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鲁洛克斯三角形上的Schwarz边值问题

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摘　要

本文给出了鲁洛克斯三角形上一类特殊函数——其实部在边界上是循环对称函数——的Schwarz边值问题的解法，首先根据这类函数的特殊性将原问题进行改造，使其分别转化为实部在边界上关于实轴对称的Schwarz边值问题和实部在边界上关于实轴反对称的Schwarz边值问题，我们发现改造后的两个问题解之和就是原问题的解，然后通过对称扩张将改造后的两个问题转换为两组Riemann边值问题，最后通过求解这两组Riemann边值问题得到原问题的解。

关键词

Schwarz边值问题，鲁洛克斯三角形，Riemann-Hilbert边值问题

Schwarz Boundary Value Problem on Reuleaux Triangle

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Abstract

This paper presents the approach to the Schwarz boundary value problem of a special function on the Reuleaux triangle, the real part of this function is a cyclosymmetric function on the boundary. According to the particularity of this type of function, the original problem is first transformed into a Schwarz boundary value problem with the real part symmetric about the real axis on the boundary and the real part antisymmetric Schwarz boundary value problem with the real axis on the boundary. The sum of the solutions of these two problems is equal to the solution of the origi-

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nal problem. After that, the two problems after the conversion are further converted into two sets of Riemann boundary value problems through the symmetric expansion, finally, the solution of the original problem is obtained by solving the solutions of the two sets of Riemann boundary value problems.

Keywords

Schwarz Boundary Value Problem, Reuleaux Triangle, Riemann-Hilbert Boundary Value Problem

1. 引言

记 $D_0=\left\{z\in\mathbb{C}\middle||z+1|<\sqrt{3},\left|z-\frac{1+i\sqrt{3}}{2}\right|<\sqrt{3},\left|z-\frac{1-i\sqrt{3}}{2}\right|<\sqrt{3}\right\}$，其边界 $L=L_1+L_2+L_3$，如图 1 所示。

其中 $a=\frac{1+i\sqrt{3}}{2},b=-1,c=\frac{1-i\sqrt{3}}{2}$，

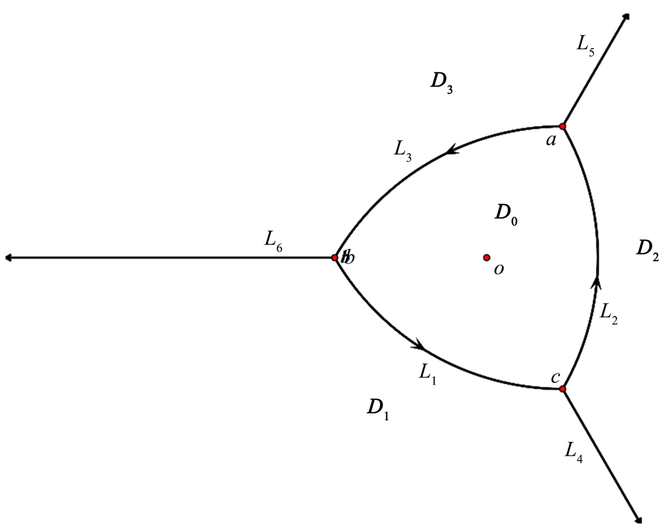

Figure 1. Reuleaux triangle
图 1. 鲁洛克斯三角形

$$D_1=\left\{z\in\mathbb{C}\middle|\left|z-\frac{1+i\sqrt{3}}{2}\right|>\sqrt{3},Im(z)<0,\mathrm{Re}\left[\left(\sqrt{3}-i\right)z\right]<0\right\}$$

$$D_2=\left\{z\in\mathbb{C}\middle||z+1|>\sqrt{3},\mathrm{Re}\left[\left(\sqrt{3}+i\right)z\right]<0<\mathrm{Re}\left[\left(\sqrt{3}-i\right)z\right]\right\}$$

$$D_3=\left\{z\in\mathbb{C}\middle|\left|z-\frac{1-i\sqrt{3}}{2}\right|>\sqrt{3},Im(z)>0,\mathrm{Re}\left[\left(\sqrt{3}+i\right)z\right]>0\right\}$$

定义 1 [1]：设 $f(t)$ 定义于(开口或封闭的)光滑曲线 L 上，若对 L 上任意两点 t',t''，恒有

$$\left|f(t')-f(t'')\right|\le A\left|t'-t''\right|^{\alpha}\ (0<\alpha\le 1)$$

其中 A,α 都是确定的常数，则称 $f(t)$ 在 L 上满足 α 阶的 Hölder 条件或 H^{α} 条件，记为 $f\in H^{\alpha}(L)$ 或简记为 $\in H^{\alpha}$，而 α 称为 Hölder 指数，若不强调指出指数 α，也可简记为 $f(t)\in H(L)$。

定义 2 [1] [2]：设 $f(t)$ 是在射线 L^* 上的连续复函数。如果：

1) 在任意有限闭区间 $I\subset L^*$ 上 $f(t)\in H^{\mu}$

2) 在 ∞ 的邻域 $U(\infty)$ 内满足条件

$$\left|f(t)-f(t')\right|\le A\left|\frac{1}{t}-\frac{1}{t'}\right|^{\mu}\ (0<\mu\le 1),t,t'\in L^*\cap U(\infty)$$

则称 $f(t)\in\hat{H}^{\mu}(L^*)$，或简记为 $\hat{H}^{\mu}$。若不强调 μ，可记为 $\hat{H}$。

注[2]：若 $f\in\hat{H}^{\mu}(L^*)$ 且 $f(\infty)=0$，我们记为 $f\in\hat{H}_0^{\mu}(L^*)$ 或简记为 $f\in\hat{H}_0(L^*)$。

2. 提出问题

求在 D_0 内的全纯函数 $\Phi_0^+(z)=u(z)+iv(z)$，连续到 D_0+L 上，满足边值条件

$$\mathrm{Re}\left[\Phi_0^+(t)\right]=f(t),t\in L \tag{1}$$

其中 $f(t)$ 是已给在 L 上 $\in H$ 的实函数。为简单起见，我们假设

$$f(t)=f\left(t\cdot \mathrm{e}^{i\frac{2\pi}{3}}\right)$$

3. 问题转化

令

$$\Psi_0(z)=\frac{1}{2}\left[\Phi_0(z)+\overline{\Phi_0(\bar{z})}\right]$$

$$\Omega_0(z)=\frac{1}{2}\left[\Phi_0(z)-\overline{\Phi_0(\bar{z})}\right]$$

代入(1)式，得到

$$\mathrm{Re}\left[\Psi_0(t)\right]=\frac{1}{2}\left[f(t)+f(\bar{t})\right]:=\tilde{f}(t) \tag{2}$$

$$\mathrm{Re}\left[\Omega_0(t)\right]=\frac{1}{2}\left[f(t)-f(\bar{t})\right]:=\tilde{\tilde{f}}(t) \tag{3}$$

由上述定义可知，$\tilde{f}$ 关于实轴对称，$\tilde{\tilde{f}}$ 关于实轴反对称，即

$$\tilde{f}(\bar{t})=\tilde{f}(t)$$

$$\tilde{\tilde{f}}(\bar{t})=-\tilde{\tilde{f}}(t)$$

再由 $f(t)=f\left(t\cdot \mathrm{e}^{i\frac{2\pi}{3}}\right)$ 可知

$$\tilde{f}\left(t\cdot \mathrm{e}^{i\frac{2\pi}{3}}\right)=\frac{1}{2}\left[f\left(t\cdot \mathrm{e}^{i\frac{2\pi}{3}}\right)+f\left(\overline{t\cdot \mathrm{e}^{i\frac{2\pi}{3}}}\right)\right]=\frac{1}{2}\left[f(t)+f\left(\overline{t\cdot \mathrm{e}^{i\frac{2\pi}{3}}}\cdot \mathrm{e}^{i\frac{2\pi}{3}}\right)\right]=\tilde{f}(t)$$

同理可知，

$$\tilde{\tilde{f}}\left(t\cdot \mathrm{e}^{i\frac{2\pi}{3}}\right)=\tilde{\tilde{f}}\left(t\right)$$

定义 3：若 z 与 $z^{\star}$ 满足

$$z^{\star}-z_0=\frac{R^2}{\overline{z}-\overline{z_0}}$$

则称 z 与 $z^{\star}$ 关于 $L:\left|z-z_0\right|=R$ 互为对称点。

我们令[3] [4] [5]

$$\tau_1\left(z\right)=\frac{\left(1+i\sqrt{3}\right)\overline{z}+4}{2\overline{z}-\left(1-i\sqrt{3}\right)}$$

$$\tau_2\left(z\right)=\frac{2-\overline{z}}{\overline{z}+1}$$

$$\tau_3\left(z\right)=\frac{\left(1-i\sqrt{3}\right)\overline{z}+4}{2\overline{z}-\left(1+i\sqrt{3}\right)}$$

则，$\tau_1\left(z\right)$ 与 z 关于 $C_a:\left|z-a\right|=\sqrt{3}$ 互为对称点，$\tau_2\left(z\right)$ 与 z 关于 $C_b:\left|z-b\right|=\sqrt{3}$ 互为对称点，$\tau_3\left(z\right)$ 与 z 关于 $C_c:\left|z-c\right|=\sqrt{3}$ 互为对称点。

由[1]可知，通过构造对称扩张可以将 Schwarz 边值问题等价转化为 Riemann 边值问题。下面我们给出 $\Psi_0\left(z\right)$ 在外域 D_1,D_2,D_3 中的对称扩张，如下

$$\Psi_1\left(z\right)=\overline{\Psi_0\left(\tau_1\left(z\right)\right)},z\in D_1$$

$$\Psi_2\left(z\right)=\overline{\Psi_0\left(\tau_2\left(z\right)\right)},z\in D_2$$

$$\Psi_3\left(z\right)=\overline{\Psi_0\left(\tau_3\left(z\right)\right)},z\in D_3$$

记

$$\Psi\left(z\right)=\begin{cases}\Psi_0\left(z\right),z\in D_0\\ \Psi_1\left(z\right),z\in D_1\\ \Psi_2\left(z\right),z\in D_2\\ \Psi_3\left(z\right),z\in D_3\end{cases}$$

易证，$\Psi\left(z\right)$ 是一分区全纯函数。

类似的，我们给出 $\Omega_0\left(z\right)$ 在外域 D_1,D_2,D_3 中的对称扩张，如下

$$\Omega_1\left(z\right)=\overline{\Omega_0\left(\tau_1\left(z\right)\right)}$$

$$\Omega_2\left(z\right)=\overline{\Omega_0\left(\tau_2\left(z\right)\right)}$$

$$\Omega_3\left(z\right)=\overline{\Omega_0\left(\tau_3\left(z\right)\right)}$$

记

$$\Omega(z)=\begin{cases}\Omega_0(z), z\in D_0\\ \Omega_1(z), z\in D_1\\ \Omega_2(z), z\in D_2\\ \Omega_3(z), z\in D_3\end{cases}$$

易证，$\Omega(z)$ 是一分区全纯函数。

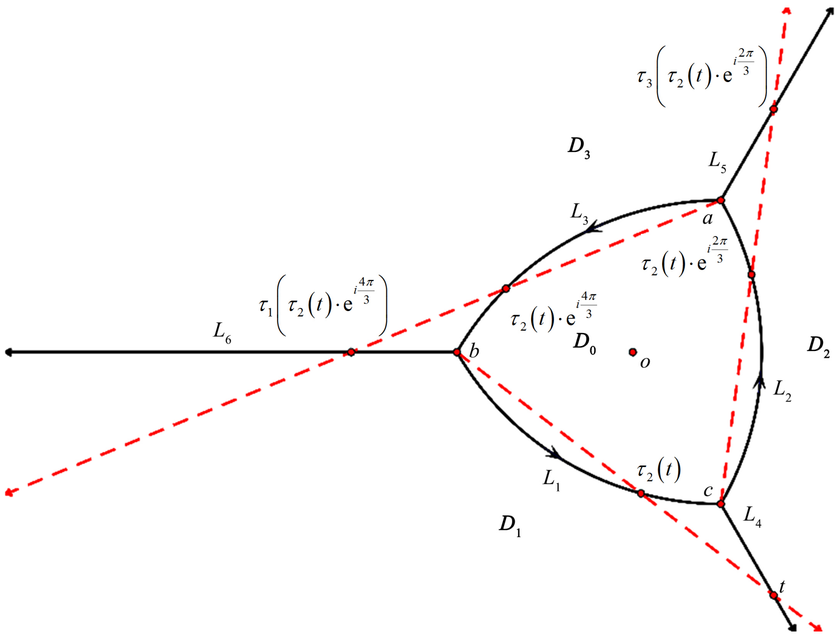

Figure 2. Positive boundary value and its equivalent relation
图 2. 正边值及其等价关系

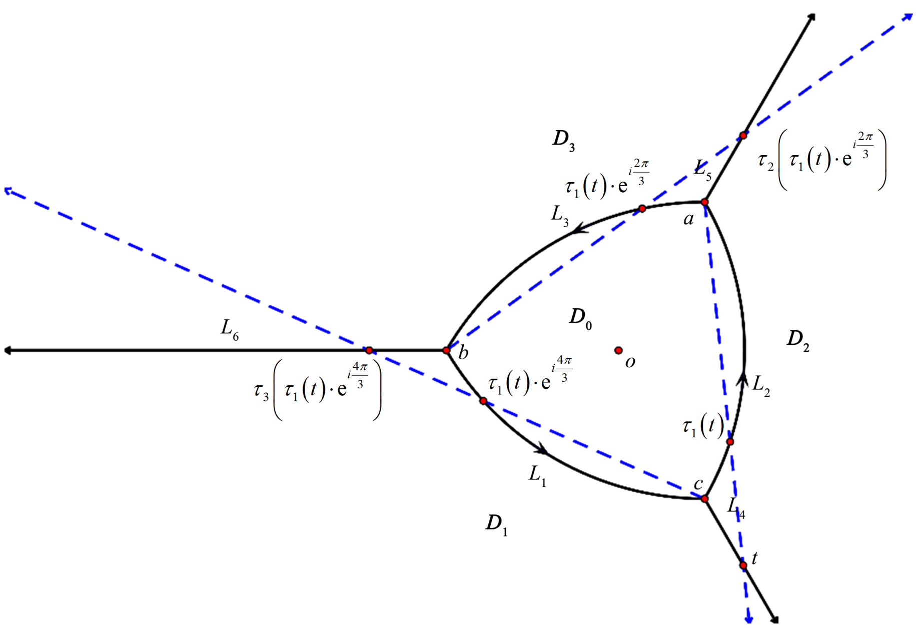

Figure 3. Negative boundary value and its equivalent relation
图 3. 负边值及其等价关系

如图 2 和图 3 所示，通过计算可以得到如下性质：

$$\tau_1\left(\overline{z}\right)=\overline{\tau_3\left(z\right)}$$

$$\tau_2\left(\overline{z}\right)=\overline{\tau_2\left(z\right)}$$

$$\tau_2\left(z\right)\cdot \mathrm{e}^{i\frac{2\pi}{3}}=\tau_3\left(z\cdot \mathrm{e}^{i\frac{2\pi}{3}}\right)$$

$$\tau_2\left(z\right)\cdot \mathrm{e}^{i\frac{4\pi}{3}}=\tau_1\left(z\cdot \mathrm{e}^{i\frac{4\pi}{3}}\right)$$

$$\tau_1\left(z\right)\cdot \mathrm{e}^{i\frac{4\pi}{3}}=\tau_3\left(z\cdot \mathrm{e}^{i\frac{4\pi}{3}}\right)$$

其中 $z\in D_0+L$。如图 4 所示，有

$$\tilde{f}\left(\tau_2\left(t\right)\right)=\tilde{f}\left(\tau_2\left(t\right)\cdot \mathrm{e}^{i\frac{4\pi}{3}}\right)=\tilde{f}\left(\tau_1\left(t\cdot \mathrm{e}^{i\frac{4\pi}{3}}\right)\right),t\in L_4$$

$$\tilde{f}\left(\tau_1\left(t\right)\right)=\tilde{f}\left(\tau_1\left(t\right)\cdot \mathrm{e}^{i\frac{4\pi}{3}}\right)=\tilde{f}\left(\tau_3\left(t\cdot \mathrm{e}^{i\frac{4\pi}{3}}\right)\right),t\in L_4$$

$$\tau_1\left(t\cdot \mathrm{e}^{i\frac{4\pi}{3}}\right)=\overline{\tau_3\left(t\cdot \mathrm{e}^{i\frac{4\pi}{3}}\right)}$$

可得

$$\tilde{f}\left(\tau_2\left(t\right)\right)=\tilde{f}\left(\tau_3\left(t\right)\right),t\in L_4$$

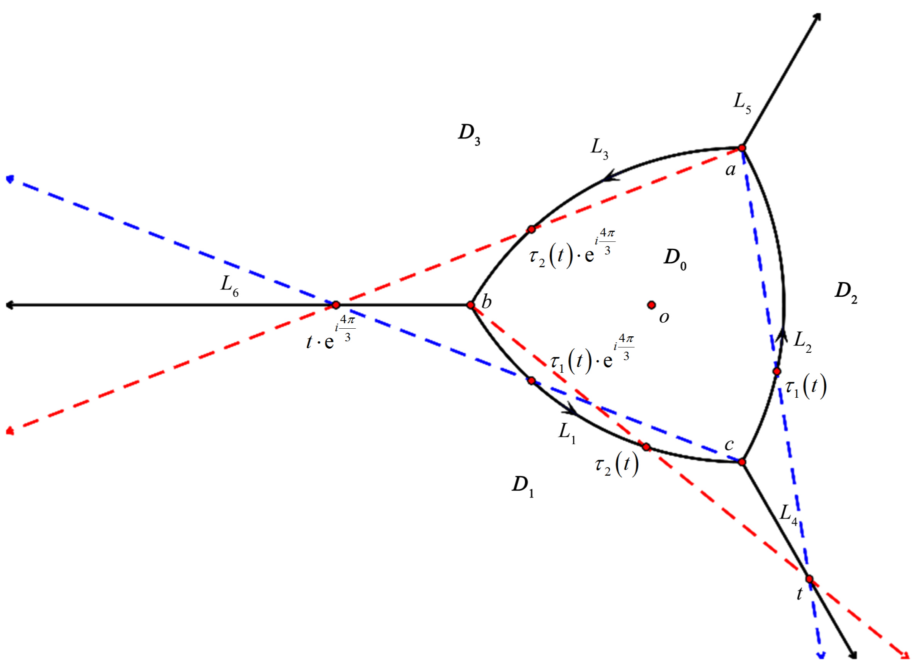

Figure 4. The boundary value problem on L_4
图 4. L_4 上的边值问题

同理，如图 5 所示，有

$$\tilde{f}\left(\tau_3(t)\right)=\tilde{f}\left(\tau_3(t)\cdot \mathrm{e}^{i\frac{2\pi}{3}}\right)=\tilde{f}\left(\tau_1\left(t\cdot \mathrm{e}^{i\frac{2\pi}{3}}\right)\right),t\in L_5$$

$$\tilde{f}\left(\tau_2(t)\right)=\tilde{f}\left(\tau_2(t)\cdot \mathrm{e}^{i\frac{2\pi}{3}}\right)=\tilde{f}\left(\tau_3\left(t\cdot \mathrm{e}^{i\frac{2\pi}{3}}\right)\right),t\in L_5$$

$$\tau_1\left(t\cdot \mathrm{e}^{i\frac{2\pi}{3}}\right)=\overline{\tau_3\left(t\cdot \mathrm{e}^{i\frac{2\pi}{3}}\right)}$$

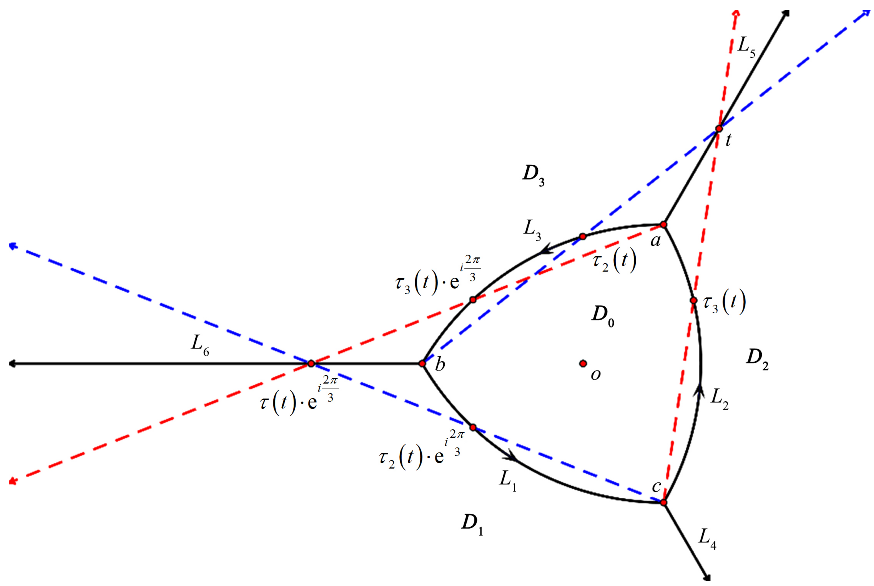

Figure 5. The boundary value problem on L_5
图 5. L_5 上的边值问题

可得

$$\tilde{f}\left(\tau_3(t)\right)=\tilde{f}\left(\tau_2(t)\right),t\in L_5$$

这就意味着，所求函数 Ψ 在 L_4,L_5,L_6 上正负边值的实部相等，接下来我们将讨论其虚部之间的关系。

设

$$\Psi_0(z)=u(z)+iv(z)$$

我们有

$$v(x_0,y_0)=\int_{(0,0)}^{(x_0,y_0)}v_x\mathrm{d}x+v_y\mathrm{d}y+v(0,0)$$

由 C-R 方程可得

$$\begin{aligned}v(x_0,y_0)&=\int_{(0,0)}^{(x_0,y_0)}-u_y\mathrm{d}x+u_x\mathrm{d}y+v(0,0)\\&=\int_{\Gamma_1}\nabla u\cdot(\mathrm{d}y,-\mathrm{d}x)+v(0,0)\\&=\int_{\Gamma_1}\nabla u\cdot\boldsymbol{n}\mathrm{d}S+v(0,0)\\&=\int_{\Gamma_1}\frac{\partial u}{\partial \boldsymbol{n}}\mathrm{d}S+v(0,0)\end{aligned}$$

记$\left(x_0', y_0'\right)$是$\left(x_0, y_0\right)$关于L_4的对称点，有

$$\begin{aligned}
v\left(x_0', y_0'\right) &= \int_{(0,0)}^{\left(x_0', y_0'\right)} v_x \mathrm{d}x + v_y \mathrm{d}y + v(0,0) \\
&= \int_{(0,0)}^{\left(x_0', y_0'\right)} -u_y \mathrm{d}x + u_x \mathrm{d}y + v(0,0) \\
&= \int_{\Gamma_2} \nabla u \cdot (\mathrm{d}y, -\mathrm{d}x) + v(0,0) \\
&= \int_{\Gamma_2} \nabla u \cdot \boldsymbol{n} \mathrm{d}S + v(0,0) \\
&= \int_{\Gamma_2} \frac{\partial u}{\partial \boldsymbol{n}} \mathrm{d}S + v(0,0) \\
&= -\int_{\Gamma_1} \frac{\partial u}{\partial \boldsymbol{n}} \mathrm{d}S + v(0,0)
\end{aligned}$$

其中Γ_1是$(0,0)$到$\left(x_0, y_0\right)$的直线段，Γ_2是$(0,0)$到$\left(x_0', y_0'\right)$的直线段。从而，有

$$\Psi_2^+(t) - \tilde{f}\left(\tau_2(t)\right) = -\left[\Psi_1^-(t) - \tilde{f}\left(\tau_1(t)\right)\right] + C, t \in L_4$$

同理，我们有

$$\Psi_3^+(t) - \tilde{f}\left(\tau_3(t)\right) = -\left[\Psi_2^-(t) - \tilde{f}\left(\tau_2(t)\right)\right] + C, t \in L_5$$

$$\Psi_1^+(t) - \tilde{f}\left(\tau_1(t)\right) = -\left[\Psi_3^-(t) - \tilde{f}\left(\tau_3(t)\right)\right] + C, t \in L_6$$

问题(2)在满足条件

$$\Psi_j^-(t) = \overline{\Psi_0^+(t)}, t \in L_j, j = 1,2,3 \tag{4}$$

时，可等价转化为

$$\begin{cases}
\Psi_0^+(t) = -\Psi_1^-(t) + 2\tilde{f}(t), t \in L_1 \\
\Psi_0^+(t) = -\Psi_2^-(t) + 2\tilde{f}(t), t \in L_2 \\
\Psi_0^+(t) = -\Psi_3^-(t) + 2\tilde{f}(t), t \in L_3 \\
\Psi_2^+(t) = -\Psi_1^-(t) + 2\tilde{f}\left(\tau_2(t)\right) + C, t \in L_4 \\
\Psi_3^+(t) = -\Psi_2^-(t) + 2\tilde{f}\left(\tau_3(t)\right) + C, t \in L_5 \\
\Psi_1^+(t) = -\Psi_3^-(t) + 2\tilde{f}\left(\tau_1(t)\right) + C, t \in L_6
\end{cases} \tag{5}$$

由

$$\tilde{\tilde{f}}\left(\tau_2(t)\right) = \tilde{\tilde{f}}\left(\tau_2(t) \cdot \mathrm{e}^{i\frac{4\pi}{3}}\right) = \tilde{\tilde{f}}\left(\tau_1\left(t \cdot \mathrm{e}^{i\frac{4\pi}{3}}\right)\right), t \in L_4$$

$$\tilde{\tilde{f}}\left(\tau_1(t)\right) = \tilde{\tilde{f}}\left(\tau_1(t) \cdot \mathrm{e}^{i\frac{4\pi}{3}}\right) = \tilde{\tilde{f}}\left(\tau_3\left(t \cdot \mathrm{e}^{i\frac{4\pi}{3}}\right)\right), t \in L_4$$

$$\tau_1\left(t \cdot \mathrm{e}^{i\frac{4\pi}{3}}\right) = \overline{\tau_3\left(t \cdot \mathrm{e}^{i\frac{4\pi}{3}}\right)}$$

可得

$$\tilde{\tilde{f}}\left(\tau_2(t)\right) = -\tilde{\tilde{f}}\left(\tau_3(t)\right), t \in L_4$$

同理，由

$$\tilde{\tilde{f}}\left(\tau_3(t)\right)=\tilde{\tilde{f}}\left(\tau_3(t)\cdot \mathrm{e}^{i\frac{2\pi}{3}}\right)=\tilde{\tilde{f}}\left(\tau_1\left(t\cdot \mathrm{e}^{i\frac{2\pi}{3}}\right)\right),t\in L_5$$

$$\tilde{\tilde{f}}\left(\tau_2(t)\right)=\tilde{\tilde{f}}\left(\tau_2(t)\cdot \mathrm{e}^{i\frac{2\pi}{3}}\right)=\tilde{\tilde{f}}\left(\tau_3\left(t\cdot \mathrm{e}^{i\frac{2\pi}{3}}\right)\right),t\in L_5$$

$$\tau_1\left(t\cdot \mathrm{e}^{i\frac{2\pi}{3}}\right)=\overline{\tau_3\left(t\cdot \mathrm{e}^{i\frac{2\pi}{3}}\right)}$$

可得

$$\tilde{\tilde{f}}\left(\tau_3(t)\right)=-\tilde{\tilde{f}}\left(\tau_2(t)\right),t\in L_5$$

设

$$\Omega_0(z)=u(z)+iv(z)$$

我们有

$$v(x_0,y_0)=\int_{(0,0)}^{(x_0,y_0)}v_x\mathrm{d}x+v_y\mathrm{d}y+v(0,0)$$

由 C-R 方程可得

$$\begin{aligned}v(x_0,y_0)&=\int_{(0,0)}^{(x_0,y_0)}-u_y\mathrm{d}x+u_x\mathrm{d}y+v(0,0)\\&=\int_{\Gamma_1}\nabla u\cdot(\mathrm{d}y,-\mathrm{d}x)+v(0,0)\\&=\int_{\Gamma_1}\nabla u\cdot \boldsymbol{n}\mathrm{d}S+v(0,0)\\&=\int_{\Gamma_1}\frac{\partial u}{\partial \boldsymbol{n}}\mathrm{d}S+v(0,0)\end{aligned}$$

记 (x_0',y_0') 是 (x_0,y_0) 关于 L_4 的对称点，有

$$\begin{aligned}v(x_0',y_0')&=\int_{(0,0)}^{(x_0',y_0')}v_x\mathrm{d}x+v_y\mathrm{d}y+v(0,0)\\&=\int_{(0,0)}^{(x_0',y_0')}-u_y\mathrm{d}x+u_x\mathrm{d}y+v(0,0)\\&=\int_{\Gamma_2}\nabla u\cdot(\mathrm{d}y,-\mathrm{d}x)+v(0,0)\\&=\int_{\Gamma_2}\nabla u\cdot \boldsymbol{n}\mathrm{d}S+v(0,0)\\&=\int_{\Gamma_2}\frac{\partial u}{\partial \boldsymbol{n}}\mathrm{d}S+v(0,0)\\&=\int_{\Gamma_1}\frac{\partial u}{\partial \boldsymbol{n}}\mathrm{d}S+v(0,0)\end{aligned}$$

其中 Γ_1 是 $(0,0)$ 到 (x_0,y_0) 的直线段，Γ_2 是 $(0,0)$ 到 (x_0',y_0') 的直线段。我们假设 $v(0,0)=0$，从而有

$$\Omega_2^+(t)-\tilde{\tilde{f}}\left(\tau_2(t)\right)=\Omega_1^-(t)-\tilde{\tilde{f}}\left(\tau_1(t)\right),t\in L_4$$

同理，我们有

$$\Omega_3^+(t)-\tilde{\tilde{f}}\left(\tau_3(t)\right)=\Omega_2^-(t)-\tilde{\tilde{f}}\left(\tau_2(t)\right),t\in L_5$$

$$\Omega_1^+(t)-\tilde{\tilde{f}}(\tau_1(t))=\Omega_3^-(t)-\tilde{\tilde{f}}(\tau_3(t)),t\in L_6$$

问题(3)在满足条件

$$\Omega_j^-(t)=\overline{\Omega_0^+(t)},t\in L_j,\ j=1,2,3 \tag{6}$$

时，可等价转化为

$$\begin{cases}\Omega_0^+(t)=-\Omega_1^-(t)+2\tilde{\tilde{f}}(t),t\in L_1\\ \Omega_0^+(t)=-\Omega_2^-(t)+2\tilde{\tilde{f}}(t),t\in L_2\\ \Omega_0^+(t)=-\Omega_3^-(t)+2\tilde{\tilde{f}}(t),t\in L_3\\ \Omega_2^+(t)=\Omega_1^-(t)+2\tilde{\tilde{f}}(\tau_2(t)),t\in L_4\\ \Omega_3^+(t)=\Omega_2^-(t)+2\tilde{\tilde{f}}(\tau_3(t)),t\in L_5\\ \Omega_1^+(t)=\Omega_3^-(t)+2\tilde{\tilde{f}}(\tau_1(t)),t\in L_6\end{cases} \tag{7}$$

4. 求解问题

首先考虑开口弧段 L_1 上的边值问题，由于 $G(t)\equiv -1$，我们取 $\log G(t)=i\pi$。引入函数

$$\Gamma_1(z)=\frac{1}{2}\int_{\Gamma_1}\frac{\mathrm{d}t}{t-z},z\notin L_1$$

由[1]可知，在端点附近有

$$\Gamma_1(z)=\begin{cases}-\dfrac{1}{2}\log(b-z)+\Delta_b(z),z\in U(b,\epsilon)\\ \dfrac{1}{2}\log(c-z)+\Delta_c(z),z\in U(c,\epsilon)\end{cases}$$

其中 $\Delta_b(z),\Delta_c(z)$ 分别在 $z=b,c$ 附近沿 L_1 剖开的邻域 N_b,N_c 中全纯。

令

$$\chi_1(z)=\mathrm{e}^{\Gamma_1(z)}$$

有

$$\chi_1(z)=\begin{cases}(b-z)^{-\frac{1}{2}}+\chi_b(z),z\in U(b,\epsilon)\\ (c-z)^{\frac{1}{2}}+\chi_c(z),\ z\in U(c,\epsilon)\end{cases}$$

其中 χ_b,χ_c 分别在 N_b,N_c 中全纯。

我们假设在 h_0 类中求解，可知 $\lambda_b=0,\lambda_c=-1$，典则函数为

$$X_1(z)=(z-c)^{-1}\chi_1(z)$$

易得 Ψ_0 在端点 $z=b,c$ 附近是 $-\frac{1}{2}$ 阶的极点。

同理可知，Ψ_0 在 L_2 的端点 $z=c,a$ 附近是 $-\frac{1}{2}$ 阶的极点，Ψ_0 在 L_3 的端点 $z=a,b$ 附近是 $-\frac{1}{2}$ 阶的极点。从而，由对称扩张的性质可以推出 Ψ 在 ∞ 处是 $-\frac{1}{2}$ 阶的极点。

记$\Sigma = L_4 + L_5 + L_6$，给出函数

$$\Gamma(z) = \frac{1}{2}\int_L \frac{\mathrm{d}\tau}{\tau - z}$$

$$W(z) = (z-a)^{\frac{1}{2}}(z-b)^{\frac{1}{2}}(z-c)^{\frac{1}{2}}\mathrm{e}^{\Gamma(z)}$$

这里引用[1]中幂函数正负边值的定义

$$W^+(t) = \begin{cases} (t-a)^{\frac{1}{2}}(t-b)^{\frac{1}{2}}(t-c)^{\frac{1}{2}}\mathrm{e}^{\Gamma^+(t)}, t \in L \\ (t-a)^{\frac{1}{2}}(t-b)^{\frac{1}{2}}(t-c)^{\frac{1}{2}}\mathrm{e}^{\Gamma(t)}, t \in \Sigma \end{cases}$$

$$W^-(t) = \begin{cases} (t-a)^{\frac{1}{2}}(t-b)^{\frac{1}{2}}(t-c)^{\frac{1}{2}}\mathrm{e}^{\Gamma^-(t)}, t \in L \\ -(t-a)^{\frac{1}{2}}(t-b)^{\frac{1}{2}}(t-c)^{\frac{1}{2}}\mathrm{e}^{\Gamma(t)}, t \in \Sigma \end{cases}$$

方程组(5)两边同时除以$W^+(t)$，得

$$\begin{cases} \dfrac{\Psi_0^+(t)}{W^+(t)} = \dfrac{\Psi_1^-(t)}{W^-(t)} + \dfrac{2\tilde{f}(t)}{W^+(t)}, t \in L_1 \\ \dfrac{\Psi_0^+(t)}{W^+(t)} = \dfrac{\Psi_2^-(t)}{W^-(t)} + \dfrac{2\tilde{f}(t)}{W^+(t)}, t \in L_2 \\ \dfrac{\Psi_0^+(t)}{W^+(t)} = \dfrac{\Psi_3^-(t)}{W^-(t)} + \dfrac{2\tilde{f}(t)}{W^+(t)}, t \in L_3 \\ \dfrac{\Psi_2^+(t)}{W^+(t)} = \dfrac{\Psi_1^-(t)}{W^-(t)} + \dfrac{2\tilde{f}(\tau_2(t))}{W^+(t)}, t \in L_4 \\ \dfrac{\Psi_3^+(t)}{W^+(t)} = \dfrac{\Psi_2^-(t)}{W^-(t)} + \dfrac{2\tilde{f}(\tau_3(t))}{W^+(t)}, t \in L_5 \\ \dfrac{\Psi_1^+(t)}{W^+(t)} = \dfrac{\Psi_3^-(t)}{W^-(t)} + \dfrac{2\tilde{f}(\tau_1(t))}{W^+(t)}, t \in L_6 \end{cases}$$

其中

$$\frac{2\tilde{f}(\tau_2(t))}{W^+(t)} \in H_\rho^*(c) \cap \hat{H}_0(L_4)$$

$$\frac{2\tilde{f}(\tau_3(t))}{W^+(t)} \in H_\rho^*(a) \cap \hat{H}_0(L_5)$$

$$\frac{2\tilde{f}(\tau_1(t))}{W^+(t)} \in H_\rho^*(b) \cap \hat{H}_0(L_6)$$

由于

$$Ord[\Psi, \infty] = -\frac{1}{2}$$

故

$$Ord\left[\frac{\Psi}{W}, \infty\right] = -1$$

从而上述方程组的解为

$$\frac{\Psi(z)}{W(z)}=\frac{1}{2\pi i}\int_{L+\Sigma}\frac{2\tilde{f}\left(\tau\left(t^{+}\right)\right)}{W^{+}(t)(t-z)}\mathrm{d}t+\frac{\lambda_a}{z-a}+\frac{\lambda_b}{z-b}+\frac{\lambda_c}{z-c}$$

$$\Psi(z)=\frac{W(z)}{2\pi i}\int_{L+\Sigma}\frac{2\tilde{f}\left(\tau\left(t^{+}\right)\right)}{W^{+}(t)(t-z)}\mathrm{d}t+W(z)\left(\frac{\lambda_a}{z-a}+\frac{\lambda_b}{z-b}+\frac{\lambda_c}{z-c}\right)$$

为了满足条件(4)，需构造$\Psi_*(z)$ [1]，我们令

$$\Gamma_{1*}(z)=\overline{\Gamma_1(\tau_1(z))}=\frac{1}{2}\int_{L_1}\frac{\mathrm{d}t}{t-z}-\int_{L_1}\frac{\mathrm{d}t}{2t-\left(1+i\sqrt{3}\right)}$$

$$\alpha_1=\frac{1}{i}\int_{L_1}\frac{\mathrm{d}t}{2t-\left(1+i\sqrt{3}\right)}=\frac{\pi}{3}$$

$$\Gamma_{1*}(z)=\Gamma_1(z)-\frac{\pi}{3}i$$

$$\Gamma_{2*}(z)=\overline{\Gamma_2(\tau_2(z))}=\frac{1}{2}\int_{L_2}\frac{\mathrm{d}t}{t-z}-\frac{1}{2}\int_{L_2}\frac{\mathrm{d}t}{t+1}$$

$$\alpha_2=\frac{1}{2i}\int_{L_2}\frac{\mathrm{d}t}{t+1}=\frac{\pi}{3}$$

$$\Gamma_{2*}(z)=\Gamma_2(z)-\frac{\pi}{3}i$$

$$\Gamma_{3*}(z)=\overline{\Gamma_3(\tau_3(z))}=\frac{1}{2}\int_{L_3}\frac{\mathrm{d}t}{t-z}-\int_{L_3}\frac{\mathrm{d}t}{2t-\left(1-i\sqrt{3}\right)}$$

$$\alpha_3=\frac{1}{i}\int_{L_3}\frac{\mathrm{d}t}{2t-\left(1-i\sqrt{3}\right)}=\frac{\pi}{3}$$

$$\Gamma_{3*}(z)=\Gamma_3(z)-\frac{\pi}{3}i$$

$$W_*(z)=\begin{cases}\left(\overline{\tau_1(z)}-\bar{a}\right)^{\frac{1}{2}}\left(\overline{\tau_1(z)}-\bar{b}\right)^{\frac{1}{2}}\left(\overline{\tau_1(z)}-\bar{c}\right)^{\frac{1}{2}}\mathrm{e}^{-i\frac{\pi}{3}}\mathrm{e}^{\Gamma(z)},z\in D_0\cup D_1\\\left(\overline{\tau_2(z)}-\bar{a}\right)^{\frac{1}{2}}\left(\overline{\tau_2(z)}-\bar{b}\right)^{\frac{1}{2}}\left(\overline{\tau_2(z)}-\bar{c}\right)^{\frac{1}{2}}\mathrm{e}^{-i\frac{\pi}{3}}\mathrm{e}^{\Gamma(z)},z\in D_0\cup D_2\\\left(\overline{\tau_3(z)}-\bar{a}\right)^{\frac{1}{2}}\left(\overline{\tau_3(z)}-\bar{b}\right)^{\frac{1}{2}}\left(\overline{\tau_3(z)}-\bar{c}\right)^{\frac{1}{2}}\mathrm{e}^{-i\frac{\pi}{3}}\mathrm{e}^{\Gamma(z)},z\in D_0\cup D_3\end{cases}$$

$$\Psi_*(z)=\overline{\Psi(\tau_j(z))}=-\frac{W_*(z)}{2\pi i}\int_{L+\Sigma}\frac{2\tilde{f}\left(\tau\left(t^{+}\right)\right)}{\overline{W^{+}(t)}\left(\bar{t}-\overline{\tau_j(z)}\right)}\mathrm{d}\bar{t}+W_*(z)\left(\frac{\overline{\lambda_a}}{\left(\overline{\tau_j(z)}-\bar{a}\right)}+\frac{\overline{\lambda_a}}{\left(\overline{\tau_j(z)}-\bar{a}\right)}+\frac{\overline{\lambda_a}}{\left(\overline{\tau_j(z)}-\bar{a}\right)}\right)$$

其中 $z\in D_0\cup D_j$，

那么

$$\Psi_0(z)=\frac{1}{2}\left[\Psi(z)+\Psi_*(z)\right]$$

是问题(2)的解，且满足条件(4)。

接下来，我们来求解方程组(7)，令

$$Q(z)=\mathrm{e}^{\Gamma(z)}$$

易得

$$\frac{Q^+(t)}{Q^-(t)}=-1,t\in L$$

$$Q^+(t)=Q^-(t)=1,t\in\Sigma$$

方程组(7)两边同时除以 $Q^+(t)$，得

$$\begin{cases}\dfrac{\Omega_0^+(t)}{Q^+(t)}=\dfrac{\Omega_1^-(t)}{Q^-(t)}+\dfrac{2\tilde{\tilde{f}}(t)}{Q^+(t)},t\in L_1\\ \dfrac{\Omega_0^+(t)}{Q^+(t)}=\dfrac{\Omega_2^-(t)}{Q^-(t)}+\dfrac{2\tilde{\tilde{f}}(t)}{Q^+(t)},t\in L_2\\ \dfrac{\Omega_0^+(t)}{Q^+(t)}=\dfrac{\Omega_3^-(t)}{Q^-(t)}+\dfrac{2\tilde{\tilde{f}}(t)}{Q^+(t)},t\in L_3\\ \dfrac{\Omega_2^+(t)}{Q^+(t)}=\dfrac{\Omega_1^-(t)}{Q^-(t)}+\dfrac{2\tilde{\tilde{f}}(\tau_2(t))}{Q^+(t)},t\in L_4\\ \dfrac{\Omega_3^+(t)}{Q^+(t)}=\dfrac{\Omega_2^-(t)}{Q^-(t)}+\dfrac{2\tilde{\tilde{f}}(\tau_3(t))}{Q^+(t)},t\in L_5\\ \dfrac{\Omega_1^+(t)}{Q^+(t)}=\dfrac{\Omega_3^-(t)}{Q^-(t)}+\dfrac{2\tilde{\tilde{f}}(\tau_1(t))}{Q^+(t)},t\in L_6\end{cases}$$

因为

$$\tilde{\tilde{f}}(\tau_2(c))=\frac{1}{2}\left[f(\tau_2(c))-f\left(\overline{\tau_2(c)}\right)\right]=\frac{1}{2}\left[f(c)-f(\bar{c})\right]=\frac{1}{2}\left[f(c)-f\left(c\cdot\mathrm{e}^{i\frac{2\pi}{3}}\right)\right]=0$$

$$\tilde{\tilde{f}}(\tau_2(\infty))=\frac{1}{2}\left[f(\tau_2(\infty))-f\left(\overline{\tau_2(\infty)}\right)\right]=\frac{1}{2}\left[f(b)-f(\bar{b})\right]=\frac{1}{2}\left[f(b)-f(b)\right]=0$$

$$\tilde{\tilde{f}}(\tau_3(a))=\frac{1}{2}\left[f(\tau_3(a))-f\left(\overline{\tau_3(a)}\right)\right]=\frac{1}{2}\left[f(a)-f(\bar{a})\right]=\frac{1}{2}\left[f(a)-f\left(a\cdot\mathrm{e}^{-i\frac{2\pi}{3}}\right)\right]=0$$

$$\tilde{\tilde{f}}(\tau_3(\infty))=\frac{1}{2}\left[f(\tau_3(\infty))-f\left(\overline{\tau_3(\infty)}\right)\right]=\frac{1}{2}\left[f(c)-f(\bar{c})\right]=\frac{1}{2}\left[f(c)-f\left(c\cdot\mathrm{e}^{i\frac{2\pi}{3}}\right)\right]=0$$

$$\tilde{\tilde{f}}(\tau_1(b))=\frac{1}{2}\left[f(\tau_1(b))-f\left(\overline{\tau_1(b)}\right)\right]=\frac{1}{2}\left[f(b)-f(\bar{b})\right]=\frac{1}{2}\left[f(b)-f(b)\right]=0$$

$$\tilde{\tilde{f}}(\tau_1(\infty))=\frac{1}{2}\left[f(\tau_1(\infty))-f\left(\overline{\tau_1(\infty)}\right)\right]=\frac{1}{2}\left[f(a)-f(\bar{a})\right]=\frac{1}{2}\left[f(a)-f\left(a\cdot\mathrm{e}^{-i\frac{2\pi}{3}}\right)\right]=0$$

所以，有

$$\tilde{\tilde{f}}\left(\tau_2\left(t\right)\right)\in H_\rho^*\left(c\right)\cap \hat{H}_0\left(L_4\right)$$

$$\tilde{\tilde{f}}\left(\tau_3\left(t\right)\right)\in H_\rho^*\left(a\right)\cap \hat{H}_0\left(L_5\right)$$

$$\tilde{\tilde{f}}\left(\tau_1\left(t\right)\right)\in H_\rho^*\left(b\right)\cap \hat{H}_0\left(L_6\right)$$

又

$$Ord\left[\frac{\Omega}{Q},\infty\right]=-\frac{1}{2}$$

故解为

$$\frac{\Omega\left(z\right)}{Q\left(z\right)}=\frac{1}{2\pi i}\int_{L+\Sigma}\frac{2\tilde{\tilde{f}}\left(\tau\left(t^+\right)\right)}{Q^+\left(t\right)\left(t-z\right)}\mathrm{d}t+\frac{\eta_a}{\sqrt{z-a}}+\frac{\eta_b}{\sqrt{z-b}}+\frac{\eta_c}{\sqrt{z-c}}$$

$$\Omega\left(z\right)=\frac{Q\left(z\right)}{2\pi i}\int_{L+\Sigma}\frac{2\tilde{\tilde{f}}\left(\tau\left(t^+\right)\right)}{Q^+\left(t\right)\left(t-z\right)}\mathrm{d}t+Q\left(z\right)\left(\frac{\eta_a}{\sqrt{z-a}}+\frac{\eta_b}{\sqrt{z-b}}+\frac{\eta_c}{\sqrt{z-c}}\right)$$

同样的，为了满足条件(6)，下面需构造$\Omega_*\left(z\right)$，通过计算可得

$$Q_*\left(z\right)=\mathrm{e}^{-i\frac{\pi}{3}}Q\left(z\right)$$

$$\Omega_*\left(z\right)=-\frac{Q_*\left(z\right)}{2\pi i}\int_{L+\Sigma}\frac{2\tilde{\tilde{f}}\left(\tau\left(t^+\right)\right)}{\overline{Q^+\left(t\right)}\left(\overline{t}-\overline{\tau_j\left(z\right)}\right)}\mathrm{d}\overline{t}+Q_*\left(z\right)\left(\frac{\overline{\eta_a}}{\sqrt{\overline{\tau_j\left(z\right)}-\overline{a}}}+\frac{\overline{\eta_b}}{\sqrt{\overline{\tau_j\left(z\right)}-\overline{b}}}\frac{\overline{\eta_c}}{\sqrt{\overline{\tau_j\left(z\right)}-\overline{c}}}\right)$$

其中$z\in D_0\cup D_j$，那么

$$\Omega_0\left(z\right)=\frac{1}{2}\left[\Omega\left(z\right)+\Omega_*\left(z\right)\right]$$

是问题(3)的解，且满足条件(6)。

5. 结论

最后，我们将问题(2)与问题(3)的解相加就得到了问题(1)的解，即

$$\Phi_0=\Psi_0\left(z\right)+\Omega_0\left(z\right)$$

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