外部无粘双曲流之间的相互作用，并证明粘性解在远离激波层区域中一致收敛于分片光滑的无粘解，这是基于匹配渐近分析法和能量估计法。文章先利用匹配渐近展开的方法构造粘性方程的近似解，再利用能量估计的方法估计近似解与粘性方程真实解之间的误差，得到误差的 H^1 估计，并用 **Sobolev** 嵌入得到 L^∞ 估计，从而证明两类方程的渐近等价性。

关键词

柯西问题，粘性激波层，匹配渐近展开，非线性稳定性，能量估计

Vanishing Viscosity Limit for a Class of Viscous Conservation Laws with Multiple Shock Layers

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Abstract

In this paper, we study the asymptotic limit of the solution of the Cauchy problem for one-dimensional quasilinear viscous equation with two non-interacting shock layers. The aim is to understand the evolution and construction of the non-interaction viscous shock layer and the interaction between the external inviscid hyperbolic flow, and to prove that the viscous solution converges uniformly to the piecewise smooth inviscid solution in the region far from the shock layer. This is based on the method of matched asymptotic expansions and energy estimates. Firstly, the approximate solution of the viscous equation is constructed by using the method of matched asymptotic expansions, and then the error between the approximate solution and the real solution of the viscous equation is estimated by the method of energy estimates, and the error estimate of H^1 is obtained by using Sobolev embedding to obtain L^∞ estimate, thus proving the asymptotic equivalence of the two kinds of equations.

Keywords

Cauchy Problem, Viscous Shock Layer, Matched Asymptotic Expansions, Nonlinear Stability, Energy Estimates

1. 引言

由可压缩流体理论, 我们知道: 一定条件下, 粘性抛物方程和对应的无粘双曲方程之间存在渐近等价性, 例如可压缩的N-S 方程 [1] [2] 和可压缩的欧拉方程, 它们之间的渐近等价性在小耗散限制下对解释许多物理现象和数值计算 [3] 有重大意义.当流体存在激波间断的情况下, 即使粘性很小, 靠近激波间断的流体表现出很多奇异性. 这种渐近等价性的严格数学证明, 在许多重要的情况下得

到了验证 [4–9]. 本文探讨在具有两个激波层条件下的柯西问题解的渐近极限, 目的是理解粘性激波层的演化与构造以及内部无粘双曲流之间的相互作用, 并证明粘性解在远离激波层区域中一致收敛于分片光滑的无粘解, 我们考虑如下方程:

$$\partial_t u^\varepsilon(x,t) + \partial_x f(u^\varepsilon(x,t)) = \varepsilon \partial_x^2 u^\varepsilon(x,t), \quad x \in \mathbb{R}, \quad t > 0, \quad \varepsilon > 0. \tag{1.1}$$

与双曲方程

$$\partial_t u(x,t) + \partial_x f(u(x,t)) = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{1.2}$$

解的渐近关系. 其中 f 是光滑的, 满足 $\partial_u^2 f(x,t) > 0$.

我们希望粘性方程的真实解 u^ε 与无粘方程的解 u 是足够靠近的, 证明过程主要分为两个部分. 首先, 由于 u^ε 解的真实形态是很难直接得到, 因此希望利用匹配渐近分析的理论来构造的粘性方程 (1.1) 的近似解 $\bar{u}_a$, 用有限个宽度为 ε 的光滑粘性激波来代替不连续激波. 在构造近似解的过程中我们会发现在远离激波区域当 $\varepsilon \to 0$ 时 u^ε 接近 u, 且需要对 u 进行高阶修正. 我们可以通过多尺度展开构造任意精度阶的近似解, 完成对匹配区内外解的估计, 这对接下来复杂的粘性激波稳定性理论相关的能量估计是非常关键的.

证明的主要部分是能量估计, 可得出存在一个精确解 u^ε 接近构造的近似解 u. 证明过程中会使用粘性激波的稳定性理论来避免 $O(\frac{1}{\varepsilon})$ 的困难, 最后来得到粘性方程的近似解与无粘方程的真实解的渐近等价性. 该方法依赖于 (1.1) 关于近似解 $\bar{u}_a$ 线性的双曲结构, 文献 [10] 对此给出了一个结论.

现在对我们的定理给出一个精确的表述. $u(x,t)$ 是方程 (1.2) 在 $[0,T](T>0)$ 上的唯一激波解, 如果满足如下条件:

(i) $u(x,t)$ 方程 (1.2) 在 $\mathbb{R} \times [0,T]$ 光滑分片解;

(ii)有两个激波层 $x = s_1(t)$ 和 $x = s_2(t)$, 其中 $0 \le t \le T$, 使得 $u(x,t)$在任何点 $x \neq s_1(t)$ 和 $x \neq s_2(t)$ 处都足够光滑;

(iii)极限存在 $i = 1, 2$ 有:

$$\begin{aligned} \partial_x^k u(s_i(t)-0,t) &= \lim_{x \to s_i(t)^-} \partial_x^k (u(x,t)), \\ \partial_x^k u(s_i(t)+0,t) &= \lim_{x \to s_i(t)^+} \partial_x^k (u(x,t)), \end{aligned} \tag{1.3}$$

(iv) Lax 熵条件 [2] 在 $x = s_i(t), i = 1, 2$, 有

$$\partial_u f(u(s_i(t)-0,t)) > \frac{d}{dt} s_i(t) > \partial_u f(u(s_i(t)+0,t)), \tag{1.4}$$

定理1: 假设无粘方程 (1.2) 是严格双曲的, 若 $u^0(x,t) \in C^2([0,T]; H^8(\mathbb{R}))$ 是无粘方程 (1.1) 的解, 对 $\forall t \in [0,T], i = 1, 2$, 满足:

$$u^0(s_i(t)+0,t) - u^0(s_i(t)-0,t) < 0. \tag{1.5}$$

则对每一个 $\varepsilon > 0$, 使得方程 (1.1) 存在唯一一个光滑解 $u^\varepsilon(x,t) \in C^1([0,T];H^2(\mathbb{R}))$ 满足:

$$\sup_{0\le t\le T}\int_{\mathbb{R}} |u^\varepsilon(\cdot,t) - u^0(\cdot,t)|^2 dx \le C\varepsilon^\sigma, \tag{1.6}$$

$$\sup_{\substack{0\le t\le T\\ |x-s_i(t)|\ge\varepsilon^\gamma}} |u^\varepsilon(\cdot,t) - u^0(\cdot,t)| \le C\varepsilon, \quad i = 1,2, \tag{1.7}$$

其中 $\gamma \in (\frac{6}{7},1), \sigma > 0$.

2. 近似解的构造

在本节中, 我们利用匹配渐近分析理论构造方程 (1.1) 的近似解 $u^\varepsilon(x,t)$. 首先, 我们先做外部展开.

2.1. 外部展开

在远离激波层 $\{x = s_1(t)\}$ 和 $\{x = s_2(t)\}$ 的区域里, 方程 (1.1) 的解可以作如下展开:

$$u^\varepsilon(x,t) \sim u^0(x,t) + \varepsilon u^1(x,t) + \varepsilon^2 u^2(x,t) + \varepsilon^3 u^3(x,t)\cdots. \tag{2.1}$$

将近似解代入 (1.1) 并根据系数 ε 的级数进行分类, 当 $x \ne s_i(t)\ \ (i = 1,2)$, 可以得到:

$$O(1): \partial_t u^0 + \partial_x f(u^0) = 0, \tag{2.2}$$

$$O(\varepsilon): \partial_t u^1 + \partial_x(f'(u^0)u^1) = \partial_x^2 u^0, \tag{2.3}$$

$$O(\varepsilon^2): \partial_t u^2 + \partial_x(f'(u^0)u^2) = \partial_x^2 u^1 - \frac{1}{2}\partial_x(f''(u^0)(u^1)^2), \tag{2.4}$$

$$O(\varepsilon^3): \partial_t u^3 + \partial_x(f'(u^0)u^3) = \partial_x^2 u^2 - \partial_x(f''(u^0)(u^1\cdot u^2)) - \frac{1}{6}\partial_x(f'''(u^0)(u^1)^3). \tag{2.5}$$

其中, 外部函数 u^0, u^1...在激波 $x = s(t)$ 处一般是不连续的, 但在激波附近是光滑的.

2.2. 激波层 $x = s_1(t)$ 附近展开

在激波层 $\{x = s_1(t)\}$ 附近, 我们得到近似解:

$$u^\varepsilon(x,t) \sim u_s^0(\xi,t) + \varepsilon u_s^1(\xi,t) + \varepsilon^2 u_s^2(\xi,t) + \varepsilon^3 u_s^3(\xi,t)\cdots, \tag{2.10}$$

其中

$$\xi = \frac{x - s_1(t)}{\varepsilon} + \delta_1(t,\varepsilon), \tag{2.11}$$

δ_1 是激波层扰动, 待定. 我们假设 $\delta_1(t,\varepsilon)$ 展开式为:

$$\delta_1(t,\varepsilon)=\delta_1^0(t)+\varepsilon\delta_1^1(t)+\varepsilon^2\delta_1^2(t)+\varepsilon^3\delta_1^3(t)\cdots. \tag{2.12}$$

将近似解代入方程 (1.1) 得到:

$$O(\frac{1}{\varepsilon}):\partial_\xi^2u_s^0+\dot{s}_1(t)\partial_\xi u_s^0-\partial_\xi f(u_s^0)=0, \tag{2.13}$$

$$O(1):\partial_\xi^2u_s^1+\dot{s}_1(t)\partial_\xi u_s^1-\partial_\xi(f'(u_s^0)u_s^1)=\dot{\delta}_1^0\partial_\xi u_s^0+\partial_t u_s^0, \tag{2.14}$$

$$O(\varepsilon):\partial_\xi^2u_s^2+\dot{s}_1(t)\partial_\xi u_s^2-\partial_\xi(f'(u_s^0)u_s^2)=\dot{\delta}_1^0\partial_\xi u_s^1+\dot{\delta}_1^1\partial_\xi u_s^0+\partial_t u_s^1+\frac{1}{2}\partial_\xi(f''(u_s^0)(u_s^1)^2), \tag{2.15}$$

$$\begin{aligned}O(\varepsilon^2):&\partial_\xi^2u_s^3+\dot{s}_1(t)\partial_\xi u_s^3-\partial_\xi(f'(u_s^0)u_s^3)\\&=\dot{\delta}_1^2\partial_\xi u_s^0+\dot{\delta}_1^1\partial_\xi u_s^1+\dot{\delta}_1^0\partial_\xi u_s^2+\partial_t u_s^2+\partial_\xi(f''(u_s^0)(u_1^1\cdot u_s^2))+\frac{1}{6}\partial_\xi(f'''(u_s^0)(u_s^1)^3),\end{aligned} \tag{2.16}$$

其中 $\dot{s}_1(t)=\frac{ds_1}{dt},\dot{\delta}_1=\frac{d\delta_1}{dt}$.在匹配区域中, 对于 $\xi\to\pm\infty$, 激波层附近展开和外部展开同时成立, 则满足如下方程:

$$u_s^0(\xi,t)=u^0(s_1(t)\pm0,t)+o(1), \tag{2.17}$$

$$u_s^1(\xi,t)=u^1(s_1(t)\pm0,t)+(\xi-\delta_1^0)\partial_x u^0(s_1(t)\pm0,t)+o(1), \tag{2.18}$$

$$\begin{aligned}u_s^2(\xi,t)=&u^2(s_1(t)\pm0,t)+(\xi-\delta_1^0)\partial_x u^1(s_1(t)\pm0,t)-\delta_1^1\partial_x u^0(s_1(t)\pm0,t)\\&+\frac{1}{2}(\xi-\delta_1^0)^2\partial_x^2u^0(s_1(t)\pm0,t)+o(1),\end{aligned} \tag{2.19}$$

$$\begin{aligned}u_s^3(\xi,t)=&u^3(s_1(t)\pm0,t)+(\xi-\delta_1^0)\partial_x u^2(s_1(t)\pm0,t)-\delta_1^1\partial_x u^1(s_1(t)\pm0,t)\\&+\frac{1}{2}(\xi-\delta_1^0)^2\partial_x^2u^1(s_1(t)\pm0,t)-\delta_1^2\partial_x u^0(s_1(t)\pm0,t)-(\xi-\delta_1^0)\delta_1^1\partial_x^2u^0(s_1(t)\pm0,t)\\&+\frac{1}{6}(\xi-\delta_1^0)^3\partial_x^3u^0(s_1(t)\pm0,t)+o(1).\end{aligned} \tag{2.20}$$

2.3. 激波层 $x=s_2(t)$ 附近展开

在激波层 $\{x=s_2(t)\}$ 附近, 我们得到近似解:

$$u^\varepsilon(x,t)\sim\bar{u}_s^0(\eta,t)+\varepsilon\bar{u}_s^1(\eta,t)+\varepsilon^2\bar{u}_s^2(\eta,t)+\varepsilon^3\bar{u}_s^3(\eta,t)+\cdots, \tag{2.21}$$

其中

$$\eta=\frac{x-s_2(t)}{\varepsilon}+\delta_2(t,\varepsilon), \tag{2.22}$$

δ_2 是激波层扰动, 待定. 我们假设 $\delta_2(t,\varepsilon)$ 展开式为:

$$\delta_2(t,\varepsilon)=\delta_2^0(t)+\varepsilon\delta_2^1(t)+\varepsilon^2\delta_2^2(t)+\varepsilon^3\delta_2^3(t)+\cdots. \tag{2.23}$$

将近似解代入方程 (1.1) 得到:

$$O(\frac{1}{\varepsilon}):\partial_\eta^2\bar{u}_s^0+\dot{s}_2(t)\partial_\eta\bar{u}_s^0-\partial_\eta f(\bar{u}_s^0)=0, \tag{2.24}$$

$$O(1):\partial_\eta^2\bar{u}_s^1+\dot{s}_2(t)\partial_\eta\bar{u}_s^1-\partial_\eta(f'(\bar{u}_s^0)\bar{u}_s^1)=\dot{\delta}_2^0\partial_\eta\bar{u}_s^0+\partial_t\bar{u}_s^0, \tag{2.25}$$

$$O(\varepsilon):\partial_\eta^2\bar{u}_s^2+\dot{s}_2(t)\partial_\eta\bar{u}_s^2-\partial_\eta(f'(\bar{u}_s^0)\bar{u}_s^2)=\dot{\delta}_2^0\partial_\eta\bar{u}_s^1+\dot{\delta}_2^1\partial_\eta\bar{u}_s^0+\partial_t\bar{u}_s^1+\frac{1}{2}\partial_\eta(f''(\bar{u}_s^0)(\bar{u}_s^1)^2), \tag{2.26}$$

$$\begin{aligned}O(\varepsilon^2):&\partial_\eta^2\bar{u}_s^3+\dot{s}_2(t)\partial_\eta\bar{u}_s^3-\partial_\eta(f'(\bar{u}_s^0)\bar{u}_s^3)\\&=\dot{\delta}_2^2\partial_\eta\bar{u}_s^0+\dot{\delta}_2^1\partial_\eta\bar{u}_s^1+\dot{\delta}_2^0\partial_\eta\bar{u}_s^2+\partial_t\bar{u}_s^2+\partial_\eta(f''(\bar{u}_s^0)(\bar{u}_1^1\cdot\bar{u}_s^2))+\frac{1}{6}\partial_\eta(f'''(\bar{u}_s^0)(\bar{u}_s^1)^3),\end{aligned} \tag{2.27}$$

其中 $\dot{s}_2(t)=\frac{ds_2}{dt},\dot{\delta}_2=\frac{d\delta_2}{dt}$. 在匹配区域中, 对于 $\eta\to\pm\infty$, 激波层附近展开和外部展开是同时成立的, 则满足如下方程:

$$\bar{u}_s^0(\eta,t)=u^0(s_2(t)\pm0,t)+o(1), \tag{2.28}$$

$$\bar{u}_s^1(\eta,t)=u^1(s_2(t)\pm0,t)+(\eta-\delta_2^0)\partial_xu^0(s_2(t)\pm0,t)+o(1), \tag{2.29}$$

$$\begin{aligned}\bar{u}_s^2(\eta,t)=&u^2(s_2(t)\pm0,t)+(\eta-\delta_2^0)\partial_xu^1(s_2(t)\pm0,t)-\delta_2^1\partial_xu^0(s_2(t)\pm0,t)+\\&\frac{1}{2}(\eta-\delta_2^0)^2\partial_x^2u^2(s_2(t)\pm0,t)+o(1),\end{aligned} \tag{2.30}$$

$$\begin{aligned}\bar{u}_s^3(\eta,t)=&u^3(s_2(t)\pm0,t)+(\eta-\delta_2^0)\partial_x\bar{u}^2(s_2(t)\pm0,t)-\delta_2^1\partial_x\bar{u}^1(s_2(t)\pm0,t)+\\&\frac{1}{2}(\eta-\delta_2^0)^2\partial_x^2\bar{u}^1(s_2(t)\pm0,t)-\delta_2^2\partial_x\bar{u}^0(s_2(t)\pm0,t)-(\eta-\delta_2^0)\delta_2^1\partial_x^2\bar{u}^0(s_2(t)\pm0,t)\\&+\frac{1}{6}(\eta-\delta_2^0)^3\partial_x^3\bar{u}^0(s_2(t)\pm0,t)+o(1).\end{aligned} \tag{2.31}$$

在 $x=s_1(t)$, 存在常微分方程,及其边界条件为:

$$\partial_\xi f(u_s^0)-\dot{s}_1\partial_\xi u_s^0=\partial_\xi^2u_s^0, \tag{2.32}$$

$$u_s^0(\xi,t)\to u_l=u^0(s_1-0,t),\quad \xi\to-\infty, \tag{2.33}$$

$$u_s^0(\xi,t)\to u_r=u^0(s_1+0,t),\quad \xi\to+\infty. \tag{2.34}$$

在 $x=s_2(t)$, 存在常微分方程,及其边界条件为:

$$\partial_\eta f(\bar{u}_s^0)-\dot{s}_2\partial_\eta\bar{u}_s^0=\partial_\eta^2\bar{u}_s^0, \tag{2.35}$$

$$\bar{u}_s^0(\eta,t)\to\bar{u}_l=u^0(s_2-0,t),\quad \eta\to-\infty, \tag{2.36}$$

$$\bar{u}_s^0(\eta,t)\to\bar{u}_r=u^0(s_2+0,t),\quad \eta\to+\infty. \tag{2.37}$$

引理2.1: 方程组 (2.32), (2.33), (2.34) 对边值问题存在唯一光滑解 $u_s^0(\xi,t)$. 另外,

$\exists\alpha_0>0$, 满足:

$$|\partial_\xi u_s^0(\xi,t)|\le Ce^{-\alpha_0|\xi|}. \tag{2.38}$$

类似地, 方程组 (2.35), (2.36), (2.37) 对边值问题存在唯一光滑解 $\bar{u}_s^0(\xi,t)$. 另外,
$\exists\bar{\alpha}_0>0$, 满足:

$$|\partial_\eta \bar{u}_s^0(\eta,t)|\le Ce^{-\bar{\alpha}_0|\eta|}. \tag{2.39}$$

证明: 我们首先考虑问题 (2.32), (2.33), (2.34), 设 $W=\partial_\xi u_s^0$, 将方程 (2.32) 变形为

$$\partial_\xi W=W(\partial_u f(u_s^0)-\dot{s}_1).$$

对等式在 $[0,\xi]$ 区间上积分得到:

$$ln\frac{W(\xi)}{W(0)}=\int_0^\xi(\partial_u f(u_s^0(y,t))-\dot{s}_1)dy.$$
$$W(\xi)=W(0)\exp\{\int_0^\xi(\partial_u f(u_s^0(y,t))-\dot{s}_1)dy\},$$
$$\partial_\xi u_s^0(\xi,t)=\partial_\xi u_s^0(0,t)\exp\{\int_0^\xi(\partial_u f(u_s^0(y,t))-\dot{s}_1)dy\}.$$

再在区间 $(-\infty,+\infty)$ 积分得到:

$$u^0(s_1(t)+0,t)-u^0(s_1(t)-0,t)=\partial_\xi u_s^0(0,t)\int_{-\infty}^{+\infty}\exp\{\int_0^\xi(\partial_u f(u_s^0(y,t))-\dot{s}_1)dy\}d\xi.$$
$$\partial_\xi u_s^0=\frac{[u^0(s_1(t)+0,t)-u^0(s_1(t)-0,t)]\exp\{\int_0^\xi(\partial_u f(u_s^0(y,t))-\dot{s}_1)dy\}}{\int_{-\infty}^{+\infty}\exp\{\int_0^\xi(\partial_u f(u_s^0(y,t))-\dot{s}_1)dy\}d\xi}.$$

这样我们就给出了 $\partial_\xi u_s^0$ 的形式，同理可得:

$$\partial_\eta \bar{u}_s^0=\frac{[u^0(s_2+0,t)-u^0(s_2-0,t)]\exp\{\int_0^\eta(\partial_u f(\bar{u}_s^0(y,t))-\dot{s}_2)dy\}}{\int_{-\infty}^{+\infty}\exp\{\int_0^\eta(\partial_u f(\bar{u}_s^0(y,t))-\dot{s}_2)dy\}d\eta}.$$

我们希望 u_s^1 满足 [3] :

$$u_s^1=\xi\cdot\partial_x u^0(s_1(t)\pm0,t)+O(1),\quad \xi\to\pm\infty.$$

所以设

$$u_s^1(\xi,t)=V_1(\xi,t)+D_1(\xi,t), \tag{2.40}$$

其中 $D_1(\xi,t)$ 是光滑函数, 满足:

$$D_1(\xi,t)=\begin{cases}\xi\cdot\partial_x u^0(s_1(t)-0,t), & \xi<-1,\\ \xi\cdot\partial_x u^0(s_1(t)+0,t), & \xi>1,\end{cases} \tag{2.41}$$

$$\frac{d}{dt}u^0(s_1(t)\pm0,t)=(\dot{s}_1(t)-f'(u^0(s_1(t)\pm0,t)))\partial_x u^0(s_1(t)\pm0,t).$$

将 (2.40) 式代入 (2.14) 式，计算可得:

$$\partial_\xi^2 V_1(\xi,t)+\dot{s}_1(t)\partial_\xi V_1(\xi,t)-\partial_\xi(f'(u_s^0(\xi,t))V_1(\xi,t))=\dot{\delta}_1^0\partial_\xi u_s^0(\xi,t)+g(\xi,t),$$

$$\begin{aligned}g(\xi,t)&=(f'(u_s^0(\xi,t)-\dot{s}_1(t)))\partial_x u^0(s_1(t)\pm 0,t)+f''(u_s^0(\xi,t))\partial_\xi u_s^0(\xi,t)\xi\partial_x u^0(s_1(t)\pm 0,t)+\partial_t u_s^0(\xi,t)\\&=-\frac{d}{dt}\int_0^\xi \partial_\xi u_s^0(y,t)dy+f''(u_s^0(\xi,t))\partial_\xi u_s^0(\xi,t)\xi\partial_x u^0(s_1(t)\pm 0,t)+\partial_t\int_0^\xi \partial_\xi u_s^0(y,t)dy,\end{aligned}$$

通过计算我们给出估计:

$$|g(\xi,t)|\le Ce^{-\alpha_0|\xi|}.$$

定义 $G(\xi,t)=\int_0^\xi g(y,t)dy$, 我们有

$$\partial_\xi V_1(\xi,t)=(f'(u_s^0(\xi,t)-\dot{s}_1(t)))V_1(\xi,t)+\dot{\delta}_1^0 u_s^0(\xi,t)+G(\xi,t)+c(t), \tag{2.42}$$

$$\partial_\xi V_1(\xi,t)+(\dot{s}_1(t)-f'(u_s^0(\xi,t)))V_1(\xi,t)=\dot{\delta}_1^0 u_s^0(\xi,t)+G(\xi,t)+c(t). \tag{2.43}$$

其中 $c(t)$ 是一些常数的积分, 待定.

引理2.2: 方程 (2.43) 存在唯一光滑解 $V_1(\xi,t)$, 满足:

$$V_1(\xi,t)=\begin{cases}(\dot{s}_1(t)-f'(u_l))^{-1}[c(t)+u_l\dot{\delta}_1^0+G_-]+O(1)\exp\{-\alpha_1|\xi|\}, & \xi\to-\infty,\\(\dot{s}_1(t)-f'(u_r))^{-1}[c(t)+u_r\dot{\delta}_1^0+G_+]+O(1)\exp\{-\alpha_1|\xi|\}, & \xi\to+\infty,\end{cases} \tag{2.44}$$

其中 $G_\pm=\lim\limits_{\xi\to\pm\infty}G(\xi,t)$, $\alpha_1>0$.

证明: 由计算可得:

$$\begin{aligned}V_1(\xi,t)=&\int_0^\xi[\dot{\delta}_1^0 u_s^0(z,t)+G(z,t)+c(t)]\exp\{-\int_z^\xi(\dot{s}_1(t)-f'(u_s^0(y,t)))dy\}dz\\&+V_1(0,t)\exp\{-\int_0^\xi(\dot{s}_1(t)-f'(u_s^0(z,t)))dz\}.\end{aligned} \tag{2.45}$$

由 Lax 熵条件 (1.4) 可得到:

$$\lim_{\xi\to\pm\infty}\partial_\xi V_1(\xi,t)\sim O(1)\exp\{-\alpha_1|\xi|\},\quad \alpha_1>0.$$

又由 (2.18), 得到:

$$[c(t)+u_r\dot{\delta}_1^0+G_+]=[u^1(s_1(t)+0,t)-\delta_1^0\partial_x u^0(s_1(t)+0,t)](\dot{s}_1(t)-f'(u_r)), \tag{2.46}$$

$$[c(t)+u_l\dot{\delta}_1^0+G_-]=[u^1(s_1(t)-0,t)-\delta_1^0\partial_x u^0(s_1(t)-0,t)](\dot{s}_1(t)-f'(u_l)). \tag{2.47}$$

$$\begin{aligned}\dot{\delta}_1^0(u_l-u_r)=&(\dot{s}_1(t)-f'(u_l))u^1(s_1(t)-0,t)-(\dot{s}_1(t)-f'(u_r))u^1(s_1(t)+0,t)\\&+\delta_1^0[(\dot{s}_1(t)-f'(u_r))\partial_x u^0(s_1(t)+0,t)-(\dot{s}_1(t)-f'(u_l))\partial_x u^0(s_1(t)-0,t)]\\&+G_+-G_-.\end{aligned} \tag{2.48}$$

(2.46)与(2.47)两式相减得到(2.18)式，所以 δ_1^0 满足一阶线性常微分方程

$$\dot{\delta}_1^0 + E_1(t)\delta_1^0 = E_2(t) + G_1(t), \tag{2.49}$$

$$\begin{aligned}
E_1(t) =& [(\dot{s}_1(t) - f'(u_r))\partial_x u^0(s_1(t)+0,t) - (\dot{s}_1(t) - f'(u_l))\partial_x u^0(s_1(t)-0,t)](u_l - u_r)^{-1},\\
E_2(t) =& [(\dot{s}_1(t) - f'(u_l))u^1(s_1(t)-0,t) - (\dot{s}_1(t) - f'(u_r))u^1(s_1(t)+0,t)](u_l - u_r)^{-1},\\
G_1(t) =& (G_+ - G_-)(u_l - u_r)^{-1},
\end{aligned}$$

其中$E_1(t)$、$E_2(t)$和$G_1(t)$都是已知光滑函数. 所以可以求得 δ_1^0 是光滑函数. 将 δ_1^0 代回 (2.46) 或 (2.47) 可解得 $c(t)$. 再将 δ_1^0 和 $c(t)$ 代入 (2.44), 满足所希望的 $V_1(\xi,t)$.

命题2.3: u_s^1 和 δ_1^0 是光滑函数, 且存在 $\alpha_1 > 0$ 和一个有界函数 $O(1)$ 使得:

$$u_s^1 = u^1(s_1(t)\pm 0,t) + (\xi - \delta_1^0)\partial_x u^0(s_1(t)\pm 0,t) + O(1)\exp\{-\alpha_1|\xi|\}, \xi \to \pm\infty. \tag{2.50}$$

$\bar{u}_s^1$ 和 δ_2^0 是光滑函数, 且存在 $\bar{\alpha}_1 > 0$ 和一个有界函数 $O(1)$ 使得:

$$\bar{u}_s^1 = u^1(s_2(t)\pm 0,t) + (\eta - \delta_2^0)\partial_x u^0(s_2(t)\pm 0,t) + O(1)\exp\{-\bar{\alpha}_1|\eta|\}, \eta \to \pm\infty. \tag{2.51}$$

类似地, 我们可以定义 u_s^2, $\bar{u}_s^1$, δ_1^1, δ_2^1, u_s^3, $\bar{u}_s^3$, δ_1^2, δ_2^2, 这里我们不做详细论述.

引理:2.4 对任意 $t\in[0,T]$, 由凸性条件 $\partial_u^2 f(x,t)$ 成立, 则

$$\partial_\xi\partial_u f(u_s^0(\xi,t)) < 0, \tag{2.52}$$

$$\partial_\eta\partial_u f(u_s^0(\eta,t)) < 0, \tag{2.53}$$

证明:因为

$$\partial_\xi u_s^0 = \frac{[u^0(s_1(t)+0,t) - u^0(s_1(t)-0,t)]\exp\{\int_0^\xi(\partial_u f(u_s^0(y,t)) - \dot{s}_1)dy\}}{\int_{-\infty}^{+\infty}\exp\{\int_0^\xi(\partial_u f(u_s^0(y,t)) - \dot{s}_1)dy\}d\xi},$$

$$\partial_\eta \bar{u}_s^0 = \frac{[u^0(s_2(t)+0,t) - u^0(s_2(t)-0,t)]\exp\{\int_0^\eta(\partial_u f(\bar{u}_s^0(y,t)) - \dot{s}_2)dy\}}{\int_{-\infty}^{+\infty}\exp\{\int_0^\eta(\partial_u f(\bar{u}_s^0(y,t)) - \dot{s}_2)dy\}d\eta},$$

利用 (1.5) 可得 $\partial_\xi u_s^0 < 0$ 和 $\partial_\eta \bar{u}_s^0 < 0$. 注意到

$$\partial_\xi\partial_u f(u_s^0(\xi,t)) = \partial_u^2 f(u_s^0(\xi,t))\partial_\xi u_s^0(\xi,t).$$

$$\partial_\eta\partial_u f(\bar{u}_s^0(\eta,t)) = \partial_u^2 f(\bar{u}_s^0(\eta,t))\partial_\eta \bar{u}_s^0(\eta,t).$$

2.4. 近似解

通过前面对激波以及内外解的理论分析, 我们可以利用截断函数构造方程 (1.3) 的近似解. 依次定义激波层内外部解为:

$$O(x,t)=u^0(x,t)+\varepsilon u^1(x,t)+\varepsilon^2u^2(x,t)+\varepsilon^3u^3(x,t),x\neq s_i(t),i=1,2, \tag{2.54}$$

$$\begin{aligned}I_1(x,t)=&u_s^0(\frac{x-s_1(t)}{\varepsilon}+\delta_1^0+\varepsilon\delta_1^1+\varepsilon^2\delta_1^2,t)+\varepsilon u_s^1(\frac{x-s_1(t)}{\varepsilon}+\delta_1^0+\varepsilon\delta_1^1+\varepsilon^2\delta_1^2,t)\\&+\varepsilon^2u_s^2(\frac{x-s_1(t)}{\varepsilon}+\delta_1^0+\varepsilon\delta_1^1+\varepsilon^2\delta_1^2,t)+\varepsilon^3u_s^3(\frac{x-s_1(t)}{\varepsilon}+\delta_1^0+\varepsilon\delta_1^1+\varepsilon^2\delta_1^2,t),\end{aligned} \tag{2.55}$$

$$\begin{aligned}I_2(x,t)=&\bar{u}_s^0(\frac{x-s_2(t)}{\varepsilon}+\delta_2^0+\varepsilon\delta_2^1+\varepsilon^2\delta_2^2,t)+\varepsilon \bar{u}_s^1(\frac{x-s_2(t)}{\varepsilon}+\delta_2^0+\varepsilon\delta_2^1+\varepsilon^2\delta_2^2,t)\\&+\varepsilon^2\bar{u}_s^2(\frac{x-s_2(t)}{\varepsilon}+\delta_2^0+\varepsilon\delta_2^1+\varepsilon^2\delta_2^2,t)+\varepsilon^3\bar{u}_s^3(\frac{x-s_2(t)}{\varepsilon}+\delta_2^0+\varepsilon\delta_2^1+\varepsilon^2\delta_2^2,t),\end{aligned} \tag{2.56}$$

设截断函数 $m(x)\in C_0^\infty(\mathbb{R})$ 满足 $0\leq m(x)\leq 1$:

$$m(x)=\begin{cases}1, & |\,x\,|\leq 1,\\ 0, & |\,x\,|\geq 2,\\ h(x), & 1<|\,x\,|<2,\end{cases} \tag{2.57}$$

这里, $h(x)$ 是一个光滑函数并满足 $0\leq h(x)\leq 1$. 接下来, 我们定义方程 (1.1) 的近似解:

$$u^a(x,t)=m_1I_1+m_2I_2+m_3I_3+(1-m_1-m_2)O+d(x,t), \tag{2.58}$$

其中, $m_1=m(\frac{x-s_1(t)}{\varepsilon^\gamma})$, $m_2=m(\frac{x-s_2(t)}{\varepsilon^\gamma})$, $d(x,t)$ 为高阶修正项, 其中 $\frac{6}{7}<\gamma<1$. 利用激波层内外解的构造原理, 近似解 $u^a(x,t)$ 是方程

$$\begin{cases}\partial_tu^a+\partial_xf(u^a)-\varepsilon\partial_x^2u^a=\sum_{i=1}^5q_i(x,t),\\ u^a(x,0)=u_0\end{cases} \tag{2.59}$$

的解. 其中

$$\begin{aligned}q_1(x,t)\;=\;&(1-m_1-m_2)\{(f(O)-f(u^0)-\varepsilon f^{'}(u^0)u^1-\varepsilon^2f^{'}(u^0)u^2-\varepsilon^3f^{'}(u^0)u^3\\&-\frac{\varepsilon^2}{2}f^{'''}(u^0)(u^1)^2-\varepsilon^3f^{''}(u^0)(u^1,u^2)--\frac{\varepsilon^2}{6}f^{''}(u^0)(u^1)^3)_x-\varepsilon^4\partial_x^2u^3\},\\
q_2(x,t)\;=\;&m_1\{(f(I_1)-f(u_s^0)-\varepsilon f^{'}(u_s^0)u_s^1-\varepsilon^2f^{'}(u_s^0)u_s^2-\varepsilon^3f^{'}(u_s^0)u_s^3-\frac{\varepsilon^2}{2}f^{'}(u_s^0)(u_s^1)^2\\&-\varepsilon^3f^{''}(u_s^0)(u_1^1,u_s^2)+\varepsilon^2\partial_tu_b^2\}-\frac{\varepsilon^2}{6}f^{''}(u_s^0)(u_s^1)^3)_x+\varepsilon^3\partial_tu_s^3+\varepsilon^4(\dot{\delta}_1^1u_s^2+\dot{\delta}_1^2u_s^1+\varepsilon\dot{\delta}_1^2u_s^2)_x\},\\
q_3(x,t)\;=\;&m_2\{(f(I_2)-f(\bar{u}_s^0)-\varepsilon f^{'}(\bar{u}_s^0)\bar{u}_s^1-\varepsilon^2f^{'}(\bar{u}_s^0)\bar{u}_s^2-\varepsilon^3f^{'}(\bar{u}_s^0)\bar{u}_s^3-\frac{\varepsilon^2}{2}f^{'}(\bar{u}_s^0)(\bar{u}_s^1)^2\\&-\varepsilon^3f^{''}(\bar{u}_s^0)(\bar{u}_1^1,\bar{u}_s^2)+\varepsilon^2\partial_t\bar{u}_b^2\}-\frac{\varepsilon^2}{6}f^{''}(\bar{u}_s^0)(\bar{u}_s^1)^3)_x+\varepsilon^3\partial_t\bar{u}_s^3+\varepsilon^4(\dot{\delta}_2^1\bar{u}_s^2+\dot{\delta}_2^2\bar{u}_s^1+\varepsilon\dot{\delta}_2^2\bar{u}_s^2)_x\},\end{aligned}$$

$$\begin{aligned} q_4(x,t) &= \partial_t m_1(I_1-O)-\partial_t m_2(I_2-O)-\varepsilon\partial_x^2 m_1(I_1-O)\\ &\quad -2\varepsilon\partial_x m_1(I_1-O)_x-\varepsilon\partial_x^2 m_2(I_2-O)-2\varepsilon\partial_x m_2(I_2-O)_x\\ &\quad +\partial_x m_1(f(I_1)-f(O))+\partial_x m_2(f(I_2)-f(O))\\ &\quad +f(m_1I_1+m_2I_2+(1-m_1-m_2)O)_x-(m_1f(I_1)+m_2f(I_2)+(1-m_1-m_2)f(O))_x,\\ q_5(x,t) &= d_t-\varepsilon d_{xx}+(f(u^a)-f(u^a-d))_x, \end{aligned}$$

$$\mathrm{supp} m_1 \subseteq \{x:|x-s_1(t)|\le 2\varepsilon^\gamma\},$$

$$\mathrm{supp} m_2 \subseteq \{x:|x-s_2(t)|\le 2\varepsilon^\gamma\},$$

根据我们的构造过程，依次得到:

$$\begin{aligned} &\mathrm{supp} q_1 \subseteq \{(x,t):|x-s_i(t)|\ge \varepsilon^\gamma, 0\le t\le T, i=1,2\},\\ &\partial_x^l q_1(x,t)=O(1)\varepsilon^{4-l\gamma},(\int_0^T\|\partial_x^l q_1(\cdot,t)\|^2dt)^{\frac12}\le O(1)\varepsilon^{4-(l-\frac12)}\gamma, l=0,1,2,\\ &\mathrm{supp} q_2 \subseteq \{(x,t):|x-s_1(t)|\le 2\varepsilon^\gamma, 0\le t\le T\}, \end{aligned} \tag{2.60}$$

$$\partial_x^l q_2(x,t)=O(1)\varepsilon^{(3-l)\gamma}, l=0,1,2, \tag{2.61}$$

$$\mathrm{supp} q_3 \subseteq \{(x,t):|x-s_2(t)|\le 2\varepsilon^\gamma, 0\le t\le T\},$$

$$\partial_x^l q_3(x,t)=O(1)\varepsilon^{(3-l)\gamma}, l=0,1,2, \tag{2.62}$$

$$\mathrm{supp} q_4 \subseteq \{(x,t):\varepsilon^\gamma\le|x-s_i(t)|\le 2\varepsilon^\gamma, 0\le t\le T\},$$

$$\partial_x^l q_4(x,t)=O(1)\varepsilon^{(3-l)\gamma}, l=0,1,2, \tag{2.63}$$

上述计算我们用到了如下结论:

$$\partial_x^l(I_1-O)=O(1)\varepsilon^{(4-l)\gamma},\mathrm{on}\{(x,t):\varepsilon^\gamma\le|x-s_1(t)|\le 2\varepsilon^\gamma, t\in[0,T], \tag{2.64}$$

$$\partial_x^l(I_2-O)=O(1)\varepsilon^{(4-l)\gamma},\mathrm{on}\{(x,t):\varepsilon^\gamma\le|x-s_2(t)|\le 2\varepsilon^\gamma, t\in[0,T], \tag{2.65}$$

令 $R^\varepsilon=\sum_{i=1}^4 q_i(x,t)$, 则 $R^\varepsilon=O(1)\varepsilon^{3\gamma}$, 现在令 $d(x,t)$ 是下列扩散问题的解

$$\begin{cases} d_t=\varepsilon d_{xx}-\sum_{i=1}^4 q_i(x,t),\\ d(x,t)=0 \end{cases} \tag{2.66}$$

因此 u^a 满足：

$$\partial_t u^a+\partial_x f(u^a)-\varepsilon\partial_x^2 u^a=(f(u^a)-f(u^a-d))_x, \tag{2.67}$$

这里需要估计 $d(x,t)$, 利用标准能量估计可得下列引理:

引理: 2.5 :令 $d(x,t)$ 是方程 (2.66) 的解，则 $\forall t\in[0,T]$ 有下列估计:

$$\sup_{0\le t\le T}\int_{\mathbb{R}}|d(x,t)|^2dx+\varepsilon\int_0^T\int_{\mathbb{R}}|\partial_x d(x,t)|^2\,dxdt\le C\varepsilon^{7\gamma},\tag{2.68}$$

$$\sup_{0\le t\le T}\int_{\mathbb{R}}|\partial_x d(x,t)|^2dx+\varepsilon\int_0^T\int_{\mathbb{R}}\left|\partial_x^2 d(x,t)\right|^2\,dxdt\le C\varepsilon^{5\gamma},\tag{2.69}$$

$$\sup_{0\le t\le T}\int_{\mathbb{R}}\left|\partial_x^2 d(x,t)\right|^2dx+\varepsilon\int_0^T\int_{\mathbb{R}}\left|\partial_x^3 d(x,t)\right|^2\,dxdt\le C\varepsilon^{3\gamma},\tag{2.70}$$

$$\sup_{0\le t\le T}\parallel d(x,t)\parallel_{L^\infty(\mathbb{R})}\le C\varepsilon^{3\gamma},\tag{2.71}$$

$$\sup_{0\le t\le T}\parallel \partial_x d(x,t)\parallel_{L^\infty(\mathbb{R})}\le C\varepsilon^{2\gamma},\tag{2.72}$$

证明:在方程 (2.66) 两边乘以 d, 再在 $\mathbb{R}$ 上积分, 并利用分部积分可得:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}d^2dx+\varepsilon\int_{\mathbb{R}}|\partial_x d|^2dx+\int_{\mathbb{R}}d\cdot R^\varepsilon dx=0.\tag{2.73}$$

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}d^2dx+\varepsilon\int_{\mathbb{R}}|\partial_x d|^2dx\le C\varepsilon^{7\gamma}+C\int_{\mathbb{R}}d^2dx.\tag{2.74}$$

再利用 Gronwall 不等式可得:

$$\sup_{0\le t\le T}\int_{\mathbb{R}}|d(x,t)|^2\,dx+\varepsilon\int_0^T\int_{\mathbb{R}}|\partial_x d(x,t)|^2\,dxdt\le C\varepsilon^{7\gamma},$$

令 $D=\partial_x d$, 则 D 满足:

$$\begin{cases}D_t=\varepsilon D_{xx}-\partial_x R^\varepsilon,\\ D(x,0)=0\end{cases}\tag{2.75}$$

同样估计可得:

$$\sup_{0\le t\le T}\int_{\mathbb{R}}|\partial_x d(x,t)|^2\,dx+\varepsilon\int_0^T\int_{\mathbb{R}}\left|\partial_x^2 d(x,t)\right|^2\,dxdt\le C\varepsilon^{5\gamma},\tag{2.76}$$

由 (2.68)，(2.69) 和 Sobolev 不等式, 得:

$$\sup_{0\le t\le T}\parallel d(x,t)\parallel_{L^\infty(\mathbb{R})}\le\sqrt{2}\parallel d(x,t)\parallel_{L^2_{\mathbb{R}}}^{\frac{1}{2}}\parallel\partial_x d(x,t)\parallel_{L^2_{\mathbb{R}}}^{\frac{1}{2}}\le C\varepsilon^{3\gamma}.$$

令 $\bar{D}=\partial_x D$, 则 $\bar{D}$ 满足:

$$\begin{cases}\bar{D}_t=\varepsilon\bar{D}_{xx}-\partial_x^2 R^\varepsilon,\\ \bar{D}(x,0)=0\end{cases}\tag{2.77}$$

同样估计可得:

$$\sup_{0\le t\le T}\int_{\mathbb{R}}\left|\partial_x^2 d(x,t)\right|^2\,dx+\varepsilon\int_0^T\int_{\mathbb{R}}\left|\partial_x^3 d(x,t)\right|^2\,dxdt\le C\varepsilon^{3\gamma},\tag{2.78}$$

由 (2.69)，(2.70) 和 Sobolev 不等式, 得:

$$\sup_{0\le t\le T} \| \partial_x d(x,t) \|_{L^\infty(\mathbb{R})}$$
$$\le \sqrt{2} \| \partial_x d(x,t) \|_{L^2_{\mathbb{R}}}^{\frac{1}{2}} \| \partial_x^2 d(x,t) \|_{L^2_{\mathbb{R}}}^{\frac{1}{2}} \le C\varepsilon^{2\gamma}.$$

引理2.6: 设 $u^a(xt)$ 是 (2.58) 定义的, 则

$$u^a(x,t) = \begin{cases} u^0(x,t) + O(1)\varepsilon, & |x - s_i(t)| \ge \varepsilon^\gamma, i = 1,2 \\ u_s^0(\xi,t) + O(1)\varepsilon^\gamma, & |x - s_1(t)| \le 2\varepsilon^\gamma \\ u_s^0(\eta,t) + O(1)\varepsilon^\gamma, & |x - s_2(t)| \le 2\varepsilon^\gamma. \end{cases} \tag{2.79}$$

证明: 由构造可得:

$$u^a(x,t) = \begin{cases} I_1 + d, & |x - s_1(t)| \le \varepsilon^\gamma, \\ O + m_1(I_1 - O) + d, & \varepsilon^\gamma \le |x - s_1(t)| \le 2\varepsilon^\gamma, \\ O + d, & |x - s_i(t)| \ge 2\varepsilon^\gamma, i = 1,2 \\ O + m_2(I_2 - O) + d, & \varepsilon^\gamma \le |x - s_2(t)| \le 2\varepsilon^\gamma, \\ I_2 + d, & |x - s_2(t)| \le \varepsilon^\gamma. \end{cases}$$

且在 $|x - s_i(t)| > \varepsilon^\gamma, i = 1,2$ 上, $O(x,t) = u^0(x,t) + O(1)\varepsilon$, 在 $|x - s_1(t)| \le \varepsilon^\gamma$ 上, $I_1(x,t) = u_s^0(\xi,t) + O(1)\varepsilon^\gamma$, 在 $|x - s_2(t)| \le \varepsilon^\gamma$ 上, $I_2(x,t) = \overline{u}_s^0(\eta,t) + O(1)\varepsilon^\gamma$. 因此, 结合 (2.57), (2.58) 和引理 2.5, (2.79) 得证.

3. 稳定性分析

假设 $u^\varepsilon(x,t)$ 是方程 (1.2) 的真实解, 令

$$u^\varepsilon(x,t) = u^a(x,t) + \varepsilon^{1/2+\delta} v(x,t), \quad \delta \in (0, \frac{1}{2}), \quad x \in \mathbb{R}, \quad t \in [0,T]. \tag{3.1}$$

因为 $u^a(x,t)$ 满足 (2.67), 计算可得:

$$\partial_t v - \varepsilon \partial_x^2 v + \varepsilon^{-(1/2+\delta)} \partial_x (f(u^a + \varepsilon^{1/2+\delta} v) - f(u^a - d)) = 0, \tag{3.2}$$

$$v(x,0) = 0. \tag{3.3}$$

令 $\varphi(x,t)=\int_{-\infty}^{x}v(z,t)dz,\ \ \forall x\in\mathbb{R}$, 从 (3.2) 可得下列积分误差方程

$$\partial_t\varphi-\varepsilon\partial_x^2\varphi+\varepsilon^{-(1/2+\delta)}(f(u^a+\varepsilon^{1/2+\delta}v)-f(u^a-d))=0. \tag{3.4}$$

$$\varphi(x,0)=0. \tag{3.5}$$

因此对问题 (3.4), (3.5) 的解 $\varphi(x,t)$ 有如下结论.

命题3.1: $\forall\varepsilon>0$, 问题 (3.4), (3.5) 有唯一解 $\varphi(x,t)\in C^1([0,T];H^2(\mathbb{R}))$, 并满足:

$$\sup_{0\le t\le T}\|\partial_x\varphi\|_{L^\infty(\mathbb{R})}\le C\varepsilon^{7\gamma/2-\delta-2}. \tag{3.6}$$

则柯西问题 (1.2) 有唯一解 $u^\varepsilon\in C^1([0,T];H^2(\mathbb{R}))$ 满足:

$$\sup_{0\le t\le T}\|u^\varepsilon(\cdot,t)-u^a(\cdot,t)\|_{L^\infty(\mathbb{R})}\le C\varepsilon^{7\gamma/2-3/2}, \tag{3.7}$$

其中 C 是正常数, $\frac{6}{7}<\gamma<1$.

对问题 (3.4), (3.5) 的解作先验估计引入以下几个引理.

引理3.2: 令 $\varphi\in C^1([0,T];H^2(\mathbb{R}))$ 是问题 (3.4), (3.5) 的解, 则存在常数 C 使得:

$$\sup_{0\le t\le T}\int_{\mathbb{R}}\varphi^2dx+\varepsilon\int_0^T\int_{\mathbb{R}}|\partial_x\varphi|^2dxdt\le C\varepsilon^{7\gamma-2\delta-1}. \tag{3.8}$$

这里假设

$$\sup_{0\le t\le T}\|\partial_x\varphi\|_{L^\infty(\mathbb{R})}\le C \tag{3.9}$$

证明: 将方程 (3.4) 的两边乘以 φ, 并在 R 上积分, 并利用分部积分可得:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\varphi^2dx+\varepsilon\int_{\mathbb{R}}|\partial_x\varphi|^2dx+\varepsilon^{-(1/2+\delta)}\int_{\mathbb{R}}(f(u^a+\varepsilon^{1/2+\delta}v)-f(u^a-d))\varphi dx=0.$$

上式方程左边第三项可变为

$$\begin{aligned}&\varepsilon^{-(1/2+\delta)}\int_{\mathbb{R}}(\partial_uf(u^a)(\varepsilon^{1/2+\delta}\varphi_x+d)+O(1)(\varepsilon^{1/2+\delta}\varphi_x+d)^2)\varphi dx\\=&\varepsilon^{-(1/2+\delta)}\int_{\mathbb{R}}\partial_uf(u^a)d\cdot\varphi dx+\int_{\mathbb{R}}\partial_uf(u^a)\varphi_x\cdot\varphi dx+O(1)\varepsilon^{-(1/2+\delta)}\int_{\mathbb{R}}(\varepsilon^{1/2+\delta}\varphi_x+d)^2\cdot\varphi dx\\=&\sum_{i=1}^{3}A_i.\end{aligned}$$

首先利用了 (2.68),我们给出估计

$$|A_1|\le C\varepsilon^{-1-2\delta}\int_{\mathbb{R}}d^2dx+C\int_{\mathbb{R}}\varphi^2dx\le C\int_{\mathbb{R}}\varphi^2dx+C\varepsilon^{7\gamma-2\delta-1},$$

$$A_2=\int_{\mathbb{R}}\partial_u f(u^a)\varphi\cdot\partial_x\varphi dx=\frac{1}{2}\int_{\mathbb{R}}\partial_u f(u^a)\partial_x\varphi^2dx=-\frac{1}{2}\int_{\mathbb{R}}\partial_x\partial_u f(u^a)\varphi^2dx$$

$$=-\frac{1}{2}\int_{-\infty}^{s_1(t)-2\varepsilon^\gamma}\partial_x\partial_u f(u^a)\varphi^2dx-\frac{1}{2}\int_{s_1(t)-2\varepsilon^\gamma}^{s_1(t)-\varepsilon^\gamma}\partial_x\partial_u f(u^a)\varphi^2dx-\frac{1}{2}\int_{s_1(t)-\varepsilon^\gamma}^{s_1(t)+\varepsilon^\gamma}\partial_x\partial_u f(u^a)\varphi^2dx$$

$$-\frac{1}{2}\int_{s_1(t)+\varepsilon^\gamma}^{s_1(t)+2\varepsilon^\gamma}\partial_x\partial_u f(u^a)\varphi^2dx-\frac{1}{2}\int_{s_1(t)+2\varepsilon^\gamma}^{s_2(t)-2\varepsilon^\gamma}\partial_x\partial_u f(u^a)\varphi^2dx-\frac{1}{2}\int_{s_2(t)-2\varepsilon^\gamma}^{s_2(t)-\varepsilon^\gamma}\partial_x\partial_u f(u^a)\varphi^2dx$$

$$-\frac{1}{2}\int_{s_2(t)-\varepsilon^\gamma}^{s_2(t)+\varepsilon^\gamma}\partial_x\partial_u f(u^a)\varphi^2dx-\frac{1}{2}\int_{s_2(t)+\varepsilon^\gamma}^{s_2(t)+2\varepsilon^\gamma}\partial_x\partial_u f(u^a)\varphi^2dx-\frac{1}{2}\int_{s_2(t)+2\varepsilon^\gamma}^{+\infty}\partial_x\partial_u f(u^a)\varphi^2dx$$

$$=\sum_{i=1}^{9}J_i.$$

然后估计 $J_i, i=1,2,3,\ldots,9$, 如下:

$$J_1=-\frac{1}{2}\int_{-\infty}^{s_1(t)-2\varepsilon^\gamma}\partial_u^2 f(u^a)(\partial_x O+\partial_x d)\varphi^2dx=O(1)\int_{-\infty}^{s_1(t)-2\varepsilon^\gamma}\varphi^2dx,$$

这里我们利用了 (2.69).

$$J_2=-\frac{1}{2}\int_{s_1(t)-2\varepsilon^\gamma}^{s_1(t)-\varepsilon^\gamma}\partial_u^2 f(u^a)(\partial_x O+\partial_x m_1(I_1-O)+m_1\partial_x(I_1-O)+\partial_x d)\varphi^2dx$$

$$=O(1)\int_{s_1(t)-2\varepsilon^\gamma}^{s_1(t)-\varepsilon^\gamma}\varphi^2dx+O(1)\varepsilon^{2\gamma}\int_{s_1(t)-2\varepsilon^\gamma}^{s_1(t)-\varepsilon^\gamma}\varphi^2dx$$

$$=O(1)\int_{s_1(t)-2\varepsilon^\gamma}^{s_1(t)-\varepsilon^\gamma}\varphi^2dx,$$

这里我们利用了在匹配区域内有$\partial_x^l(I_1-O)=O(1)\varepsilon^{(4-l)\gamma}$.

$$J_3=-\frac{1}{2}\int_{s_1(t)-\varepsilon^\gamma}^{s_1(t)+\varepsilon^\gamma}\frac{1}{\varepsilon}\partial_\xi\partial_u f(I_1+d)\varphi^2dx=\frac{1}{2\varepsilon}\int_{s_1(t)-\varepsilon^\gamma}^{s_1(t)+\varepsilon^\gamma}|\partial_\xi\partial_u f(u_s^0)\varphi^2|dx+O(1)\int_{s_1(t)-\varepsilon^\gamma}^{s_1(t)+\varepsilon^\gamma}\varphi^2dx,$$

这里我们利用了 $\partial_\xi\partial_u f(u_s^0)<0$. 类似地

$$\sum_{i=4}^{9}J_i=O(1)\int_{s_1(t)+\varepsilon^\gamma}^{s_1(t)+2\varepsilon^\gamma}\varphi^2dx+O(1)\int_{s_1(t)+2\varepsilon^\gamma}^{s_2(t)-2\varepsilon^\gamma}\varphi^2dx,$$

$$+O(1)\int_{s_2(t)-2\varepsilon^\gamma}^{s_2(t)-\varepsilon^\gamma}\varphi^2dx,+\frac{1}{2\varepsilon}\int_{s_2(t)-\varepsilon^\gamma}^{s_2(t)+\varepsilon^\gamma}|\partial_\eta\partial_u f(\bar{u}_s^0)\varphi^2|dx+O(1)\int_{s_2(t)-\varepsilon^\gamma}^{s_2(t)+\varepsilon^\gamma}\varphi^2dx,$$

$$+O(1)\int_{s_2(t)+\varepsilon^\gamma}^{s_2(t)+2\varepsilon^\gamma}\varphi^2dx,+O(1)\int_{s_2(t)+2\varepsilon^\gamma}^{+\infty}\varphi^2dx.$$

最后给出估计

$$
\begin{aligned}
|A_3| &\leq C\varepsilon^{-(1/2+\delta)}\int_{\mathbb{R}}|d^2\cdot\varphi|dx+\varepsilon^{1/2+\delta}|(\varphi_x)^2\varphi|dx\\
&\leq C\varepsilon^{4\gamma-1-2\delta}\int_{\mathbb{R}}d^2dx+C\int_{\mathbb{R}}\varphi^2dx+C\varepsilon^{1+2\delta-1}\cdot\varepsilon\int_{\mathbb{R}}(\varphi_x)^2dx\\
&\leq C\varepsilon^{11\gamma-2\delta-1}+\beta\varepsilon\int_{\mathbb{R}}(\varphi_x)^2dx+C\int_{\mathbb{R}}\varphi^2dx,
\end{aligned}
$$

这里, 对某个 $\beta > 0$, 有,我们这里利用了 (2.71) 和 (3.9). 综合以上所有的估计可得:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\varphi^2dx+(1-\beta)\varepsilon\int_{\mathbb{R}}|\partial_x\varphi|dx+\frac{1}{2\varepsilon}\int_{s_1(t)-\varepsilon^\gamma}^{s_1(t)+\varepsilon^\gamma}|\partial_\xi\partial_u f(u_s^0(x,t))\varphi^2|dx\\
&+\frac{1}{2\varepsilon}\int_{s_2(t)-\varepsilon^\gamma}^{s_2(t)+\varepsilon^\gamma}|\partial_\eta\partial_u f(\bar{u}_s^0)\varphi^2|dx\leq C\varepsilon^{7\gamma-2\delta-1}+C\int_{\mathbb{R}}\varphi^2dx.
\end{aligned}
\tag{3.10}
$$

选择适当 β, 使得 $1-\beta > 0$, 且利用 Gronwall 不等式, 有

$$
\sup_{0\leq t\leq T}\int_{\mathbb{R}}\varphi^2dx+\varepsilon\int_0^T\int_{\mathbb{R}}|\partial_x\varphi|^2dxdt\leq C\varepsilon^{7\gamma-2\delta-1}.
$$

引理3.3: 与引理 3.2 同样的假设条件, 则存在正常数 C, 使得:

$$
\sup_{0\leq t\leq T}\int_{\mathbb{R}}|\partial_x\varphi|^2dx+\varepsilon\int_0^T\int_{\mathbb{R}}|\partial_x^2\varphi|^2dxdt\leq C\varepsilon^{7\gamma-2\delta-3}.
\tag{3.11}
$$

证明: 将 (3.2) 两边同乘以 v, 并在 R 上积分, 并利用分部积分可得:

$$
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}v^2dx+\varepsilon\int_{\mathbb{R}}|\partial_x v|^2dx-\varepsilon^{-(1/2+\delta)}\int_{\mathbb{R}}(f(u^a+\varepsilon^{1/2+\delta}v)-f(u^a-d)\partial_x vdx=0.
$$

方程左边第三项可变为

$$
\begin{aligned}
&\varepsilon^{-(1/2+\delta)}|\int_{\mathbb{R}}(\partial_u f(u^a)(\varepsilon^{1/2+\delta}v+d))\partial_x vdx|\\
\leq &O(1)\varepsilon^{-1-\delta-1}\int_{\mathbb{R}}d^2dx+\beta\varepsilon\int_{\mathbb{R}}|\partial_x v|^2dx+O(1)\varepsilon^{-1}\int_{\mathbb{R}}v^2dx+\beta\varepsilon\int_{\mathbb{R}}|\partial_x v|^2dx\\
\leq &C\varepsilon^{7\gamma-2\delta-2}+2\beta\varepsilon\int_{\mathbb{R}}|\partial_x v|^2dx+C\varepsilon^{7\gamma-2\delta-3},
\end{aligned}
$$

这里我们利用了 (2.68) 和引理 3.2. 所以

$$
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}|\partial_x\varphi|^2dx+(1-2\beta)\varepsilon\int_{\mathbb{R}}|\partial_x^2\varphi|^2dx\leq C\varepsilon^{7\gamma-2\delta-3}.
\tag{3.12}
$$

通过选择适当 β, 使得 $1-2\beta>0$ 且利用 Gronwall 不等式, 可得:

$$\sup_{0\le t\le T}\|\partial_x\varphi\|^2_{L^2(\mathbb{R})}+\varepsilon\int_0^T\|\partial_x^2\varphi\|^2_{L^2(\mathbb{R})}dt\le C\varepsilon^{7\gamma-2\delta-3}.$$

最后, 我们对 φ 的二阶导数进行估计.

引理3.4: 与引理 3.2 同样的假设条件, 则存在正常数 C, 使得:

$$\sup_{0\le t\le T}\int_{\mathbb{R}}|\partial_x v|^2dx+\varepsilon\int_0^T\int_{\mathbb{R}}|\partial_x^2 v|^2dxdt\le C\varepsilon^{7\gamma-2\delta-5}. \tag{3.13}$$

证明: 令 $\theta(x,t)=\partial_x v(x,t)=\partial_x^2\varphi(x,t)$, 则 θ 满足:

$$\partial_t\theta-\varepsilon\partial_x^2\theta+\varepsilon^{-(1/2+\delta)}\partial_x^2(f(u^a+\varepsilon^{1/2+\delta}v)-f(u^a-d))=0. \tag{3.14}$$

$$\theta(x,0)=0. \tag{3.15}$$

将方程 (3.14) 的两边乘以 θ, 然后在 R 上积分并利用分部积分可得:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\theta^2dx+\varepsilon\int_{\mathbb{R}}|\partial_x\theta|^2dx=\varepsilon^{-(1/2+\delta)}\int_{\mathbb{R}}\partial_x(f(u^a+\varepsilon^{1/2+\delta}v)-f(u^a-d))\theta_x dx.$$

上面方程右边项是有界的, 因为

$$\begin{aligned}
&\varepsilon^{-(1/2+\delta)}|\int_{\mathbb{R}}\partial_x(f(u^a+\varepsilon^{1/2+\delta}v)-f(u^a-d))\theta_x dx|\\
=&\varepsilon^{-(1/2+\delta)}|\int_{\mathbb{R}}\partial_x(\partial_u f(u^a)(\varepsilon^{1/2+\delta}v+d))\partial_x\theta dx|\\
\le&O(1)\varepsilon^{-(1/2+\delta)}\int_{\mathbb{R}}|(\varepsilon^{1/2+\delta}v+d)\partial_x\theta|dx+\varepsilon^{-(1/2+\delta)}\int_{\mathbb{R}}|\partial_u f(u^a)(\varepsilon^{-(1/2+\delta)}\partial_x v+\partial_x d)\partial_x\theta|dx\\
\le&O(1)\varepsilon^{-1}\int_{\mathbb{R}}v^2dx+\beta\varepsilon\int_{\mathbb{R}}|\partial_x\theta|^2dx+O(1)\varepsilon^{-2-2\delta}\int_{\mathbb{R}}d^2dx+\beta\varepsilon\int_{\mathbb{R}}|\partial_x\theta|^2dx\\
&+O(1)\varepsilon^{-2-2\delta}\int_{\mathbb{R}}|\partial_x d|^2dx+\beta\varepsilon\int_{\mathbb{R}}|\partial_x\theta|^2dx+O(1)\varepsilon^{-1}\int_{\mathbb{R}}|\partial_x v|^2dx+\beta\varepsilon\int_{\mathbb{R}}|\partial_x\theta|^2dx\\
\le&C\varepsilon^{7\gamma-2\delta-4}+4\beta\varepsilon\int_{\mathbb{R}}|\partial_x\theta|^2dx+C\varepsilon^{7\gamma-2\delta-2}+C\varepsilon^{5\gamma-2\delta-2}+C\varepsilon^{7\gamma-2\delta-5},
\end{aligned}$$

这里利用了 (2.68), (2.69)和引理 3.3. 因此

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\theta^2dx+(1-4\beta)\varepsilon\int_{\mathbb{R}}|\partial_x\theta|^2dx\le C\varepsilon^{7\gamma-2\delta-5}. \tag{3.16}$$

选择适当的 β, 使得 $1-4\beta>0$ 并利用 Gronwall 不等式可得:

$$\sup_{0\le t\le T}\int_{\mathbb{R}}\theta^2dx+\varepsilon\int_0^T\int_{\mathbb{R}}|\partial_x\theta|^2dxdt\le C\varepsilon^{7\gamma-2\delta-5}.$$

命题 3.1 的证明: 为证明问题 (3.4), (3.5) 的解 $\varphi \in C^1([0,T];H^2(\mathbb{R}))$ 的存在性, 我们需要证明假设 (3.9) 成立, 由 (3.11), (3.13) 和 Sobolev不等式, 得:

$$\begin{aligned}\sup_{0\le t\le T}\|\partial_x\varphi(x,t)\|_{L^\infty(\mathbb{R})} &\le \sqrt{2}\sup_{0\le t\le T}\|\partial_x\varphi(x,t)\|_{L^2(\mathbb{R})}^{1/2}\cdot\sup_{0\le t\le T}\|\partial_x^2\varphi(x,t)\|_{L^2(\mathbb{R})}^{1/2} \\ &\le C\varepsilon^{7\gamma/2-\delta-2}.\end{aligned} \tag{3.17}$$

(3.17) 表明问题 (1.2) 存在唯一解 $u^\varepsilon \in C^1([0,T];H^2(\mathbb{R}))$, 且

$$\begin{aligned}\sup_{0\le t\le T}\|u^\varepsilon(\cdot,t)-u^a(\cdot,t)\|_{L^\infty(\mathbb{R})} &= \sup_{0\le t\le T}\|\varepsilon^{1/2+\delta}\partial_x\varphi(x,t)\|_{L^\infty(\mathbb{R})} \\ &\le C\varepsilon^{7\gamma/2-3/2}.\end{aligned}$$

由此 (3.7) 成立, 则命题 3.1 得证. 下面证明定理 1. 由引理 3.3, 可得:

$$\begin{aligned}\sup_{0\le t\le T}\|u^\varepsilon(\cdot,t)-u^a(\cdot,t)\|_{L^2(\mathbb{R})} &= \sup_{0\le t\le T}\|\varepsilon^{1/2+\delta}\partial_x\varphi(x,t)\|_{L^2(\mathbb{R})} \\ &\le C\varepsilon^{7\gamma/2-1}.\end{aligned}$$

这样得到 (1.6). 接下来证明 (1.7), 利用 (3.7) 和引理 2.5, 对 $|x-s_i(t)|\ge\varepsilon^\gamma, i=1,2,\gamma\in(\frac{6}{7},1)$,

$$\begin{aligned}\sup_{\substack{0\le t\le T\\|x-s_i(t)|\ge\varepsilon^\gamma}}|u^\varepsilon(\cdot,t)-u^0(\cdot,t)| &\le \sup_{0\le t\le T}|u^\varepsilon(\cdot,t)-u^a(\cdot,t)| + \sup_{\substack{0\le t\le T\\|x-s_i(t)|\ge\varepsilon^\gamma}}|u^a(\cdot,t)-u^0(\cdot,t)| \\ &\le C\varepsilon^{7\gamma/2-3/2}+C\varepsilon \\ &\le C\varepsilon.\end{aligned}$$

因此可得出结论.

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