Pure Mathematics 理论数学, 2021, 11(5), 739-751
Published Online May 2021 in Hans. http://www.hanspub.org/journal/pm
https://doi.org/10.12677/pm.2021.115089

三维有界域上不可压缩无磁耗散MHD方程弱解的能量守恒

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收稿日期：2021年3月15日；录用日期：2021年4月9日；发布日期：2021年5月18日

摘 要

本文研究了三维有界域上不可压缩无磁耗散MHD方程弱解的能量守恒问题. 先对方程进行整体磨光, 再取截断函数, 然后关于 $\delta, \varepsilon, \tau$ 取极限, 从而得到能量等式. 为了得到能量守恒, 对弱解 $(\mathbf{u}, \mathbf{b}, P)$ 加条件：$\mathbf{u} \in L_t^p L_x^q$，$\mathbf{b} \in L_t^4 L_x^4$ 且 $\nabla \mathbf{b} \in L_t^2 L_x^2, P \in L_t^2 L_x^2$.

关键词

不可压缩MHD方程，无磁耗散，有界区域，弱解，能量守恒

Energy Conservation for the Weak Solutions to the Three-Dimensional Incompressible Magnetohydrodynamic Equations of Viscous Non-Resistive Fluids in a Bounded Domain

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文章引用: 汪雄. 三维有界域上不可压缩无磁耗散MHD方程弱解的能量守恒[J]. 理论数学, 2021, 11(5): 739-751.
DOI: 10.12677/pm.2021.115089

Received: Mar. 15^{th}, 2021; accepted: Apr. 9^{th}, 2021; published: May 18^{th}, 2021

Abstract

In this paper, we mainly study the energy conservation for the weak solutions to the three-dimensional incompressible magnetohydrodynamic equations of viscous non-resistive fluids in a bounded domain. To get energy conservation, we first use the global mollification method to the equation, next take cut-off function, then get the limit of $\delta, \varepsilon, \tau$. We propose a condition for $(\mathbf{u}, \mathbf{b}, P)$: $\mathbf{u} \in L_t^p L_x^q$, $\mathbf{b} \in L_t^4 L_x^4$, $\nabla \mathbf{b} \in L_t^2 L_x^2$ *and* $P \in L_t^2 L_x^2$.

Keywords

Incompressible Magnetohydrodynamic Equations, Viscous Non-Resistive Fluids, Bounded Domain, Weak Solutions, Energy Conservation

1. 研究背景及主要结论

三维不可压缩无磁耗散MHD方程

$$\begin{cases} \mathrm{div}\mathbf{u} = 0, \\ \mathbf{u}_t + \mathrm{div}(\mathbf{u} \otimes \mathbf{u}) + \nabla P = (\nabla \times \mathbf{b}) \times \mathbf{b} + \mu \Delta \mathbf{u}, \\ \mathbf{b}_t - \nabla \times (\mathbf{u} \times \mathbf{b}) = 0, \\ \mathrm{div}\mathbf{b} = 0, \end{cases} \tag{1.1}$$

其中, $\mathbf{u} = (u_1, u_2, u_3)(x, t)$ 表示流体的速度, $\mathbf{b} = (b_1, b_2, b_3)(x, t)$ 表示磁场, $P = P(x, t)$ 表示压力. 本文考虑的是上述三维不可压缩无磁耗散MHD方程组(1.1)在有界开区域 $\Omega \subset \mathbb{R}^3$具有下述边值条件

$$\mathbf{u} \mid_{\partial\Omega} = 0, \tag{1.2}$$

和初值条件

$$\mathbf{u}(x,0)=\mathbf{u}_0(x),\ \mathbf{b}(x,0)=\mathbf{b}_0(x),\ x\in\Omega. \tag{1.3}$$

的弱解的能量守恒问题.

定义 1.1. 对于给定的 $T>0$, 若初值满足 $\mathbf{u}_0$, $\mathbf{b}_0\in L^2(\Omega)$, 如果有以下几点成立

- 问题 (1.1)-(1.3) 在空间 $\mathcal{D}'(\Omega\times[0,T))$ 中成立且满足

$$\begin{cases}\mathbf{u}\in L^\infty(0,T;L^2(\Omega)),\\ \mathbf{u}\in L^2(0,T;H_0^1(\Omega)),\\ \mathbf{b}\in L^\infty(0,T;L^2(\Omega)),\\ P\in L^\infty(0,T;L^1_{loc}(\Omega));\end{cases} \tag{1.4}$$

- 对任意 $\varphi\in\mathcal{C}_0^\infty\big(\Omega\times[0,T)\big)$

$$\int_0^T\int_\Omega\big(\mathbf{u}\cdot\varphi_t+\mathbf{u}\otimes\mathbf{u}:\nabla\varphi+P\mathrm{div}\varphi-\mathbf{b}\otimes\mathbf{b}:\nabla\varphi+\frac{|\mathbf{b}|^2}{2}\mathrm{div}\varphi-\mu\nabla\mathbf{u}:\nabla\varphi\big)\mathrm{d}x\mathrm{d}t$$
$$+\int_\Omega\mathbf{u}_0(x)\varphi(x,0)\mathrm{d}x=0;$$

- 下述能量不等式对几乎所有的 $t\in[0,T]$ 成立

$$\begin{aligned}&\int_\Omega(\frac{1}{2}|\mathbf{u}|^2+\frac{1}{2}|\mathbf{b}|^2)\mathrm{d}x+\int_0^t\int_\Omega\mu|\nabla\mathbf{u}|^2\mathrm{d}x\mathrm{d}s\\ \le&\int_\Omega(\frac{1}{2}|\mathbf{u}_0|^2+\frac{1}{2}|\mathbf{b}_0|^2)\mathrm{d}x;\end{aligned} \tag{1.5}$$

则称 $(\mathbf{u},\mathbf{b},P)$ 是问题(1.1)-(1.3) 在 $\Omega\times[0,T]$ 上的一个弱解.

事实上, 如果解充分光滑, 比如强解或者经典解, 会有下述能量等式成立

$$\begin{aligned}&\int_\Omega(\frac{1}{2}|\mathbf{u}|^2+\frac{1}{2}|\mathbf{b}|^2)\mathrm{d}x+\int_0^t\int_\Omega\mu|\nabla\mathbf{u}|^2\mathrm{d}x\mathrm{d}s\\ =&\int_\Omega(\frac{1}{2}|\mathbf{u}_0|^2+\frac{1}{2}|\mathbf{b}_0|^2)\mathrm{d}x.\end{aligned} \tag{1.6}$$

在问题 (1.1)中, 取 $\mathbf{b}=0$, 则该问题转化成不可压缩的 Navier-Stokes 方程在有界域上的能量守恒问题, 该问题已在文章 [1] 的附录中给出了答案. 受文章 [1] 启发, 我们研究不可压缩无磁耗散MHD 方程在有界域上的能量守恒问题.

本文的主要结果如下:

定理 1.1. 令 Ω 是有界开集, 边界 $\partial\Omega$ 满足 $\mathcal{C}^1$ 光滑性条件. 假设

$$\begin{cases} \mathbf{u}(x,0)=\mathbf{u}_0(x),\ \operatorname{div}\mathbf{u}_0=0 \text{ 在空间}\mathcal{D}'(\Omega)\text{中成立;} \\ \mathbf{b}(x,0)=\mathbf{b}_0(x),\ \operatorname{div}\mathbf{b}_0=0 \text{ 在空间}\mathcal{D}'(\Omega)\text{中成立.} \end{cases} \tag{1.7}$$

记 $(\mathbf{u},\mathbf{b},P)$ 是方程组满足定义 1.1. 的弱解. 如果

$$P\in L^2\big(0,T;L^2(\Omega)\big), \tag{1.8}$$

$$\mathbf{u}\in L^p(0,T;L^q(\Omega)),\ p\geq 4,\ q\geq 6, \tag{1.9}$$

且

$$\mathbf{b}\in L^4\big(0,T;L^4(\Omega)\big),\ \nabla\mathbf{b}\in L^2\big(0,T;L^2(\Omega)\big), \tag{1.10}$$

则对任意 $t\in[0,T]$, 能量等式 (1.6) 成立.

2. 记号说明及常用引理

引理 2.1.(Hardy型嵌入不等式 [2]) 如果$f\in W_0^{1,p}(\Omega)$, $p\in[1,\infty)$, 且存在一个常数 $C=C(p,\Omega)$ 与 p 和 Ω 有关, 那么

$$\left\|\frac{f(x)}{dist(x,\partial\Omega)}\right\|_{L^p(\Omega)}\leq C\|f\|_{W_0^{1,p}(\Omega)}.$$

下面引入与磨光有关的记号和结论, 参考Chen-Liang-Wang-Xu的文章 [1]和Evans的书 [3]. 对任意的 $(x,t)\in\Omega'\times(\varepsilon,T-\varepsilon)$, 其中 $\Omega'\subset\subset\Omega$, 定义卷积

$$\mathbf{u}^\varepsilon(x,t)=\int_0^T\int_\Omega\mathbf{u}(y,s)\eta_\varepsilon(x-y,t-s)\mathrm{d}y\mathrm{d}s,\ \eta_\varepsilon(x,t)=\frac{1}{\varepsilon^4}\eta(\frac{x}{\varepsilon},\frac{t}{\varepsilon}), \tag{2.1}$$

这里 $\eta(x,t)$ 是支集在单位球上的标准磨光核.

命题 2.1. 在空间 $L^q_{loc}\big(0,T;W^{1,p}(\Omega')\big)$ 中, 当 $\varepsilon\to 0$时有

$$\mathbf{u}^\varepsilon(x,t)\to\mathbf{u}(x,t),\quad\forall p,\,q\in[1,+\infty). \tag{2.2}$$

由于 $\partial\Omega\in\mathcal{C}^1$, 对固定的点 $x_1\in\partial\Omega$,存在实数 $r_1>0$ 及函数 $h:\mathbb{R}^2\to\mathbb{R}$ 记 $V_1=\Omega\cap B(x_1,\frac{r_1}{2})=\{x\in B(x_1,r_1):x^3>h(x^1,x^2)\}$ 对于一个很小的正数 ε, $0<\varepsilon<\frac{r_1}{8}$. 定义平移点 $x_1^\varepsilon:=x-\varepsilon\vec{n}(x_1)$, $\forall x\in V_1$, 其中 $\vec{n}(x_1)$ 是 $\partial\Omega$ 在点 x_1 处的单位外法向量, 则 $B(x^\varepsilon,\varepsilon)\subset B(x_1,r_1)\cap\Omega$. 定义平移函数

$$\tilde{\mathbf{u}}_1(x,t)=\mathbf{u}(x_1^\varepsilon,t),\ \forall x\in V_1.$$

对任意的 $(x,t)\in V_1\times(\varepsilon,T-\varepsilon)$ 且 $\mathcal{V}_1:=B(x_1,\frac{r_1}{2}+2\varepsilon)\cap\Omega$, 有

$$\begin{aligned}\tilde{\mathbf{u}}_1^{\varepsilon}(x,t)&=\int_0^T\int_{\mathcal{V}_1}\tilde{\mathbf{u}}(y,s)\eta_\varepsilon(x-y,t-s)\mathrm{d}y\mathrm{d}s\\&=\int_0^T\int_{\mathcal{V}_1-\varepsilon\vec{n}(x_1)}\mathbf{u}(y,s)\eta_\varepsilon(x^\varepsilon-y,t-s)\mathrm{d}y\mathrm{d}s.\end{aligned}\tag{2.3}$$

命题 2.2. 在空间 $L^q_{loc}\big(0,T;W^{1,p}(V_1)\big)$ 中, 当 $\varepsilon\to0$时有

$$\tilde{\mathbf{u}}_1^{\varepsilon}(x,t)\to\mathbf{u}(x,t),\quad\forall p,q\in[1,+\infty).$$

由于 $\partial\Omega$ 的紧性, $\partial\Omega$ 的开覆盖能找到有限子覆盖, 即能找到有限个点 $x_i\in\partial\Omega$, 半径 $r_i>0$ 及对应的集合 $V_i=\Omega\cap B(x_i,\frac{r_i}{2})$, $i\in\{1,2,...,k\}$, 使得 $\partial\Omega\subset\bigcup_{i=1}^k\overline{V_i}$ 且 $\tilde{\mathbf{u}}_i^\varepsilon\in\mathcal{C}^\infty\big(\overline{V_i}\big)$. 存在 $V_0\subset\subset\Omega$, 使得 $\Omega\subset\bigcup_{i=0}^k\overline{V_i}$.
令 $\{\psi_i\}_{i=0}^k$是从属于开集族 $\{V_0,B(x_1,\frac{r_1}{2}),...,B(x_k,\frac{r_k}{2})\}$ 的单位分解, 即

$$\begin{cases}0\le\psi_i\le1,\ i\in\{0,1,2,...,k\}\\\psi_0\in\mathcal{C}^\infty\big(V_0\big),\ \operatorname{supp}\psi_0\subset V_0,\\\psi_i\in\mathcal{C}^\infty\big(B(x_i,\dfrac{r_i}{2})\big),\ \operatorname{supp}\psi_i\subset B(x_i,\dfrac{r_i}{2}),\ i\in\{1,2,...,k\},\\\displaystyle\sum_{i=0}^k\psi_i=1.\end{cases}\tag{2.4}$$

定义

$$[\mathbf{u}]^\varepsilon(x,t):=\psi_0(x)\mathbf{u}^\varepsilon(x,t)+\sum_{i=1}^k\psi_i(x)\tilde{\mathbf{u}}_i^\varepsilon(x,t),\quad\forall x\in\Omega.\tag{2.5}$$

则 $[\mathbf{u}]^\varepsilon(x,t)\in\mathcal{C}^\infty\big(0,T;\mathcal{C}^\infty(\Omega)\big)$.

命题 2.3. 在空间 $L^q_{loc}\big(0,T;W^{1,p}(\Omega)\big)$ 中, 当 $\varepsilon\to0$时

$$[\mathbf{u}]^\varepsilon\to\mathbf{u}\quad\forall p,\ q\in[1,+\infty).\tag{2.6}$$

进一步, 当 $\varepsilon\to0$时有

$$\begin{aligned}&\text{在空间}L^q_{loc}\big(0,T;W^{1,p}(V_0)\big)\text{中},\ [\mathbf{u}]^\varepsilon-\mathbf{u}^\varepsilon\to0;\\&\text{在空间}L^q_{loc}\big(0,T;W^{1,p}(V_i)\big)\text{中},\ [\mathbf{u}]^\varepsilon-\tilde{\mathbf{u}}_i^\varepsilon\to0.\end{aligned}\tag{2.7}$$

引理 2.2.(文献 [4]中引理2.3) 如果 $f\in W^{1,r_1}(\Omega\times[0,T])$, $g\in L^{r_2}(\Omega\times[0,T])$, 其中 $1\le r,r_1,r_2\le\infty$, $\frac{1}{r_1}+\frac{1}{r_2}=\frac{1}{r}$, 则对于某个与 ε, f , g 无关的常数 $C>0$, 有下述结论成立

$$\|\partial(fg)^\varepsilon-\partial(fg^\varepsilon)\|_{L^r_{loc}(\Omega\times(0,T))}\le C\|g\|_{L^{r_2}(\Omega\times[0,T])}\left(\|\partial_tf\|_{L^{r_1}(\Omega\times[0,T])}+\|\nabla f\|_{L^{r_1}(\Omega\times[0,T])}\right),\tag{2.8}$$

此处 $\partial=\partial_t$ 或 $\partial=\partial_x$, g^ε 的定义是由 (2.1) 给出的. 更进一步, 当 $\varepsilon\to 0$ 时, 有

$$\text{在空间 } L^{\underline{r}}_{loc}\big(\Omega\times(0,T)\big)\ \text{中},\quad \partial(fg)^\varepsilon-\partial(fg^\varepsilon)\to 0.$$

此处, 当 $r_2<\infty$ 时, $\underline{r}=r$; 当 $r_2=\infty$ 时, $\underline{r}<r$.

引理 2.3.(文章 [1]中推论2.1) 如果 $f\in W^{1,r_1}(\Omega\times[0,T])$, $g\in L^{r_2}(\Omega\times[0,T])$ 其中 $1\le r,r_1,r_2\le\infty$, $\frac{1}{r_1}+\frac{1}{r_2}=\frac{1}{r}$. 则

$$\|\partial_t((\tilde f\tilde g)^\varepsilon_i-f\tilde g^\varepsilon_i)\|_{L^r_{loc}(0,T;L^r(V_i))}\le C\|g\|_{L^{r_2}(\Omega\times[0,T])}\left(\|\partial_t f\|_{L^{r_1}(\Omega\times[0,T])}+\|\nabla f\|_{L^{r_1}(\Omega\times[0,T])}\right)$$

且

$$\|\partial_x((\tilde f\tilde g)^\varepsilon_i-f\tilde g^\varepsilon_i)\|_{L^r_{loc}(0,T;L^r(V_i))}\le C\|g\|_{L^{r_2}(\Omega\times[0,T])}\left(\|\partial_t f\|_{L^{r_1}(\Omega\times[0,T])}+\|\nabla f\|_{L^{r_1}(\Omega\times[0,T])}\right),$$

此处 $\tilde g^\varepsilon_i$ 的定义是由 (2.3) 给出的. 更进一步, 当 $\varepsilon\to 0$时,有

$$\text{在空间}L^{r}_{loc}(0,T;L^{\underline{r}}(V_i))\text{中},\ \partial((\tilde f\tilde g)^\varepsilon_i-f\tilde g^\varepsilon_i)\to 0,$$

此处, 当 $r_2<\infty$ 时, $\underline{r}=r$; 当 $r_2=\infty$ 时, $\underline{r}<r$.

引理 2.4.(Aubin-Lions引理, 文章 [4]) 如果 X 是自反的 Banach 空间, Y 是 Banach 空间, $X\hookrightarrow Y$, Y' 可分且在 X' 中稠密的. 假设函数序列 $\{f_n\}$ 满足

$$\begin{cases} f_n\in L^\infty(0,T;X),\ \partial_t f_n\in L^p(0,T;Y),\ 1<p\le\infty,\\ \|f_n\|_{L^\infty(0,T;X)},\ \|\partial_t f_n\|_{L^p(0,T;Y)}\le C,\ \forall n\ge 1.\end{cases}$$

则 f_n 在 $C^0([0,T],X_\omega)$ 中相对紧.

3. 定理1.1的证明

为了证明定理 1.1. 得到能量守恒的结论, 下面我们采用文章 [1]中的方法将方程 $(1.1)_2$ 磨光, 得

$$\begin{aligned}&\partial_t\big(\psi_0\mathbf{u}^\varepsilon+\sum_{i=1}^k\psi_i\tilde{\mathbf{u}}^\varepsilon_i\big)+\big(\psi_0\mathrm{div}(\mathbf{u}\otimes\mathbf{u})^\varepsilon+\sum_{i=1}^k\psi_i\mathrm{div}(\tilde{\mathbf{u}}\otimes\tilde{\mathbf{u}})^\varepsilon_i\big)\\&+\big(\psi_0\nabla P^\varepsilon+\sum_{i=1}^k\psi_i\nabla\tilde P^\varepsilon_i\big)\\&-\big(\psi_0\mathrm{div}(\mathbf{b}\otimes\mathbf{b})^\varepsilon+\sum_{i=1}^k\psi_i\mathrm{div}(\tilde{\mathbf{b}}\otimes\tilde{\mathbf{b}})^\varepsilon_i\big)+\big(\psi_0\nabla(\frac{|\mathbf{b}|^2}{2})^\varepsilon+\sum_{i=1}^k\psi_i\nabla(\frac{|\tilde{\mathbf{b}}|^2}{2})^\varepsilon_i\big)\\&-\mu\big(\psi_0\Delta\mathbf{u}^\varepsilon+\sum_{i=1}^k\psi_i\Delta\tilde{\mathbf{u}}^\varepsilon_i\big)=0.\end{aligned}\tag{3.1}$$

取截断函数 $\xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon$, 其中 $\xi_\tau(t)\in\mathcal{C}_0^1\big((\tau,T-\tau)\big)$, $\eta_\delta(x)\in\mathcal{C}_0^1(\Omega)$且

$$\begin{cases} 0\le\eta_\delta(x)\le 1,\ \eta_\delta(x)=1 \text{ 当 } x\in\Omega \text{ 且 } dist(x,\partial\Omega)\ge\delta\text{时};\\ \eta_\delta\to 1 \text{ 当 } \delta\to 0\text{时, 且 } |\nabla\eta_\delta|\le\dfrac{2}{dist(x,\partial\Omega)}. \end{cases}$$

将方程(3.1)两边乘上 $\xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon$ 并在 $\Omega\times(0,T)$ 上积分, 得到

$$\begin{aligned} &\int_0^T\int_\Omega \xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon\cdot\partial_t\big(\psi_0\mathbf{u}^\varepsilon+\sum_{i=1}^k\psi_i\tilde{\mathbf{u}}_i^\varepsilon\big)\mathrm{d}x\mathrm{d}t\\ &+\int_0^T\int_\Omega \xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon\cdot\big(\psi_0\mathrm{div}(\mathbf{u}\otimes\mathbf{u})^\varepsilon+\sum_{i=1}^k\psi_i\mathrm{div}(\tilde{\mathbf{u}}\otimes\tilde{\mathbf{u}})_i^\varepsilon\big)\mathrm{d}x\mathrm{d}t\\ &+\int_0^T\int_\Omega \xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon\cdot\big(\psi_0\nabla P^\varepsilon+\sum_{i=1}^k\psi_i\nabla\tilde{P}_i^\varepsilon\big)dxdt\\ &-\int_0^T\int_\Omega \xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon\cdot\big(\psi_0\mathrm{div}(\mathbf{b}\otimes\mathbf{b})^\varepsilon+\sum_{i=1}^k\psi_i\mathrm{div}(\tilde{\mathbf{b}}\otimes\tilde{\mathbf{b}})_i^\varepsilon\big)\mathrm{d}x\mathrm{d}t\\ &+\int_0^T\int_\Omega \xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon\cdot\big(\psi_0\nabla(\frac{|\mathbf{b}|^2}{2})^\varepsilon+\sum_{i=1}^k\psi_i\nabla(\frac{|\tilde{\mathbf{b}}|^2}{2})_i^\varepsilon\big)\mathrm{d}x\mathrm{d}t\\ &-\mu\int_0^T\int_\Omega \xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon\cdot\big(\psi_0\Delta\mathbf{u}^\varepsilon+\sum_{i=1}^k\psi_i\Delta\tilde{\mathbf{u}}_i^\varepsilon\big)\mathrm{d}x\mathrm{d}t=0. \end{aligned} \tag{3.2}$$

接下来对(3.2)中的各项关于 δ 和 ε 取极限. 其中第二项, 第三项, 第六项分别参考文章 [1]中的引理A.1, 引理A.2, 引理3.3和章节3.2, 得到

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon\cdot\big(\psi_0\mathrm{div}(\mathbf{u}\otimes\mathbf{u})^\varepsilon+\sum_{i=1}^k\psi_i\mathrm{div}(\tilde{\mathbf{u}}\otimes\tilde{\mathbf{u}})_i^\varepsilon\big)\mathrm{d}x\mathrm{d}t=0; \tag{3.3}$$

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon\cdot\big(\psi_0\nabla P^\varepsilon+\sum_{i=1}^k\psi_i\nabla\tilde{P}_i^\varepsilon\big)\mathrm{d}x\mathrm{d}t=0; \tag{3.4}$$

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon\cdot\big(\psi_0\Delta\mathbf{u}^\varepsilon+\sum_{i=1}^k\psi_i\Delta\tilde{\mathbf{u}}_i^\varepsilon\big)\mathrm{d}x\mathrm{d}t=-\int_0^T\int_\Omega\xi_\tau|\nabla\mathbf{u}|^2\mathrm{d}x\mathrm{d}t. \tag{3.5}$$

(3.2)中的第一项, 第四项, 第五项关于 δ 和 ε 取极限的结果由下列各引理给出.

引理 3.1. (3.2)中的第一项满足

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon\cdot\partial_t\big(\psi_0\mathbf{u}^\varepsilon+\sum_{i=1}^k\psi_i\tilde{\mathbf{u}}_i^\varepsilon\big)\mathrm{d}x\mathrm{d}t=-\frac{1}{2}\int_0^T\int_\Omega\xi_\tau'|\mathbf{u}|^2\mathrm{d}x\mathrm{d}t.$$

证明 由定义式(2.5)知

$$\begin{aligned}&\int_0^T\int_\Omega \xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon\cdot\partial_t\big(\psi_0\mathbf{u}^\varepsilon+\sum_{i=1}^k\psi_i\tilde{\mathbf{u}}_i^\varepsilon\big)\mathrm{d}x\mathrm{d}t\\=&\int_0^T\int_\Omega \xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon\cdot\partial_t[\mathbf{u}]^\varepsilon\mathrm{d}x\mathrm{d}t\\=&\frac{1}{2}\int_0^T\int_\Omega \xi_\tau\eta_\delta\partial_t|[\mathbf{u}]^\varepsilon|^2\mathrm{d}x\mathrm{d}t\\=&-\frac{1}{2}\int_0^T\int_\Omega \xi_\tau'\eta_\delta|[\mathbf{u}]^\varepsilon|^2\mathrm{d}x\mathrm{d}t.\end{aligned}$$

对上式插项, 并令 $\varepsilon\to 0,\delta\to 0$, 根据 ξ_τ',η_δ 的有界性及命题2.3得

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{2}\int_0^T\int_\Omega \xi_\tau'\eta_\delta\big(|[\mathbf{u}]^\varepsilon|^2-|\mathbf{u}|^2\big)\mathrm{d}x\mathrm{d}t=0.$$

从而,

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon\cdot\partial_t\big(\psi_0\mathbf{u}^\varepsilon+\sum_{i=1}^k\psi_i\tilde{\mathbf{u}}_i^\varepsilon\big)\mathrm{d}x\mathrm{d}t=-\frac{1}{2}\int_0^T\int_\Omega \xi_\tau'|\mathbf{u}|^2\mathrm{d}x\mathrm{d}t.$$

引理3.1.证毕.

引理 3.2. (3.2)中的第四项满足

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon\cdot\big(\psi_0\mathrm{div}(\mathbf{b}\otimes\mathbf{b})^\varepsilon+\sum_{i=1}^k\psi_i\mathrm{div}(\tilde{\mathbf{b}}\otimes\tilde{\mathbf{b}})_i^\varepsilon\big)\mathrm{d}x\mathrm{d}t=-\int_0^T\int_\Omega \xi_\tau\mathbf{b}\cdot\mathrm{div}(\mathbf{b}\otimes\mathbf{u})\mathrm{d}x\mathrm{d}t.$$

证明 先分部积分, 有

$$\begin{aligned}&\int_0^T\int_\Omega \xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon\cdot\big(\psi_0\mathrm{div}(\mathbf{b}\otimes\mathbf{b})^\varepsilon+\sum_{i=1}^k\psi_i\mathrm{div}(\tilde{\mathbf{b}}\otimes\tilde{\mathbf{b}})_i^\varepsilon\big)\mathrm{d}x\mathrm{d}t\\=&-\int_0^T\int_\Omega \xi_\tau\nabla\eta_\delta\otimes[\mathbf{u}]^\varepsilon:\big(\psi_0(\mathbf{b}\otimes\mathbf{b})^\varepsilon+\sum_{i=1}^k\psi_i(\tilde{\mathbf{b}}\otimes\tilde{\mathbf{b}})_i^\varepsilon\big)\mathrm{d}x\mathrm{d}t\\&-\int_0^T\int_\Omega \xi_\tau\eta_\delta\nabla[\mathbf{u}]^\varepsilon:\big(\psi_0(\mathbf{b}\otimes\mathbf{b})^\varepsilon+\sum_{i=1}^k\psi_i(\tilde{\mathbf{b}}\otimes\tilde{\mathbf{b}})_i^\varepsilon\big)\mathrm{d}x\mathrm{d}t\\&-\int_0^T\int_\Omega \xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon\big(\nabla\psi_0:(\mathbf{b}\otimes\mathbf{b})^\varepsilon+\sum_{i=1}^k\nabla\psi_i:(\tilde{\mathbf{b}}\otimes\tilde{\mathbf{b}})_i^\varepsilon\big)\mathrm{d}x\mathrm{d}t\\:=&I_{41}+I_{42}+I_{43}.\end{aligned}$$

先看 I_{41}

$$I_{41}=-\int_0^T\int_\Omega \xi_\tau\nabla\eta_\delta\otimes[\mathbf{u}]^\varepsilon:\Big(\psi_0\big((\mathbf{b}\otimes\mathbf{b})^\varepsilon-(\mathbf{b}\otimes\mathbf{b})\big)+\sum_{i=1}^k\psi_i\big((\tilde{\mathbf{b}}\otimes\tilde{\mathbf{b}})_i^\varepsilon-(\mathbf{b}\otimes\mathbf{b})\big)\Big)\mathrm{d}x\mathrm{d}t$$
$$-\int_0^T\int_\Omega \xi_\tau\nabla\eta_\delta\otimes([\mathbf{u}]^\varepsilon-\mathbf{u}):(\mathbf{b}\otimes\mathbf{b})\mathrm{d}x\mathrm{d}t$$
$$-\int_0^T\int_\Omega \xi_\tau\nabla\eta_\delta\otimes\mathbf{u}:(\mathbf{b}\otimes\mathbf{b})\mathrm{d}x\mathrm{d}t.$$

用 Hölder 不等式及 (1.4), (1.9), (1.10), (2.4), 引理 2.1 及命题 2.1, 命题 2.2, 命题 2.3, 得

$$\lim_{\delta\to0}\lim_{\varepsilon\to0}I_{41}=0. \tag{3.6}$$

再看 I_{42}

$$I_{42}=-\int_0^T\int_\Omega \xi_\tau\eta_\delta\nabla[\mathbf{u}]^\varepsilon:\Big(\psi_0\Big((\mathbf{b}\otimes\mathbf{b})^\varepsilon-(\mathbf{b}\otimes\mathbf{b})\Big)+\sum_{i=1}^k\psi_i\big((\tilde{\mathbf{b}}\otimes\tilde{\mathbf{b}})_i^\varepsilon-(\mathbf{b}\otimes\mathbf{b})\big)\Big)\mathrm{d}x\mathrm{d}t$$
$$-\int_0^T\int_\Omega \xi_\tau\eta_\delta(\nabla[\mathbf{u}]^\varepsilon-\nabla\mathbf{u}):(\mathbf{b}\otimes\mathbf{b})\mathrm{d}x\mathrm{d}t-\int_0^T\int_\Omega \xi_\tau\eta_\delta\nabla\mathbf{u}:(\mathbf{b}\otimes\mathbf{b})\mathrm{d}x\mathrm{d}t.$$

由 Hölder 不等式及 (1.4), (1.10), (2.4), 命题 2.1, 命题 2.2 和命题 2.3, 得

$$\lim_{\delta\to0}\lim_{\varepsilon\to0}I_{42}=-\int_0^T\int_\Omega \xi_\tau\nabla\mathbf{u}:(\mathbf{b}\otimes\mathbf{b})\mathrm{d}x\mathrm{d}t=-\int_0^T\int_\Omega \xi_\tau\mathbf{b}\cdot\mathrm{div}(\mathbf{b}\otimes\mathbf{u})\mathrm{d}x\mathrm{d}t. \tag{3.7}$$

最后看 I_{43}

$$I_{43}=-\int_0^T\int_\Omega \xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon\Big(\nabla\psi_0:\big((\mathbf{b}\otimes\mathbf{b})^\varepsilon-\mathbf{b}\otimes\mathbf{b}\big)+\sum_{i=1}^k\nabla\psi_i:\big((\tilde{\mathbf{b}}\otimes\tilde{\mathbf{b}})_i^\varepsilon-\mathbf{b}\otimes\mathbf{b}\big)\Big)\mathrm{d}x\mathrm{d}t$$
$$-\int_0^T\int_\Omega \xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon\sum_{i=0}^k\nabla\psi_i:(\mathbf{b}\otimes\mathbf{b})\mathrm{d}x\mathrm{d}t$$
$$=-\int_0^T\int_\Omega \xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon\Big(\nabla\psi_0:\big((\mathbf{b}\otimes\mathbf{b})^\varepsilon-\mathbf{b}\otimes\mathbf{b}\big)+\sum_{i=1}^k\nabla\psi_i:\big((\tilde{\mathbf{b}}\otimes\tilde{\mathbf{b}})_i^\varepsilon-\mathbf{b}\otimes\mathbf{b}\big)\Big)\mathrm{d}x\mathrm{d}t$$
$$-\int_0^T\int_\Omega \xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon\nabla\big(\sum_{i=0}^k\psi_i\big):(\mathbf{b}\otimes\mathbf{b})\mathrm{d}x\mathrm{d}t$$

由 Hölder 不等式及 (1.4), (1.10), (2.4), 命题 2.1, 命题 2.2 和命题 2.3, 得

$$\lim_{\delta\to0}\lim_{\varepsilon\to0}I_{43}=0. \tag{3.8}$$

故根据 (3.6), (3.7), (3.8) 有

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}(I_{41}+I_{42}+I_{43}) = -\int_0^T\int_\Omega \xi_\tau \mathbf{b}\cdot \mathrm{div}(\mathbf{b}\otimes\mathbf{u})\mathrm{d}x\mathrm{d}t.$$

引理3.2.证毕.

引理 3.3. (3.2)中的第五项满足

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon\cdot\big(\psi_0\nabla(\frac{|\mathbf{b}|^2}{2})^\varepsilon+\sum_{i=1}^k\psi_i\nabla(\frac{|\tilde{\mathbf{b}}|^2}{2})_i^\varepsilon\big)\mathrm{d}x\mathrm{d}t=0.$$

证明 先分部积分, 有

$$\begin{aligned}
&\int_0^T\int_\Omega \xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon\cdot\big(\psi_0\nabla(\frac{|\mathbf{b}|^2}{2})^\varepsilon+\sum_{i=1}^k\psi_i\nabla(\frac{|\tilde{\mathbf{b}}|^2}{2})_i^\varepsilon\big)\mathrm{d}x\mathrm{d}t\\
=&-\int_0^T\int_\Omega \xi_\tau\nabla\eta_\delta\cdot[\mathbf{u}]^\varepsilon\big(\psi_0(\frac{|\mathbf{b}|^2}{2})^\varepsilon+\sum_{i=1}^k\psi_i(\frac{|\tilde{\mathbf{b}}|^2}{2})_i^\varepsilon\big)\mathrm{d}x\mathrm{d}t\\
&-\int_0^T\int_\Omega \xi_\tau\eta_\delta\mathrm{div}[\mathbf{u}]^\varepsilon\big(\psi_0(\frac{|\mathbf{b}|^2}{2})^\varepsilon+\sum_{i=1}^k\psi_i(\frac{|\tilde{\mathbf{b}}|^2}{2})_i^\varepsilon\big)\mathrm{d}x\mathrm{d}t\\
&-\int_0^T\int_\Omega \xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon\cdot\big(\nabla\psi_0\nabla(\frac{|\mathbf{b}|^2}{2})^\varepsilon+\sum_{i=1}^k\nabla\psi_i\nabla(\frac{|\tilde{\mathbf{b}}|^2}{2})_i^\varepsilon\big)\mathrm{d}x\mathrm{d}t\\
:=&I_{51}+I_{52}+I_{53}.
\end{aligned}$$

先看 I_{51}

$$\begin{aligned}
&-\int_0^T\int_\Omega \xi_\tau\nabla\eta_\delta\cdot[\mathbf{u}]^\varepsilon\big(\psi_0(\frac{|\mathbf{b}|^2}{2})^\varepsilon+\sum_{i=1}^k\psi_i(\frac{|\tilde{\mathbf{b}}|^2}{2})_i^\varepsilon\big)\mathrm{d}x\mathrm{d}t\\
=&-\int_0^T\int_\Omega \xi_\tau\nabla\eta_\delta\cdot[\mathbf{u}]^\varepsilon\Big(\psi_0\big((\frac{|\mathbf{b}|^2}{2})^\varepsilon-\frac{|\mathbf{b}|^2}{2}\big)+\sum_{i=1}^k\psi_i\big((\frac{|\tilde{\mathbf{b}}|^2}{2})_i^\varepsilon-\frac{|\mathbf{b}|^2}{2}\big)\Big)\mathrm{d}x\mathrm{d}t\\
&-\int_0^T\int_\Omega \xi_\tau\nabla\eta_\delta\cdot([\mathbf{u}]^\varepsilon-\mathbf{u})\Big(\psi_0\frac{|\mathbf{b}|^2}{2}+\sum_{i=1}^k\psi_i\frac{|\mathbf{b}|^2}{2}\Big)\mathrm{d}x\mathrm{d}t\\
&-\int_0^T\int_\Omega \xi_\tau\nabla\eta_\delta\cdot\mathbf{u}\frac{|\mathbf{b}|^2}{2}\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

用 Hölder 不等式及 (1.4), (1.9), (1.10), (2.4), 引理 2.1 及命题 2.1, 命题 2.2, 命题 2.3, 得

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}I_{51}=0. \tag{3.9}$$

再看 I_{52},

$$
\begin{aligned}
&-\int_0^T\int_\Omega \xi_\tau\eta_\delta \mathrm{div}[\mathbf{u}]^\varepsilon\big(\psi_0(\frac{|\mathbf{b}|^2}{2})^\varepsilon+\sum_{i=1}^k\psi_i(\frac{|\tilde{\mathbf{b}}|^2}{2})_i^\varepsilon\big)\mathrm{d}x\mathrm{d}t\\
=&-\int_0^T\int_\Omega \xi_\tau\eta_\delta \mathrm{div}[\mathbf{u}]^\varepsilon\Big(\psi_0\big((\frac{|\mathbf{b}|^2}{2})^\varepsilon-\frac{|\mathbf{b}|^2}{2}\big)+\sum_{i=1}^k\psi_i\big((\frac{|\tilde{\mathbf{b}}|^2}{2})_i^\varepsilon-\frac{|\mathbf{b}|^2}{2}\big)\Big)\mathrm{d}x\mathrm{d}t\\
&-\int_0^T\int_\Omega \xi_\tau\eta_\delta(\mathrm{div}[\mathbf{u}]^\varepsilon-\mathrm{div}\mathbf{u})(\frac{|\mathbf{b}|^2}{2})\mathrm{d}x\mathrm{d}t-\int_0^T\int_\Omega \xi_\tau\eta_\delta \mathrm{div}\mathbf{u}(\frac{|\mathbf{b}|^2}{2})\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

由 Hölder 不等式及 (1.4), (1.10), (2.4), 命题 2.1, 命题 2.2 和命题 2.3, 得

$$
\lim_{\delta\to 0}\lim_{\varepsilon\to 0} I_{52}=-\int_0^T\int_\Omega \xi_\tau\eta_\delta \mathrm{div}\mathbf{u}(\frac{|\mathbf{b}|^2}{2})\mathrm{d}x\mathrm{d}t=0. \tag{3.10}
$$

最后看 I_{53}

$$
\begin{aligned}
&-\int_0^T\int_\Omega \xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon\cdot\big(\nabla\psi_0\nabla(\frac{|\mathbf{b}|^2}{2})^\varepsilon+\sum_{i=1}^k\nabla\psi_i\nabla(\frac{|\tilde{\mathbf{b}}|^2}{2})_i^\varepsilon\big)\mathrm{d}x\mathrm{d}t\\
=&-\int_0^T\int_\Omega \xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon\cdot\Big(\nabla\psi_0\big(\nabla(\frac{|\mathbf{b}|^2}{2})^\varepsilon-\frac{|\mathbf{b}|^2}{2}\big)+\sum_{i=1}^k\nabla\psi_i\big(\nabla(\frac{|\tilde{\mathbf{b}}|^2}{2})_i^\varepsilon-\frac{|\mathbf{b}|^2}{2}\big)\Big)\mathrm{d}x\mathrm{d}t\\
&-\int_0^T\int_\Omega \xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon\cdot\sum_{i=0}^k\nabla\psi_i\frac{|\mathbf{b}|^2}{2}\mathrm{d}x\mathrm{d}t\\
=&-\int_0^T\int_\Omega \xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon\cdot\Big(\nabla\psi_0\big(\nabla(\frac{|\mathbf{b}|^2}{2})^\varepsilon-\frac{|\mathbf{b}|^2}{2}\big)+\sum_{i=1}^k\nabla\psi_i\big(\nabla(\frac{|\tilde{\mathbf{b}}|^2}{2})_i^\varepsilon-\frac{|\mathbf{b}|^2}{2}\big)\Big)\mathrm{d}x\mathrm{d}t\\
&-\int_0^T\int_\Omega \xi_\tau\eta_\delta[\mathbf{u}]^\varepsilon\cdot\nabla(\sum_{i=0}^k\psi_i)\frac{|\mathbf{b}|^2}{2}\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

由 Hölder 不等式及 (1.4), (1.10), (2.4), 命题 2.1, 命题 2.2 和命题 2.3, 得

$$
\lim_{\delta\to 0}\lim_{\varepsilon\to 0} I_{53}=0. \tag{3.11}
$$

故根据 (3.9), (3.10), (3.11) 有

$$
\lim_{\delta\to 0}\lim_{\varepsilon\to 0}(I_{51}+I_{52}+I_{53})=0.
$$

引理3.3.证毕.

由(3.3), (3.4), (3.5), 引理3.1, 引理3.2 及引理3.3知, 令方程(3.2)中 $\varepsilon\to 0$, $\delta\to 0$ 得

$$
-\frac{1}{2}\int_0^T\int_\Omega \xi_\tau'|\mathbf{u}|^2\mathrm{d}x\mathrm{d}t+\int_0^T\int_\Omega \xi_\tau\mathbf{b}\cdot\mathrm{div}(\mathbf{b}\otimes\mathbf{u})\mathrm{d}x\mathrm{d}t+\mu\int_0^T\int_\Omega \xi_\tau|\nabla\mathbf{u}|^2\mathrm{d}x\mathrm{d}t=0. \tag{3.12}
$$

DOI: 10.12677/pm.2021.115089

结合方程 (1.1)$_3$, 得到其中

$$\begin{aligned}
&\int_0^T\int_\Omega \xi_\tau \mathbf{b}\cdot \mathrm{div}(\mathbf{b}\otimes\mathbf{u})\mathrm{d}x\mathrm{d}t\\
=&\int_0^T\int_\Omega \xi_\tau \mathbf{b}\cdot \mathrm{div}(\mathbf{b}\otimes\mathbf{u})\mathrm{d}x\mathrm{d}t-\int_0^T\int_\Omega \xi_\tau \mathbf{b}\cdot \mathrm{div}(\mathbf{u}\otimes\mathbf{b})\mathrm{d}x\mathrm{d}t\\
=&\int_0^T\int_\Omega \xi_\tau \mathbf{b}\cdot \big(\mathrm{div}(\mathbf{b}\otimes\mathbf{u})-\mathrm{div}(\mathbf{u}\otimes\mathbf{b})\big)\mathrm{d}x\mathrm{d}t\\
=&\int_0^T\int_\Omega \xi_\tau \mathbf{b}\cdot \partial_t\mathbf{b}\mathrm{d}x\mathrm{d}t\\
=&-\frac{1}{2}\int_0^T\int_\Omega \xi_\tau^{'}|\mathbf{b}|^2\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

从而, (3.12) 化简成

$$-\frac{1}{2}\int_0^T\int_\Omega \xi_\tau^{'}(|\mathbf{u}|^2+|\mathbf{b}|^2)\mathrm{d}x\mathrm{d}t+\mu\int_0^T\int_\Omega \xi_\tau|\nabla\mathbf{u}|^2\mathrm{d}x\mathrm{d}t=0. \tag{3.13}$$

由于 $\xi_\tau(t)\in\mathcal{C}_0^1((\tau,T-\tau))$, 对于任意 $t_0>0$, 取任意小 τ,α , 使得 $\tau+\alpha<t_0$, 取

$$\xi_\tau(t)=\begin{cases}0, & 0\le t\le\tau,\\ \frac{t-\tau}{\alpha}, & \tau<t\le\tau+\alpha,\\ 1, & \tau+\alpha<t\le t_0,\\ \frac{t_0+\alpha-t}{\alpha}, & t_0<t\le t_0+\alpha,\\ 0, & t\ge t_0+\alpha.\end{cases} \tag{3.14}$$

将(3.14)代入(3.13), 得到

$$\begin{aligned}
&-\frac{1}{2\alpha}\int_\tau^{\tau+\alpha}\int_\Omega(|\mathbf{b}|^2+|\mathbf{u}|^2)\mathrm{d}x\mathrm{d}t+\frac{1}{2\alpha}\int_{t_0}^{t_0+\alpha}\int_\Omega(|\mathbf{b}|^2+|\mathbf{u}|^2)\mathrm{d}x\mathrm{d}t\\
&+\mu\int_\tau^{\tau+\alpha}\frac{t-\tau}{\alpha}\int_\Omega|\nabla\mathbf{u}|^2\mathrm{d}x\mathrm{d}t+\mu\int_{\tau+\alpha}^{t_0}\int_\Omega|\nabla\mathbf{u}|^2\mathrm{d}x\mathrm{d}t\\
&+\mu\int_{t_0}^{t_0+\alpha}\frac{-t+t_0+\alpha}{\alpha}\int_\Omega|\nabla\mathbf{u}|^2\mathrm{d}x\mathrm{d}t=0.
\end{aligned} \tag{3.15}$$

由积分的绝对连续性,

$$\lim_{\alpha\to0}\int_\tau^{\tau+\alpha}\frac{t-\tau}{\alpha}\int_\Omega|\nabla\mathbf{u}|^2\mathrm{d}x\mathrm{d}t=0;$$

$$\lim_{\alpha\to0}\int_{t_0}^{t_0+\alpha}\frac{-t+t_0+\alpha}{\alpha}\int_\Omega|\nabla\mathbf{u}|^2\mathrm{d}x\mathrm{d}t=0.$$

记

$$E(t) = \frac{1}{2}\int_{\Omega}(|\mathbf{b}|^2 + |\mathbf{u}|^2)\mathrm{d}x,$$
$$F(t) = \mu\int_0^t\int_{\Omega}|\nabla\mathbf{u}|^2\mathrm{d}x\mathrm{d}s.$$

断言:

$$\mathbf{u},\ \mathbf{b} \in \mathcal{C}^0\big([0,T];L^2(\Omega)\big). \tag{3.16}$$

事实上, 由 $(1.1)_2$, (1.4), (1.8), (1.9), (1.10), 知 $\mathbf{u}_t \in L^2\big(0,T;H^{-1}(\Omega)\big)$. 在引理2.4. 中, 取 $X = L^2(\Omega)$, $Y = H^{-1}(\Omega)$, 从而 $\mathbf{u} \in \mathcal{C}^0\big([0,T];L^2_w(\Omega)\big)$. 同理, 由 $(1.1)_3$, (1.4), (1.9), (1.10) 和引理2.4., 得到 $\mathbf{b} \in \mathcal{C}^0\big([0,T];L^2_w(\Omega)\big)$. 结合能量不等式 (1.6) 和初值条件, 得到 $\mathbf{u}$, $\mathbf{b} \in \mathcal{C}^0\big([0,T];L^2(\Omega)\big)$.

在(3.15)中, 由于(3.16)和勒贝格微分定理, 令 $\alpha \to 0$, 得到

$$(E+F)(t_0) = (E+F)(\tau). \tag{3.17}$$

在 (3.17) 中, 由于(3.16), 令 $\tau \to 0^+$, 得到

$$(E+F)(t_0) = (E+F)(0),\ t_0 \in (0,T),$$

此即(1.6). 由 t_0 的任意性, 已完成定理1.1.的证明.

参考文献

[1] Chen, R.M., Liang, Z.L., Wang, D.H. and Xu, R.Z. (2020) Energy Equality in Compressible Fluids with Physical Boundaries. *SIAM Journal on Mathematical Analysis*, **52**, 1363-1385. https://doi.org/10.1137/19M1287213

[2] Kufner, A., John, O. and Fuk, S. (1977) Function Spaces. Academia, Prague.

[3] Evans, L.C. (1998) Partial Differential Equations. American Mathematical Society, Providence, RI.

[4] Lions, P.L. (1996) Mathematical Topics in Fluid Mechanics. Vol. 1, Incompressible Models. Oxford University Press, New York.