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一类广义压力下的二维可压缩欧拉方程的特征分解

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摘 要

本文利用直接方法讨论了一类压力下二维可压缩欧拉系统的特征分解存在的充分条件，得到了压强p与特征值$\Lambda_\pm$的特征分解，证明了与常状态相邻的任何波都是简单波。

关键词

欧拉方程，特征分解，简单波

Characteristic Decomposition of the Two-Dimensional Compressible Euler System for a Class of Pressure Laws

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Abstract

This paper is considering a two-dimensional compressible Euler system for a class of pressure laws, a direct method is used to discuss the sufficient conditions for the existence of the characteristic decomposition and the decompositions of the pressure p and the characteristics $\Lambda_\pm$ are obtained. It is found that any wave adjacent to a constant state is a simple wave.

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Keywords

Euler Equations, Characteristic Decomposition, Simple Wave

1. 引言

拟线性双曲守恒律是现代数学研究的主要内容之一。空气动力学，气象学，水波等离子物理学和燃烧理论都涉及拟线性双曲守恒律。因此，研究拟线性双曲守恒律具有非常重要的现实意义。本文针对一类压力定律[1]考虑二维可压缩的欧拉系统，对 Courant 和 Friedrichs 的《超音速流与激波》[2]一书中关于可约方程的著名结果进行了推广。

特征分解法可以非常有效的处理拟线性双曲方程的一些问题[3] [4] [5] [6] [7] [8]。Dai 和 Zhang 在压力梯度系统中揭示了特征分解是构建整体光滑解的有力工具[9]，随后 Li、Zhang 和 Zheng 针对理想气体可压缩欧拉方程证明了该特征分解并将其用于讨论简单波[10]。之后，Hu 和 Sheng 为一般 2 × 2 拟线性严格双曲型方程的特征分解的存在建立了一个充分条件[11]。

同时，特征分解法也是研究简单波的一个重要技术。简单波被定义为区域内的流动，其解仅取决于单个参数。它在描述和建立空气动力学和流体力学中的流动问题的解决方案中起着重要的作用。本文的主要目的是扩展[1]中的定理，即在一类压力定律下的二维等熵可压缩欧拉方程中，与常状态流动相邻的任何流动都是简单波。并且在这个状态下，简单波的流动区域被一族直线所覆盖，速度，压力和密度沿每条直线是恒定的。

考虑二维等熵可压缩欧拉方程：

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0, \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = 0, \\ (\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y = 0, \end{cases} \tag{1}$$

其中 ρ 是密度，(u,v) 是速度，$p(\rho) = k_1^{\gamma_1} + k_2^{\gamma_2}$ 是压力，$\gamma_1, \gamma_2 \in (1,3)$，$k_1, k_2 > 0$ 是任意常数[2]，$c(\rho) = \sqrt{p'(\rho)}$ 是声速。由此，我们可得

$$\rho'(c) = \frac{2c}{p''(\rho)},\ k'(\rho) = \frac{2\rho p''(\rho)^2 - 2\rho'(c) p'(\rho) p''(\rho) - 2\rho p'(\rho) p'''(\rho)}{\rho^2 p''(\rho)^2}.$$

为了方便下文中的研究，我们引入新的概念：

$$\Lambda_+ = \tan\alpha,\ \Lambda_- = \tan\beta,\ U = u - \xi,\ V = v - \eta,$$

$$\sigma = \frac{\alpha+\beta}{2},\ \omega = \frac{\alpha-\beta}{2},\ k(\rho) = \frac{2p'(\rho)}{\rho p''(\rho)},\ m = \frac{2p'(\rho) - \rho p''(\rho)}{2p'(\rho) + \rho p''(\rho)},$$

$$\partial_\pm = \partial_\xi + \partial_\eta,\ \bar{\partial}_+ = \cos\alpha\partial_\xi + \sin\alpha\partial_\eta,\ \bar{\partial}_- = \cos\beta\partial_\xi + \sin\beta\partial_\eta.$$

考虑只依赖于自相似变量 $(\xi,\eta) = (x/t, y/t)$ 的自相似流，方程(1)可以写成

$$\begin{cases}(\rho U)_\xi+(\rho V)_\eta+2\rho=0,\\ UU_\xi+VU_\eta+\dfrac{1}{\rho}p_\xi+U=0,\\ UV_\xi+VV_\eta+\dfrac{1}{\rho}p_\eta+V=0.\end{cases}\tag{2}$$

考虑无旋流：$u_y=v_x\Rightarrow u_\eta=v_\xi$，化简方程(2)，可以得到

$$\begin{cases}\left(c^2-U^2\right)u_\xi-UV\left(u_\eta+v_\xi\right)+\left(c^2-V^2\right)v_{\eta=0},\\ u_\eta=v_\xi.\end{cases}\tag{3}$$

和拟定流的伯努利定律：

$$\frac{U^2+V^2}{2}+\int_{\rho_0}^{\rho}\frac{1}{\rho}p'(\rho)\mathrm{d}\rho+\varphi=Const,\ \varphi_\xi=U,\ \varphi_\eta=V.$$

上式的特征值为和特征线为

$$\Lambda_\pm=\frac{UV+c\sqrt{U^2+V^2-c^2}}{U^2-c^2},\ C_\pm:\frac{\mathrm{d}\eta}{\mathrm{d}\xi}=\Lambda_\pm.$$

利用左特征向量，可将方程(3)写成

$$\partial_\pm u+\Lambda_\mp\partial_\pm v=0.\tag{4}$$

根据[10]中的思想，引入特征角(α,β)的概念，则u,v,c可以被重新表示成：

$$u-\xi=c\frac{\cos\sigma}{\sin\omega},\ v-\eta=c\frac{\sin\sigma}{\sin\omega}.\tag{5}$$

2. u，v，c 的特征方程

在这一部分，我们将得出变量α,β,c的特征方程。首先，分别对方程(5)进行求导

$$\bar\partial_\pm u=\cos(\sigma+\omega)+\frac{\cos\sigma}{\sin\omega}\bar\partial_\pm c+\frac{c\cos\alpha\bar\partial_\pm\beta-c\cos\beta\bar\partial_\pm\alpha}{2\sin^2\omega},\tag{6}$$

$$\bar\partial_\pm v=\sin(\sigma+\omega)+\frac{\sin\sigma}{\sin\omega}\bar\partial_\pm c+\frac{c\sin\alpha\bar\partial_\pm\beta-c\sin\beta\bar\partial_\pm\alpha}{2\sin^2\omega}.\tag{7}$$

将(6)式和(7)式分别代入(4)式，可以得到

$$\bar\partial_+c=-\frac{\cos2\omega}{\cot\omega}+\frac{c}{\sin2\omega}\left(\bar\partial_+\alpha-\cos2\omega\bar\partial_+\beta\right),\tag{8}$$

$$\bar\partial\_c=-\frac{\cos2\omega}{\cot\omega}+\frac{c}{\sin2\omega}\left(\cos2\omega\bar\partial\_\alpha-\bar\partial\_\beta\right).\tag{9}$$

将伯努利定律求导并将(6)式和(7)式代入，计算可得

$$\left(\frac{1}{\sin^2\omega}+k(\rho)\right)\bar\partial_\pm c=\frac{c\cos\omega}{2\sin^3\omega}\left(\bar\partial_\pm\alpha-\bar\partial_\pm\beta\right)-\cot\omega.\tag{10}$$

将上式分别代入(8)式和(9)式，可以获得

$$c\bar\partial\_\beta=\Omega\cos^2\omega\left(c\bar\partial\_\alpha-2\sin^2\omega\right),\tag{11}$$

$$c\bar{\partial}_+\alpha = \Omega\cos^2\omega\left(c\bar{\partial}_+\beta + 2\sin^2\omega\right). \tag{12}$$

其中

$$\Omega = \frac{k(\rho)\cos 2\omega - 1}{\left(1+k(\rho)\right)\cos^2\omega} = m(\rho) - \tan^2\omega. \tag{13}$$

结合方程(9)方程和(11)有

$$\bar{\partial}_-c = \left(\frac{\cot\omega}{1+k(\rho)}\right)\left(c\bar{\partial}_-\alpha - 2\sin^2\omega\right), \tag{14}$$

$$c\bar{\partial}_-\beta = \left(\frac{1+k(\rho)}{2}\right)\Omega\sin 2\omega\bar{\partial}_-c. \tag{15}$$

同理，由方程(8)和方程(12)，我们有

$$\bar{\partial}_+c = -\left(\frac{\cot\omega}{1+k(\rho)}\right)\left(c\bar{\partial}_+\beta + 2\sin^2\omega\right), \tag{16}$$

$$c\bar{\partial}_+\alpha = -\left(\frac{1+k(\rho)}{2}\right)\Omega\sin 2\omega\bar{\partial}_+c. \tag{17}$$

将方程(14)~(17)代入(6)式和(7)式，可以得到

$$\bar{\partial}_\pm u = \pm k(\rho)\sin(\sigma\mp\omega)\bar{\partial}_\pm c, \tag{18}$$

$$\bar{\partial}_\pm v = \mp k(\rho)\cos(\sigma\mp\omega)\bar{\partial}_\pm c. \tag{19}$$

3. 特征分解

这一部分，我们介绍 α,β 和 $\alpha+\beta$ 的特征分解。首先，引用[12]中的交换子关系式。

性质 1 对任意的变量 $I(\xi,\eta)$，有下式成立

$$\partial_-\partial_+I - \partial_+\partial_-I = \frac{\partial_-\Lambda_+ - \partial_+\Lambda_-}{\Lambda_- - \Lambda_+}\left(\partial_-I - \partial_+I\right).$$

直接证明可得。

性质 2 对任意的变量 $I(\xi,\eta)$，有下式成立

$$\bar{\partial}_-\bar{\partial}_+I - \bar{\partial}_+\bar{\partial}_-I = \frac{1}{\sin 2\omega}\left[\left(\cos(2\omega)\bar{\partial}_+\beta - \bar{\partial}_-\alpha\right)\bar{\partial}_-I - \left(\bar{\partial}_+\beta - \cos(2\omega)\bar{\partial}_-\alpha\right)\bar{\partial}_+I\right]$$

由性质 1 可以验证。

性质 3 对变量 c，我们有以下特征分解

$$\begin{cases} c\bar{\partial}_-\bar{\partial}_+c = \bar{\partial}_+c\left\{\sin 2\omega + \dfrac{1+k(\rho)}{2\cos^2\omega}\bar{\partial}_+c + \left[\left(\dfrac{1+k(\rho)}{2}\right)\Omega\cos 2\omega + 1 - k'(\rho)\rho\right]\bar{\partial}_-c\right\}, \\ c\bar{\partial}_+\bar{\partial}_-c = \bar{\partial}_+c\left\{\sin 2\omega + \dfrac{1+k(\rho)}{2\cos^2\omega}\bar{\partial}_-c + \left[\left(\dfrac{1+k(\rho)}{2}\right)\Omega\cos 2\omega + 1 - k'(\rho)\rho\right]\bar{\partial}_+c\right\}. \end{cases}$$

证明：对 u 运用性质 2 的关系式并将(18)代入其中可得

$$\bar{\partial}_+\left[k(\rho)\sin\alpha\bar{\partial}_-c\right]+\bar{\partial}_-\left[k(\rho)\sin\beta\bar{\partial}_+c\right]$$
$$=-\frac{k(\rho)}{\sin 2\omega}\left[\sin\beta\bar{\partial}_+c\left(\bar{\partial}_+\beta-\cos 2\omega\bar{\partial}_-\alpha\right)-\sin\alpha\bar{\partial}_-c\left(\bar{\partial}_-\alpha-\cos 2\omega\bar{\partial}_+\beta\right)\right].$$

对上式求导化简整理可得

$$\begin{aligned}&(\sin\alpha+\sin\beta)\frac{c\rho k'(\rho)}{p'(\rho)}\bar{\partial}_+c\bar{\partial}_-c+\sin\alpha\bar{\partial}_+\bar{\partial}_-c+\sin\beta\bar{\partial}_-\bar{\partial}_+c\\&=-\frac{1}{\sin 2\omega}\Big[\left(\sin\beta\bar{\partial}_+\beta-\sin\beta\cos 2\omega\bar{\partial}_-\alpha+\cos\beta\sin 2\omega\bar{\partial}_-\beta\right)\bar{\partial}_+c\\&\quad-\left(\sin\alpha\bar{\partial}_-\alpha-\sin\alpha\cos 2\omega\bar{\partial}_+\beta-\cos\alpha\sin 2\omega\bar{\partial}_+\alpha\right)\bar{\partial}_-c\Big].\end{aligned}\tag{20}$$

因为 $\mathrm{d}\rho/\mathrm{d}c=2c/p''(\rho)$，对 c 运用性质 2 的关系式，我们有

$$\bar{\partial}_-\bar{\partial}_+c-\bar{\partial}_+\bar{\partial}_-c=\frac{1}{\sin 2\omega}\left[\left(\cos(2\omega)\bar{\partial}_+\beta-\bar{\partial}_-\alpha\right)\bar{\partial}_-c-\left(\bar{\partial}_+\beta-\cos(2\omega)\bar{\partial}_-\alpha\right)\bar{\partial}_+c\right]$$

将上式代入(20)式，我们可以得到

$$\begin{aligned}&(\sin\alpha+\sin\beta)\frac{c\rho k'(\rho)}{p'(\rho)}\bar{\partial}_+c\bar{\partial}_-c+(\sin\alpha+\sin\beta)\bar{\partial}_-\bar{\partial}_+c\\&=\frac{1}{\sin 2\omega}\Big[-(\sin\alpha+\sin\beta)\bar{\partial}_+\beta\bar{\partial}_+c+(\sin\alpha+\sin\beta)\cos 2\omega\bar{\partial}_-\alpha\bar{\partial}_+c\\&\quad-\cos\beta\sin 2\omega\bar{\partial}_-\beta\bar{\partial}_+c+\cos\alpha\sin 2\omega\bar{\partial}_+\alpha\bar{\partial}_-c\Big].\end{aligned}$$

将方程(14)~(17)代入上式即可获得性质 3 的第一式，另一个式子的证明与其相似。

性质 4 对变量 α,β，我们有以下特征分解

$$\begin{cases}c\bar{\partial}_+\bar{\partial}_-\alpha+M_1\bar{\partial}_-\alpha=\left[\dfrac{\sin(2\omega)}{2}\left(1-3\tan^2\omega\right)-\dfrac{2k'\rho\tan\omega}{(1+k)^2\Omega}\right]\bar{\partial}_+\alpha,\\ c\bar{\partial}_-\bar{\partial}_+\beta+M_2\bar{\partial}_+\beta=\left[\dfrac{\sin(2\omega)}{2}\left(1-3\tan^2\omega\right)-\dfrac{2k'\rho\tan\omega}{(1+k)^2\Omega}\right]\bar{\partial}_-\beta,\end{cases}$$

其中

$$\begin{aligned}M_1=\frac{1}{\sin(2\omega)}\Bigg[&4\sin^4\omega\left(1-\Omega\cos^2\omega\right)-\frac{k'\rho\sin^2(2\omega)}{(1+k)^2}-c\bar{\partial}_-\alpha\\&+\left(1-\frac{1}{2}\Omega\sin^2(2\omega)-\frac{2k'\rho\cos^2\omega}{(1+k)^2}\right)c\bar{\partial}_+\beta\Bigg],\end{aligned}$$

$$\begin{aligned}M_2=\frac{1}{\sin(2\omega)}\Bigg[&4\sin^4\omega\left(1-\Omega\cos^2\omega\right)-\frac{k'\rho\sin^2(2\omega)}{(1+k)^2}+c\bar{\partial}_+\beta\\&-\left(1-\frac{1}{2}\Omega\sin^2(2\omega)-\frac{2k'\rho\cos^2\omega}{(1+k)^2}\right)c\bar{\partial}_-\alpha\Bigg].\end{aligned}$$

证明：我们使用方程(14)~(17)，用 α,β 代替 c 然后代入性质 3 的第一式，得到

$$
\begin{aligned}
&c\bar{\partial}_+\left[\frac{\cot\omega}{1+k(\rho)}c\bar{\partial}\_\alpha-\frac{\sin(2\omega)}{1+k(\rho)}\right]\\
&=\left(\frac{\cot\omega}{1+k(\rho)}c\bar{\partial}\_\alpha-\frac{\sin(2\omega)}{1+k(\rho)}\right)\left\{\sin 2\omega+\frac{1+k(\rho)}{2\cos^2\omega}\left[\frac{\cot\omega}{1+k(\rho)}c\bar{\partial}\_\alpha-\frac{\sin(2\omega)}{1+k(\rho)}\right]\right.\\
&\quad\left.+\left[-\frac{2}{1+k(\rho)}\frac{1}{\Omega\sin(2\omega)}c\bar{\partial}_+\alpha\right]\left[\frac{1+k(\rho)}{2}\Omega\cos 2\omega+1-k'(\rho)\rho\right]\right\},
\end{aligned}
$$

推导整理之后，可以得到性质 4 的第一式，同理可证明第二式。

下面我们用同样的方法可以得出 σ 的特征分解。

性质 5 对变量 σ，我们有以下特征分解

$$
\begin{cases}
c\bar{\partial}_+\bar{\partial}\_\sigma+N_1\bar{\partial}\_\sigma=\dfrac{k(\rho)+1}{2k(\rho)}\tan\omega\left(\Omega\cos(2\omega)-2\tan^2\omega\right)\bar{\partial}_+\sigma,\\
c\bar{\partial}\_\bar{\partial}_+\sigma+N_2\bar{\partial}_+\sigma=\dfrac{k(\rho)+1}{2k(\rho)}\tan\omega\left(\Omega\cos(2\omega)-2\tan^2\omega\right)\bar{\partial}\_\sigma,
\end{cases}
$$

其中

$$
N_1=\frac{k(\rho)+1}{2k(\rho)\cos^2\omega}\left[\tan\omega\left(1+2\cos^2\omega\right)-\frac{2c}{\sin(2\omega)}\left(\bar{\partial}\_\sigma-\cos 2\omega\bar{\partial}_+\sigma\right)\right]+\tan\omega\left(1-4\cos^2\omega\right)
$$

$$
N_2=\frac{k(\rho)+1}{2k(\rho)\cos^2\omega}\left[\tan\omega\left(1+2\cos^2\omega\right)+\frac{2c}{\sin(2\omega)}\left(\bar{\partial}_+\sigma-\cos 2\omega\bar{\partial}\_\sigma\right)\right]+\tan\omega\left(1-4\cos^2\omega\right)
$$

4. 结论

对于气体动力学中的二维可压缩欧拉方程，压强 p 的特征分解为 $\partial_\mp\partial_\pm p=m_\pm\partial_\pm p$，特征值 $\Lambda_\pm$ 的特征分解为 $\partial_\mp\partial_\pm\Lambda_\pm=n_\pm\partial_\pm\Lambda_\pm$。

定理：对于气体动力学中的二维可压缩欧拉方程，与常状态相邻的是简单波，且沿着正(负)特征线，物理变量 (u,v,c) 是恒定的。

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