Pure Mathematics 理论数学, 2021, 11(11), 1850-1870
Published Online November 2021 in Hans. http://www.hanspub.org/journal/pm
https://doi.org/10.12677/pm.2021.1111208

均值-方差准则下再保险双方联合收益的最优投资再保险策略

孙婷婷，王慧慧，舒慧生*

东华大学理学院，上海

收稿日期：2021年10月15日；录用日期：2021年11月15日；发布日期：2021年11月22日

摘 要

在两类保险业务相关的风险模型背景下，研究了保险公司和再保险公司的联合收益的最优投资再保险问题。假设保险公司可以向再保险公司购买比例再保险和投资于一个由无风险资产和风险资产组成的金融市场，再保险公司采用期望保费原理收取保费并可以通过投资无风险资产来降低风险。在均值-方差准则下，通过求解扩展的**Hamilton-Jacobi-Bellman**方程组，得到了联合收益的最优投资策略和最优再保险策略的表达式以及最优值函数，并通过实例验证了结果的有效性。

关键词

期望保费原理，再保险，均值-方差准则，联合收益，**HJB**方程

Optimal Investment Reinsurance Strategy for the Joint Benefits of the Insurer and the Reinsurer under the Mean-Variance

Tingting Sun, Huihui Wang, Huisheng Shu*

College of Science and Technology, Donghua University, Shanghai

Received: Oct. 15^{th}, 2021; accepted: Nov. 15^{th}, 2021; published: Nov. 22^{nd}, 2021

* 通讯作者。

文章引用: 孙婷婷, 王慧慧, 舒慧生. 均值-方差准则下再保险双方联合收益的最优投资再保险策略[J]. 理论数学, 2021, 11(11): 1850-1870. DOI: 10.12677/pm.2021.1111208

Abstract

Under the background of the risk model related to two types of insurance business, the optimal investment reinsurance of the joint benefits of the insurer and the reinsurer is studied. Assuming that the insurer can buy proportional reinsurance from the reinsurer and invest in a financial market consisting of risk-free and risk assets, the reinsurer can use the expected premium principle to charge premiums and reduce risk by investing in risk-free assets. Under the mean-variance criterion, the expressions of the optimal investment strategy and the optimal reinsurance strategy and the optimal value function of the combined returns are obtained by solving the extended Hamilton-Jacobi-Bellman equation system, and the validity of the result is verified by example.

Keywords

Expected Premium Principle, Reinsurance, Mean-Variance Criterion, Joint Benifits, HJB Equation

1. 介绍

现代保险市场由保险公司和再保险公司组成，两者通过再保险相互联系。对于保险公司来说，风险管理是一个重要的问题。如何选择投资多少股票和债券，以及购买多少和什么类型的再保险被归结为最优投资再保险问题，为保险公司寻求最优的投资再保险策略成为一个备受关注的热点。

在最优投资再保险问题中，最常被使用的目标函数有破产概率最小化准则、期望效用最大化准则以及均值-方差准则。文献 [1–4]都是在破产概率最小化准则下研究保险公司最优策略问题。对于期望效用最大化准则，文献 [5]对其进行了研究；文献 [6]假设保险公司的基本索赔过程遵循漂移的布朗运动，在不做空约束下，研究了终端财富期望指数效用最大化问题。文献 [7] 在CEV模型中，考虑终端财富中最大化预期的指数效用，研究了再保险投资问题和仅投资问题。对于均值-方差准则下保险公司的最优投资再保险问题，文献 [8] 首次给出了研究；文献 [9]在均值-方差准则下研究了投资再保险问题和纯投资问题的最优策略；文献 [10]研究了跳跃扩散金融市场中具有投资和

再保险的保险公司的均值- 方差最优问题。

上述都是从保险公司的角度出发研究最优策略的，但是在现实生活中，存在同时控股保险公司和再保险公司的集团，如中国再保险(集团)股份有限公司等，正如文献 [11]陈述一份再保险合约涉及双方，即保险公司和再保险公司，在再保险最优形式的标准下，对其中一方最优的再保险策略，对另一方来说可能不是最优的，所以，在最优投资再保险的研究中，考虑保险公司和再保险公司双方是合理的。文献 [12,13]考虑了保险公司和再保险公司双方的利益，对两者共同生存和盈利的概率进行了研究。文献 [14]研究了由CEV模型建模的风险资产，以最大化保险双方的终端财富指数效用为准则，得出了最优投资再保险策略；在此基础上，文献 [15,16]研究了不同优化准则下，保险公司和再保险公司的最优策略。文献 [17]应用动态规划方法和对偶理论，研究了同时考虑保险公司和再保险公司利益的最优再保险投资策略。对于保险公司和再保险公司双方博弈的问题，文献 [18]给出了研究。文献 [19]研究了保险公司和再保险公司纳什均衡投资再保险策略和跨期限制对纳什均衡投资再保险策略的影响。

此外，随着人们保险意识的不断增强，以及对保险的配置有更多的个性化需求，保险公司的险种业务越来越复杂，险种的类型也越来越多，如重症险、寿险、意外险、医疗险等。这些险种可能会同时生效，比如意外险和医疗险。随着对再保险的深入研究，文献 [20]考虑了两种具有相关性的险种业务，在方差保费原理下，利用随机控制理论，导出了终端财富期望指数效用最大化准则下复合泊松风险模型和布朗运动风险模型的最优比例再保险策略和值函数的封闭表达式。文献 [21]还在均值-方差准则下，研究了具有跳跃扩散风险资产的金融市场的最优再保险投资问题，其中保险风险由复合泊松过程建模，两个跳跃过程由一个共同冲击关联，基于随机线性二次控制理论，给出了HJB方程粘性解的最优策略和值函数的显式表达式。基于以上参考文献的研究和探讨，本文将同时考虑保险公司和再保险公司双方，在期望保费原理下研究具有两类相依险种业务的风险模型，目标函数采用均值-方差准则，对最优投资再保险问题进行了探讨。

2. 模型和假设

假设$(\Omega, \mathbb{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$是一个带流$\{\mathcal{F}_t\}_{t\geq 0}$的完备概率空间，其中流$\{\mathcal{F}_t\}_{t\geq 0}$满足通常条件：右连续、递增、$\{\mathcal{F}_t\}_{t\geq 0}$包含所有P-可略集。在本文中，所有的随机变量都定义在概率空间中。

2.1. 盈余过程

令$X = \{X_i, i = 1, 2, ...\}$，$Y = \{Y_i, i = 1, 2, ...\}$是保险公司的两类不同的险种，且具有相关性。$X = \{X_i, i = 1, 2, ...\}$和$Y = \{Y_i, i = 1, 2, ...\}$是独立同分布的正随机变量，分布函数分别为$F_1(x)$，$F_2(y)$，当$X \leq 0$ 时，$F_1(x) = 0$，当$X > 0$时，$0 \leq F_i(x) \leq 1$；当$Y \leq 0$ 时，$F_2(y) = 0$，当$Y > 0$时，$0 \leq F_2(y) \leq 1$。X_i 是第一类险种第i次的索赔大小，其均值为$E(X_i) = \mu_{11}$，二阶矩为$E(X_i^2) = \mu_{12}$；Y_i是第一类险种第i次的索赔大小，其均值为$E(Y_i) = \mu_{21}$，二阶矩为$E(Y_i^2) = \mu_{22}$。两类险种到时刻t 的总索赔额$L(t)$为

$$L(t) = \sum_{i=1}^{\widetilde{N}_1(t)} X_i + \sum_{i=1}^{\widetilde{N}_2(t)} Y_i = L_1(t) + L_2(t).$$

其中$\widetilde{N}_1(t)$和$\widetilde{N}_2(t)$为$(0,T]$内第一类险种和第二类险种索赔发生的次数，与X，Y独立，且索赔次数的相关性如下：

$$\widetilde{N}_1(t)=N_1(t)+N(t),\widetilde{N}_2(t)=N_2(t)+N(t),$$

其中，$N_1(t)$，$N_2(t)$，$N(t)$服从Poisson分布且相互独立，强度分别为$\lambda_1>0$，$\lambda_2>0$，$\lambda>0$，$N_i(t)$，$i=1,2$表示只索赔第i类险种的索赔次数，$N(t)$ 表示同一事故中同时索赔两类险种的索赔次数，即索赔发生的次数$(\widetilde{N}_1(t),\widetilde{N}_2(t))$ 通过共同的Poisson变量$N(t)$相关。于是，由上述设定，保险公司的盈余过程为

$$\begin{aligned}R(t)&=x_0+ct-L(t)\\&=x_0+ct-L_1(t)-L_2(t)\\&=x_0+ct-\sum_{i=1}^{N_1(t)+N(t)}X_i-\sum_{i=1}^{N_2(t)+N(t)}Y_i\end{aligned}\tag{2.1}$$

其中$x_0\geq0$是保险公司的初始盈余，c是保费率(单位时间内保险公司的保费收入)。

2.2. 再保险模型

为了降低经营风险，稳定公司经营，允许保险公司在原保险合同的基础上，通过签订分保合同，将其所承担的部分风险和责任向其他保险公司进行转移，即购买再保险。假设保险公司向再保险公司购买比例再保险，对于两类不同的险种业务，保险公司的自留比例分别为q_{1t}和q_{2t}，这里$q_{it}\in[0,1]$，$i=1,2$，即对于时刻t发生的索赔额X_i和Y_i，保险公司赔付$q_{1t}X_i$和$q_{2t}Y_i$，再保险公司赔付剩余的理赔额$(1-q_{1t})X_i$和$(1-q_{2t})Y_i$。假设再保险公司的保费率为$\delta(q_{1t},q_{2t})$，按照期望保费原理收取保费，则

$$\delta(q_{1t},q_{2t})=(1+\eta_1)(1-q_1)a_1+(1+\eta_2)(1-q_2)a_2.\tag{2.2}$$

其中$a_1=(\lambda_1+\lambda)E(X)$，$a_2=(\lambda_2+\lambda)E(Y)$，$\eta_i$为再保险公司对于第$i$类险种的安全负荷$(i=1,2)$。

2.3. 金融市场

保险公司除了通过购买再保险转移风险外，还可以在金融市场进行投资，来提高赔偿的能力。假设金融市场由一个无风险资产(债券)和一个风险资产(股票)组成。在t时刻，无风险资产的价格$S_0(t)$为

$$\begin{cases}dS_0(t)=rS_0(t)dt,\\S_0(t)=S_0.\end{cases}\tag{2.3}$$

其中$r>0$是无风险资产(债券)的利率；风险资产的价格$S_1(t)$为

$$\begin{cases}dS_1(t)=S_1(t)[\beta dt+\sigma dW(t)],\\S_1(t)=S_1.\end{cases}\tag{2.4}$$

其中β为风险资产(股票)的回报率，σ为风险资产(股票)的波动率，$W(t)$是一个标准的布朗运动，与X_i，Y_i，$N_1(t)$，$N_2(t)$，$N(t)$独立。

2.4. 财富过程

本文中，保险公司可以投资于无风险资产和风险资产，再保险公司可以投资于无风险资产来规避风险。在t时刻，设$\pi(t)$为保险公司投资于风险资产的金额，保险公司的财富过程R_{1t}为

$$dR_{1t} = [c + r(R_{1t} - \pi) - \delta(q_{1t}, q_{2t})]dt + \pi[\beta dt + \sigma dW(t)] - q_{1t}dL_1(t) - q_{2t}dL_2(t). \tag{2.5}$$

再保险公司的财富过程R_{2t}为

$$dR_{2t} = [rR_{2t} + \delta(q_{1t}, q_{2t})]dt - (1 - q_{1t})dL_1(t) - (1 - q_{2t})dL_2(t). \tag{2.6}$$

本文为了同时兼顾保险公司和再保险公司的利益，将两者的财富过程以$\alpha \in [0,1]$加权，记为$R_t = \alpha R_{1t} + (1-\alpha)R_{2t}$。当加权系数$\alpha = 0$时，$R_t$刻画了再保险公司的财富过程，当加权系数$\alpha = 1$时，$R_t$ 刻画了保险公司的财富过程，且α的大小刻画了最优策略的寻找过程中保险公司的利益比重，越大则更侧重保险公司的利益，反之则更侧重再保险公司的利益。另一方面，我们可以理解成一个保险公司和一个再保险公司属于同一个保险集团，α 和$(1-\alpha)$分别为两者所持有的股份比例，R_t为保险集团的财富过程。因此，由伊藤公式，我们有

$$\begin{aligned} dR_t &= [rR_t + \alpha c + \alpha(\beta - r)\pi + (1-2\alpha)\delta(q_{1t}, q_{2t})] + \alpha\sigma\pi dW(t) - \sum_{i=1}^{2}[1 - \alpha - (1-2\alpha)q_{it}]dL_i(t) \\ &= \{rR_t + \alpha c + \alpha(\beta - r)\pi + (1-2\alpha)[(1+\eta_1)(1-q_1)a_1 + (1+\eta_2)(1-q_2)a_2]\}dt \\ &+ \alpha\sigma\pi dW(t) - \sum_{i=1}^{2}[1 - \alpha - (1-2\alpha)q_{it}]dL_i(t). \end{aligned} \tag{2.7}$$

由上式可知，当$\alpha = \frac{1}{2}$时，财富过程与q_{1t}，q_{2t}无关，其最优再保险策略为任意值，故本文在$\alpha \neq \frac{1}{2}$且$\alpha \in (0,1)$的基础上寻求最优策略。

定义 2.1 如果策略$Q = (q_{1t}, q_{2t}, \pi(t))$是关于流$\{\mathcal{F}_t\}_{t\geq 0}$-适应的随机过程，且满足以下条件：

1. 对于流$\{\mathcal{F}_t\}_{t\geq 0}$，$(q_{1s}, q_{2t}, \pi(t))$是循序可测的；

2. 对$\forall s \in [t,T]$，$0 \leq q_{is} \leq 1, i = 1,2$，$E[\int_t^T (q_{1s}^2 + q_{2s}^2 + \pi(s)^2)] < +\infty$；

3. (Q, R_t^Q)是随机微分方程(2.7)的唯一解。

则策略$Q = (q_{1t}, q_{2t}, \pi(t))$为可容许策略，所有可容许策略的集合记为$\mathbb{Q}$。

3. 定义和引理

根据文献 [22]，我们在博弈论框架下为保险集团构建了一个投资再保险问题，保险集团的目标是最大化终端时刻T的财富期望，假设其效用函数采用均值-方差准则，即对$\forall(t,x) \in ([0,T] \times R)$，

目标函数为

$$\sup_{Q\in\mathbb{Q}} J(t,x,Q)=\sup_{Q\in\mathbb{Q}}\{E_{t,x}[R_T^Q]-\frac{\gamma}{2}Var_{t,x}[R_T^Q]\}. \tag{3.1}$$

其中$E_{t,x}[\cdot]=E[\cdot|R_t^Q=x]$，$Var_{t,x}[\cdot]=Var[\cdot|R_t^Q=x]$，$x$是初始盈余，$\gamma>0$为风险厌恶系数。

记$\mathcal{C}^{1,2}$是一个函数空间，满足对$\forall\psi(t,x)\in\mathcal{C}^{1,2}$，$\psi$，$\psi_t$，$\psi_x$和$\psi_{xx}$是$[0,T]\times R$上的连续函数，其中$\psi_t$为$\psi$对$t$的一阶导，$\psi_x$和$\psi_{xx}$为$\psi$对$x$的一阶导和二阶导，且

$$\begin{aligned}\mathcal{A}^Q\psi(t,x)=&\psi_t+\{rx+c\alpha+\alpha(\beta-r)\pi+(1-2\alpha)[(1+\eta_1)(1-q_1)a_1+(1+\eta_2)(1-q_2)a_2]\}\psi_x\\&+\frac{1}{2}\alpha^2\sigma^2\pi^2V_{xx}+\lambda_1E[\psi(t,x-(1-\alpha-(1-2\alpha)q_1)X)-\psi(t,x)]\\&+\lambda_2E[\psi(t,x-(1-\alpha-(1-2\alpha)q_2)Y)-\psi(t,x)]\\&+\lambda E[\psi(t,x-(1-\alpha-(1-2\alpha)q_1)X-(1-\alpha-(1-2\alpha)q_2)Y)-\psi(t,x)],\end{aligned} \tag{3.2}$$

定义 3.1 *(均衡策略)* 如果$Q\in R^+\times R^+\times R$，$h>0$和$(t,x)\in[0,T]\times R$，

$$\liminf_{h\to 0}\frac{J(t,x,Q^*)-J(t,x,Q_h)}{h}\geq 0.$$

其中

$$Q_h(s,\widetilde{x})=\begin{cases}Q,s\in[t,t+h],\widetilde{x}\in R,\\Q^*(s,\widetilde{x}),s\in[t+h,T],\widetilde{x}\in R,\end{cases}$$

则$Q^*(t,x)$为均衡策略，相应的均衡值函数为

$$V(t,x)=J(t,x,Q^*)=E_{t,x}[R_T^{Q^*}]-\frac{\gamma}{2}Var_{t,x}[R_T^{Q^*}]. \tag{3.3}$$

引理 3.1 *(验证定理)* 针对问题(3.1)，如果存在两个实函数$U(t,x),g(t,x)\in C^{1,2}([0,T]\times\mathbb{R})$满足下列HJB方程组：

$$\sup_{Q\in\mathbb{Q}}\{\mathcal{A}^QU(t,x)-\mathcal{A}^Q\frac{\gamma}{2}g(t,x)^2+\gamma g(t,x)\mathcal{A}^Qg(t,x)\}=0, \tag{3.4}$$

$$U(T,x)=x, \tag{3.5}$$

$$\mathcal{A}^{Q^*}g(t,x)=0, \tag{3.6}$$

$$g(T,x)=x. \tag{3.7}$$

其中

$$Q^*=arg\sup_{Q\in\mathbb{Q}}\{\mathcal{A}^QU(t,x)-\mathcal{A}^Q\frac{\gamma}{2}g(t,x)^2+\gamma g(t,x)\mathcal{A}^Qg(t,x)\}, \tag{3.8}$$

则$V(t,x)=U(t,x)$，$E_{t,x}[R_T^{Q^*}]=g(t,x)$，最优策略是$Q^*$。

上述定理的证明，见文献 [22]中的定理4.1。

4. 模型求解

本节，我们在均值-方差准则下去求解保险集团的最优投资和再保险策略。假设$U(t,x)$和$g(t,x)$是两个满足定理3.1中条件的函数，于是(3.4)和(3.6)可以写成如下形式：

$$\begin{aligned}
&\sup_{Q\in\mathbb{Q}}\{U_t+\{rx+c\alpha+\alpha(\beta-r)\pi+(1-2\alpha)[(1+\eta_1)(1-q_1)a_1+(1+\eta_2)(1-q_2)a_2]\}U_x\\
&+\frac{1}{2}(U_{xx}-\gamma g_x(t,x)^2)\alpha^2\sigma^2\pi^2-(\lambda_1+\lambda_2+\lambda)[U(t,x)+\frac{\gamma}{2}g(t,x)^2]\\
&+\lambda_1E[U(t,x-\theta_1X)-\frac{\gamma}{2}g(t,x-\theta_1X)(g(t,x-\theta_1X)-2g(t,x))]\\
&+\lambda_2E[U(t,x-\theta_2Y)-\frac{\gamma}{2}g(t,x-\theta_2Y)(g(t,x-\theta_2Y)-2g(t,x))]\\
&+\lambda E[U(t,x-\theta_1q_1X-\theta_2Y)-\frac{\gamma}{2}g(t,x-\theta_1X-\theta_2Y)(g(t,x-\theta_1X-\theta_2Y)-2g(t,x))]=0, \quad (4.1)
\end{aligned}$$

$$\begin{aligned}
&g_t+\{rx+c\alpha+\alpha(\beta-r)\pi+(1-2\alpha)[(1+\eta_1)(1-q_1)a_1+(1+\eta_2)(1-q_2)a_2]\}g_x\\
&+\frac{1}{2}\alpha^2\sigma^2\pi^2g_{xx}+\lambda_1E[g(t,x-\theta_1X)-g(t,x)]\\
&+\lambda_2E[g(t,x-\theta_2Y)-g(t,x)]\\
&+\lambda E[g(t,x-\theta_1X-\theta_2Y)-g(t,x)]=0, \quad (4.2)
\end{aligned}$$

其中$\theta_i=1-\alpha-(1-2\alpha)q_i, i=1,2$。根据Bjork和Murgoci (2010)以及定理3.1中条件，我们猜测解如下：

$$U(T,x)=A(t)x+B(t), A(T)=1, B(T)=0,$$

$$g(T,x)=a(t)x+b(t), a(T)=1, b(T)=0,$$

对其求微分可得：

$$U_t(T,x)=\dot{A}(t)x+\dot{B}(t), U_x(T,x)=A(t), U_{xx}(T,x)=0, \quad (4.3)$$

$$g_t(T,x)=\dot{a}(t)x+\dot{b}(t), g_x(T,x)=a(t), g_{xx}(T,x)=0, \quad (4.4)$$

其中，$\dot{A}(t)=dA(t)/dt$，$\dot{B}(t)=dB(t)/dt$，$\dot{a}(t)=da(t)/dt$，$\dot{b}(t)=db(t)/dt$，代入(4.1)得

$$A(t)[rx+c\alpha+\alpha(\beta-r)\pi+(1-2\alpha)((1+\eta_1)(1-q_1)a_1+(1+\eta_2)(1-q_2)a_2)]+\dot{B}(t)$$
$$\dot{A}(t)x-A(t)\theta_1a_1-A(t)\theta_2a_2-\frac{\gamma}{2}a(t)^2(\theta_1^2b_1^2+\theta_2^2b_2^2+2\theta_1\theta_2\lambda\mu_{11}\mu_{21}+\alpha^2\sigma^2\pi^2)=0. \tag{4.5}$$

其中$b_1^2=(\lambda_1+\lambda)E(X^2)$，$b_2^2=(\lambda_2+\lambda)E(Y^2)$。令

$$f(q_1,q_2)=-\frac{\gamma}{2}a(t)^2\theta_1^2b_1^2-\frac{\gamma}{2}a(t)^2\theta_2^2b_2^2-A(t)\theta_1a_1-A(t)\theta_2a_2-\gamma a(t)^2\theta_1\theta_2\lambda\mu_{11}\mu_{21}$$
$$+A(t)(1-2\alpha)(1+\eta_1)(1-q_1)a_1+A(t)(1-2\alpha)(1+\eta_2)(1-q_2)a_2. \tag{4.6}$$

则

$$\begin{cases}\frac{\partial f(q_1,q_2)}{\partial q_1}=(2\alpha-1)(-\gamma a(t)^2\theta_1b_1^2+A(t)\eta_1a_1-\gamma a(t)^2\theta_2\lambda\mu_{11}\mu_{21}),\\ \frac{\partial f(q_1,q_2)}{\partial q_2}=(2\alpha-1)(-\gamma a(t)^2\theta_2b_2^2+A(t)\eta_2a_2-\gamma a(t)^2\theta_1\lambda\mu_{11}\mu_{21}),\\ \frac{\partial^2 f(q_1,q_2)}{\partial q_1^2}=-\gamma a(t)^2(2\alpha-1)^2b_1^2,\\ \frac{\partial^2 f(q_1,q_2)}{\partial q_2^2}=-\gamma a(t)^2(2\alpha-1)^2b_2^2,\\ \frac{\partial^2 f(q_1,q_2)}{\partial q_1q_2}=-\gamma a(t)^2(2\alpha-1)^2\lambda\mu_{11}\mu_{21}.\end{cases} \tag{4.7}$$

$f(q_1,q_2)$的Hessian矩阵可以表示为：

$$\begin{pmatrix}\frac{\partial^2 f(q_1,q_2)}{\partial q_1^2} & \frac{\partial^2 f(q_1,q_2)}{\partial q_1\partial q_2}\\ \frac{\partial^2 f(q_1,q_2)}{\partial q_1\partial q_2} & \frac{\partial^2 f(q_1,q_2)}{\partial q_1^2}\end{pmatrix}=\begin{pmatrix}-\gamma a(t)^2(2\alpha-1)^2b_1^2 & -\gamma a(t)^2(2\alpha-1)^2\lambda\mu_{11}\mu_{21}\\ -\gamma a(t)^2(2\alpha-1)^2\lambda\mu_{11}\mu_{21} & -\gamma a(t)^2(2\alpha-1)^2b_2^2\end{pmatrix}$$

根据柯西-施瓦茨不等式，我们知道$b_1^2b_2^2=(\lambda_1+\lambda)E(X^2)(\lambda_2+\lambda)E(Y^2)>\lambda^2\mu_{11}^2\mu_{21}^2$，故$\gamma^2a(t)^4(2\alpha-1)^2(b_1^2b_2^2-\lambda^2\mu_{11}^2\mu_{21}^2)>0$，所以$f(q_1,q_2)$的Hessian 矩阵正定，即$f(q_1,q_2)$是关于$q_1$，$q_2$的凸函数。于是

$$\begin{cases}-\gamma a(t)^2\theta_1b_1^2+A(t)\theta_1a_1-\gamma a(t)^2\theta_2\lambda\mu_{11}\mu_{21}=0,\\ -\gamma a(t)^2\theta_2b_2^2+A(t)\theta_2a_2-\gamma a(t)^2\theta_1\lambda\mu_{11}\mu_{21}=0.\end{cases} \tag{4.8}$$

则

$$\begin{cases}\overline{\theta}_1=\frac{A(t)(-a_2\eta_2\lambda\mu_{11}\mu_{21}+a_1b_2^2\eta_1)}{\gamma a(t)^2(b_1^2b_2^2-\lambda^2\mu_{11}^2\mu_{21}^2)},\\ \overline{\theta}_2=\frac{A(t)(-a_1\eta_1\lambda\mu_{11}\mu_{21}+a_2b_1^2\eta_2)}{\gamma a(t)^2(b_1^2b_2^2-\lambda^2\mu_{11}^2\mu_{21}^2)}.\end{cases} \tag{4.9}$$

由$\theta_i=1-\alpha-(1-2\alpha)q_i, i=1,2$，我们有$\overline{q}_i=\frac{\overline{\theta}_i-(1-\alpha)}{2\alpha-1}, i=1,2$。根据(4.5)，得到

$$\widetilde{\pi}=\frac{A(t)(\beta-r)}{\alpha\gamma a(t)^2\sigma^2}, \tag{4.10}$$

将$\overline{q}_1,\overline{q}_2,\widetilde{\pi}$代入(4.5)和(4.2)，有

$$(\dot{A}(t)+rA(t))x+\dot{B}(t)+A(t)(\alpha c-(1+\eta_1)\alpha a_1-(1+\eta_2)\alpha a_2)+\frac{A(t)^2}{2\gamma a(t)^2}\xi=0, \tag{4.11}$$

$$(\dot{a}(t)+ra(t))x+\dot{b}(t)+a(t)(\alpha c-(1+\eta_1)\alpha a_1-(1+\eta_2)\alpha a_2)+\frac{A(t)^2}{\gamma a(t)^2}\xi=0, \tag{4.12}$$

其中

$$\xi=a_1^2\eta_1^2b_2^2+a_2^2\eta_2^2b_2^1-2a_1a_2\eta_1\eta_2\lambda\mu_{11}\mu_{21}+\frac{(\beta-r)^2}{\sigma^2}, \tag{4.13}$$

要想(4.11)和(4.12)成立，则

$$\begin{aligned}
&\dot{A}(t)+rA(t)=0, A(T)=1,\\
&\dot{B}(t)+A(t)(\alpha c-(1+\eta_1)\alpha a_1-(1+\eta_2)\alpha a_2)+\frac{A(t)^2}{2\gamma a(t)^2}\xi(t)=0,\\
&\dot{a}(t)+ra(t)=0, a(T)=1,\\
&\dot{b}(t)+a(t)(\alpha c-(1+\eta_1)\alpha a_1-(1+\eta_2)\alpha a_2)+\frac{A(t)^2}{\gamma a(t)^2}\xi(t)=0.
\end{aligned}$$

解上述微分方程，有

$$A(t)=e^{r(T-t)}, \tag{4.14}$$

$$B(t)=(\alpha c-(1+\eta_1)\alpha a_1-(1+\eta_2)\alpha a_2)\frac{1}{r}(e^{r(T-t)}-1)+\frac{1}{2\gamma}\xi(T-t), \tag{4.15}$$

$$a(t)=e^{r(T-t)}, \tag{4.16}$$

$$b(t)=(\alpha c-(1+\eta_1)\alpha a_1-(1+\eta_2)\alpha a_2)\frac{1}{r}(e^{r(T-t)}-1)+\frac{1}{\gamma}\xi(T-t). \tag{4.17}$$

将(4.14)和(4.16)代入(4.9)，可得

$$\overline{q}_1=\frac{1}{2\alpha-1}[\frac{-a_2\eta_2\lambda\mu_{11}\mu_{21}+a_1b_2^2\eta_1}{\gamma e^{r(T-t)}(b_1^2b_2^2-\lambda^2\mu_{11}^2\mu_{21}^2)}-(1-\alpha)], \tag{4.18}$$

$$\overline{q}_2=\frac{1}{2\alpha-1}[\frac{-a_1\eta_1\lambda\mu_{11}\mu_{21}+a_2b_1^2\eta_2}{\gamma e^{r(T-t)}(b_1^2b_2^2-\lambda^2\mu_{11}^2\mu_{21}^2)}-(1-\alpha)]. \tag{4.19}$$

记$m_1=-a_2\eta_2\lambda\mu_{11}\mu_{21}+a_1b_2^2\eta_1, m_2=-a_1\eta_1\lambda\mu_{11}\mu_{21}+a_2b_1^2\eta_2$，又简单验证可知$\frac{a_2\lambda\mu_{11}\mu21}{a_1b_2^2}<\frac{a_2b_1^2}{a_1\lambda\mu_{11}\mu_{21}}$。令$t_{i0}(\widehat{t}_{i0},\widetilde{t}_{i0})$是使得$\overline{q}_i=0(\widehat{q}_i=0,\widetilde{q}_i=0)$成立的时间点，$t_{i1}(\widehat{t}_{i1},\widetilde{t}_{i1})$是使得$\overline{q}_i=1(\widehat{q}_i=1,\widetilde{q}_i=1)$ 成立的时间点，t_{0i} 是使得$q_i(T-t)=0$ 成立的时间点，$i=1,2$，$t_{\widetilde{0}i}$是使得$\widetilde{q}_i(T-t)=0$

DOI: 10.12677/pm.2021.1111208

成立的时间点$i=1,2$，于是我们可以得到如下两个定理：

定理 4.1 若$\frac{1}{2}<\alpha<1$，则最优再保险策略(q_1^*,q_2^*)为

（1）当$\eta_1>\frac{a_2b_1^2}{a_1\lambda\mu_{11}\mu 21}\eta_2>\frac{a_2\lambda\mu_{11}\mu_{21}}{a_1b_2^2}\eta_2$ 时，即（$m_1>0,m_2<0$）

$$(q_1^*,q_2^*)=\begin{cases}(0,0),t\leq\widehat{t}_{10}\\(\widehat{q}_1,0),\widehat{t}_{10}<t<\widehat{t}_{11}\\(1,0),t\geq\widehat{t}_{11}\end{cases}\tag{4.20}$$

其中$\widehat{q}_1=\frac{1}{2\alpha-1}[\frac{-\gamma e^{r(T-t)}(1-\alpha)\lambda\mu_{11}\mu_{21}+\eta_1a_1}{\gamma e^{r(T-t)}b_1^2}-(1-\alpha)]$

（2）当$\frac{a_2b_1^2}{a_1\lambda\mu_{11}\mu 21}\eta_2>\eta_1>\frac{a_2\lambda\mu_{11}\mu_{21}}{a_1b_2^2}\eta_2$ 时，即（$m_1>0,m_2>0$）

若$m_1\geq m_2$，有

$$(q_1^*,q_2^*)=\begin{cases}(0,0),t\leq\widehat{t}_{10}\\(0,\widehat{q}_2),\widehat{t}_{10}<t\leq t_{20}\\(\overline{q}_1,\overline{q}_2),t_{20}<t<t_{11}\\(1,\widetilde{q}_2),t_{11}\leq t<\widetilde{t}_{21}\\(1,1),t\geq\widetilde{t}_{21}\end{cases}\tag{4.21}$$

其中$\widehat{q}_2=\frac{1}{2\alpha-1}[\frac{-\gamma e^{r(T-t)}(1-\alpha)\lambda\mu_{11}\mu_{21}+\eta_1a_1}{\gamma e^{r(T-t)}b_2^2}-(1-\alpha)]$，$\widetilde{q}_2=\frac{1}{2\alpha-1}[\frac{-\gamma e^{r(T-t)}\alpha\lambda\mu_{11}\mu_{21}+\eta_2a_2}{\gamma e^{r(T-t)}b_2^2}-(1-\alpha)]$；

若$m_1<m_2$，有

$$(q_1^*,q_2^*)=\begin{cases}(0,0),t\leq\widehat{t}_{20}\\(0,\widehat{q}_2),\widehat{t}_{20}<t\leq t_{10}\\(\overline{q}_1,\overline{q}_2),t_{10}<t<t_{21}\\(\widetilde{q}_1,1),t_{21}\leq t<\widetilde{t}_{11}\\(1,1),t\geq\widetilde{t}_{11}\end{cases}\tag{4.22}$$

其中$\widehat{q}_2=\frac{1}{2\alpha-1}[\frac{-\gamma e^{r(T-t)}(1-\alpha)\lambda\mu_{11}\mu_{21}+\eta_2a_2}{\gamma e^{r(T-t)}b_2^2}-(1-\alpha)]$，$\widetilde{q}_1=\frac{1}{2\alpha-1}[\frac{-\gamma e^{r(T-t)}\alpha\lambda\mu_{11}\mu_{21}+\eta_1a_1}{\gamma e^{r(T-t)}b_1^2}-(1-\alpha)]$；

（3）当$\frac{a_2b_1^2}{a_1\lambda\mu_{11}\mu 21}\eta_2>\frac{a_2\lambda\mu_{11}\mu_{21}}{a_1b_2^2}\eta_2>\eta_1$ 时，即$(m_1<0,m_2>0)$

$$(q_1^*,q_2^*)=\begin{cases}(0,0),t\leq\widehat{t}_{20}\\(0,\widehat{q}_2),\widehat{t}_{20}<t<\widehat{t}_{21}\\(0,1),t\geq\widehat{t}_{21}\end{cases}\tag{4.23}$$

定理 4.2 若$0 < \alpha < \frac{1}{2}$，则最优再保险策略(q_1^*, q_2^*)为

（1）当$\eta_1 > \frac{a_2 b_1^2}{a_1 \lambda \mu_{11} \mu 21}\eta_2 > \frac{a_2 \lambda \mu_{11}\mu_{21}}{a_1 b_2^2}\eta_2$ 时，即（$m_1 > 0, m_2 < 0$）

$$(q_1^*, q_2^*) = \begin{cases} (1,1), t \leq \widetilde{t}_{11} \\ (\widetilde{q}_1, 1), \widetilde{t}_{11} < t < \widetilde{t}_{10} \\ (0,1), t \geq \widetilde{t}_{10} \end{cases} \tag{4.24}$$

其中$\widetilde{q}_1 = \frac{1}{2\alpha-1}[\frac{-\gamma e^{r(T-t)}\alpha\lambda\mu_{11}\mu_{21}+\eta_1 a_1}{\gamma e^{r(T-t)} b_1^2} - (1-\alpha)]$

（ii）当$\frac{a_2 b_1^2}{a_1 \lambda \mu_{11} \mu 21}\eta_2 > \eta_1 > \frac{a_2 \lambda \mu_{11}\mu_{21}}{a_1 b_2^2}\eta_2$ 时，即（$m_1 > 0, m_2 > 0$）

若$m_1 \geq m_2$，有

$$(q_1^*, q_2^*) = \begin{cases} (1,1), t \leq \widetilde{t}_{11} \\ (1, \widetilde{q}_2), \widetilde{t}_{11} < t \leq t_{21} \\ (\overline{q}_1, \overline{q}_2), t_{21} < t < t_{10} \\ (0, \widehat{q}_2), t_{10} \leq t < \widehat{t}_{20} \\ (0,0), t \geq \widehat{t}_{20} \end{cases} \tag{4.25}$$

其中$\widetilde{q}_2 = \frac{1}{2\alpha-1}[\frac{-\gamma e^{r(T-t)}\alpha\lambda\mu_{11}\mu_{21}+\eta_1 a_1}{\gamma e^{r(T-t)} b_2^2} - (1-\alpha)]$，$\widehat{q}_2 = \frac{1}{2\alpha-1}[\frac{-\gamma e^{r(T-t)}(1-\alpha)\lambda\mu_{11}\mu_{21}+\eta_2 a_2}{\gamma e^{r(T-t)} b_2^2} - (1-\alpha)]$

若$m_1 < m_2$，有

$$(q_1^*, q_2^*) = \begin{cases} (1,1), t \leq \widetilde{t}_{21} \\ (1, \widetilde{q}_2), \widetilde{t}_{21} < t \leq t_{11} \\ (\overline{q}_1, \overline{q}_2), t_{11} < t < t_{20} \\ (\widehat{q}_1, 0), t_{20} \leq t < \widehat{t}_{10} \\ (0,0), t \geq \widehat{t}_{10} \end{cases} \tag{4.26}$$

其中$\widetilde{q}_2 = \frac{1}{2\alpha-1}[\frac{-\gamma e^{r(T-t)}\alpha\lambda\mu_{11}\mu_{21}+\eta_2 a_2}{\gamma e^{r(T-t)} b_2^2} - (1-\alpha)]$，$\widehat{q}_1 = \frac{1}{2\alpha-1}[\frac{-\gamma e^{r(T-t)}(1-\alpha)\lambda\mu_{11}\mu_{21}+\eta_1 a_1}{\gamma a(t)^2 b_1^2} - (1-\alpha)]$

（iii）当$\frac{a_2 b_1^2}{a_1 \lambda \mu_{11} \mu 21}\eta_2 > \frac{a_2 \lambda \mu_{11}\mu_{21}}{a_1 b_2^2}\eta_2 > \eta_1$ 时，即（$m_1 < 0, m_2 > 0$）

$$(q_1^*, q_2^*) = \begin{cases} (1,1), t \leq \widetilde{t}_{21} \\ (1, \widetilde{q}_2), \widetilde{t}_{21} < t < \widetilde{t}_{20} \\ (1,1), t \geq \widetilde{t}_{20} \end{cases} \tag{4.27}$$

注 4.1 如果复合$Possion$过程$L_1(t)$和$L_2(t)$分布相同，即$\lambda_1=\lambda_2$，$\mu_{11}=\mu_{21}$，$\mu_{12}=\mu_{22}$，于是$a_1=a_2$，$b_1^2=b_2^2$，则*(4.9)*中当再保险公司对于第1类险种的安全负荷η_1和对于第2类险种的安全负荷η_2相等时，有$\overline{\theta}_1$=$\overline{\theta}_2$，进一步$q_1^*=q_2^*$；当再保险公司对于第1类险种的安全负荷η_1和对于第2 类险种的安全负荷η_2不相等时，有$\eta_1\overline{\theta}_1=\eta_2\overline{\theta}_2$

定理 4.3 针对问题*(3.1)*，我们由*(4.10)*式可知最优投资策略为$\pi^*=\frac{(\beta-r)}{\alpha\gamma e^{r(T-t)}\sigma^2}$，最优再保险策略由定理*4.1*和定理*4.2*给出，且最优值函数$V(t,x)$ 为

（*1*）当最优再保险策略$(q_1^*,q_2^*)=(\overline{q}_1,\overline{q}_2)$时，

$$V(t,x)=e^{r(T-t)}x+(\alpha c-(1+\eta_1)\alpha a_1-(1+\eta_2)\alpha a_2)\frac{1}{r}(e^{r(T-t)}-1)+\frac{1}{2\gamma}\xi(T-t) \tag{4.28}$$

其中$\xi=a_1^2\eta_1^2b_2^2+a_2^2\eta_2^2b_2^1-2a_1a_2\eta_1\eta_2\lambda\mu_{11}\mu_{21}+\frac{(\beta-r)^2}{\sigma^2}$

（*2*）当最优再保险策略$(q_1^*,q_2^*)=(0,0)$时，

$$\begin{aligned}V(t,x)=&((1-2\alpha)(1+\eta_1)a_1+(1-2\alpha)(1+\eta_2)a_2-(1-\alpha)a_1-(1-\alpha)a_2)\frac{1}{r}(e^{r(T-t)}-1)\\&\alpha c\frac{1}{r}(e^{r(T-t)}-1)+e^{r(T-t)}x-\frac{\gamma}{4r}(e^{2r(T-t)}-1)(1-\alpha)^2(b_1^2+b_2^2+2\lambda\mu_{11}\mu_{21})\\&+\frac{1}{2\gamma}\frac{(\beta-r)^2}{\sigma^2}(T-t)\end{aligned} \tag{4.29}$$

（*3*）当最优再保险策略$(q_1^*,q_2^*)=(0,\widehat{q}_2)$时，

$$\begin{aligned}V(t,x)=&(\alpha c-(1+\eta_2)\alpha a_2+(1-2\alpha)(1+\eta_1)a_1-\frac{(1-\alpha)\lambda\mu_{11}\mu_{21}\eta_2a_2}{b_2^2})\frac{1}{r}(e^{r(T-t)}-1)\\&-(1-\alpha)a_1\frac{1}{r}(e^{r(T-t)}-1)+e^{r(T-t)}x-\frac{\gamma}{4r}(e^{2r(T-t)}-1))(1-\alpha)^2[b_1^2-\frac{\lambda^2\mu_{11}^2\mu_{21}^2}{b_2^2}]\\&+\frac{1}{2\gamma}[\frac{(\beta-r)^2}{\sigma^2}+\frac{\eta_2^2a_2^2}{b_2^2}](T-t)\end{aligned} \tag{4.30}$$

（*4*）当最优再保险策略$(q_1^*,q_2^*)=(\widehat{q}_1,0)$时，

$$\begin{aligned}V(t,x)=&(\alpha c-(1+\eta_1)\alpha a_1+(1-2\alpha)(1+\eta_2)a_2-\frac{(1-\alpha)\lambda\mu_{11}\mu_{21}\eta_1a_1}{b_1^2})\frac{1}{r}(e^{r(T-t)}-1)\\&-(1-\alpha)a_2\frac{1}{r}(e^{r(T-t)}-1)+e^{r(T-t)}x-\frac{\gamma}{4r}(e^{2r(T-t)}-1))(1-\alpha)^2[b_2^2-\frac{\lambda^2\mu_{11}^2\mu_{21}^2}{b_1^2}]\\&+\frac{1}{2\gamma}[\frac{(\beta-r)^2}{\sigma^2}+\frac{\eta_1^2a_1^2}{b_1^2}](T-t)\end{aligned} \tag{4.31}$$

（5）当最优再保险策略$(q_1^*, q_2^*) = (\widetilde{q}_1, 1)$时，

$$V(t,x) = e^{r(T-t)}x + (\alpha c - (1+\eta_1)\alpha a_1 - \alpha a_2 - \frac{\alpha\lambda\mu_{11}\mu_{21}\eta_1 a_1}{b_1^2})\frac{1}{r}(e^{r(T-t)} - 1) - \frac{\gamma}{4r}(e^{2r(T-t)} - 1))\alpha^2[b_2^2 - \frac{\lambda^2\mu_{11}^2\mu_{21}^2}{b_1^2}] + \frac{1}{2\gamma}[\frac{(\beta-r)^2}{\sigma^2} + \frac{\eta_1^2 a_1^2}{b_1^2}](T-t) \tag{4.32}$$

（6）当最优再保险策略$(q_1^*, q_2^*) = (1, \widetilde{q}_2)$时，

$$V(t,x) = e^{r(T-t)}x + (\alpha c - (1+\eta_2)\alpha a_2 - \alpha a_1 - \frac{\alpha\lambda\mu_{11}\mu_{21}\eta_2 a_2}{b_2^2})\frac{1}{r}(e^{r(T-t)} - 1) - \frac{\gamma}{4r}(e^{2r(T-t)} - 1))\alpha^2[b_1^2 - \frac{\lambda^2\mu_{11}^2\mu_{21}^2}{b_2^2}] + \frac{1}{2\gamma}[\frac{(\beta-r)^2}{\sigma^2} + \frac{\eta_2^2 a_2^2}{b_2^2}](T-t) \tag{4.33}$$

（7）当最优再保险策略$(q_1^*, q_2^*) = (1, 1)$时，

$$V(t,x) = e^{r(T-t)}x + (\alpha c - \alpha a_1 - \alpha a_2)\frac{1}{r}(e^{r(T-t)} - 1) + \frac{1}{2\gamma}\frac{(\beta-r)^2}{\sigma^2}(T-t) - \frac{\gamma}{4r}(e^{2r(T-t)} - 1)\alpha^2(b_1^2 + b_2^2 + 2\lambda\mu_{11}\mu_{21}) \tag{4.34}$$

5. 数值分析

这一部分中，我们将通过数值例子去说明参数对最优策略的影响。假设$\lambda = 1, \lambda_1 = 2, \lambda_2 = 3, T = 10, r = 0.03, \alpha = 0.6, \mu_{11} = 0.045, \mu_{11} = 0.053, \mu_{21} = 0.045, \mu_{22} = 0.055, \eta_1 = 0.35, \eta_2 = 0.35, \gamma = 0.5, \sigma = 0.25$。参数如有变化，另作说明。

图 1说明最优再保险策略(q_1^*, q_2^*)随着时间t的增大而增大，在没有顾虑的情况下，保险公司会对两类险种的再保险投入增加。图 2中三条曲线分别为$\sigma = 0.25$，0.35，0.45时最优投资策略曲线，最优投资策略π^*随着波动率σ的增大而减小，σ越大时，风险资产的收益波动就会越大，所以，当风险厌恶相同时，保险公司在风险资产上的投资金额会减少，且最优再保险策略与波动率σ无关，最优再保险策略与风险资产参数无关。

下面我们只分析参数对最优再保险策略q_1^*的影响，对q_2^*的分析可以类似得出。图 3的三条曲线表示风险厌恶系数$\gamma = 0.5$，0.51，0.52时的最优再保险策略q_1^*的曲线，当γ越大时，最优再保险策略越小，自留比例越小，即保险公司越厌恶风险时，更想买再保险，让再保险公司承担更多的风险。图 4的三条曲线表示风险厌恶系数$\gamma = 0.5$，0.51，0.52时的最优投资策略曲线，当γ越大，即风险厌恶程度更高时，最优投资策略π^*越小，保险公司会减少在风险资产上的投资。

图 5表示最优再保险策略q_1^*与参数λ的关系，三条曲线分别为$\lambda = 1$，1.5，2，当λ越大时，最优再保险策略q_1^*越小。由于本文考虑到保险公司和再保险公司的联合收益，故当保险公司索赔次数增多时，再保险公司就会拒绝接受保险公司购买更多的再保险，且最优投资策略与参数λ 无关最优投资策略与保险业务参数无关。图 6中的三条曲线表示参数$\alpha = 0.6$，0.7，0.8时最优再保险策略q_1^*的曲线，当α越大时，保险公司所占的财富比重越高，对再保险公司的投入比重减少。

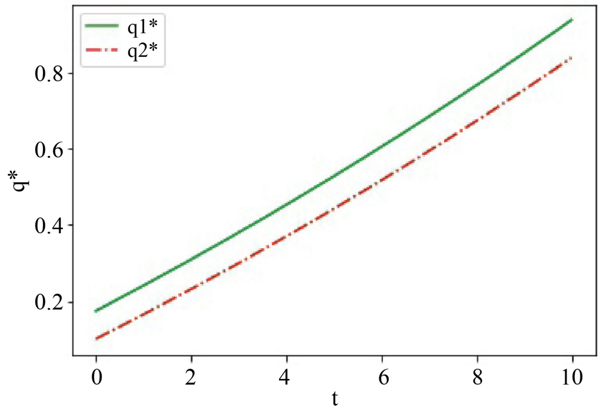

Figure 1. The effect of t on optimal reinsurance strategy
图 1. t对最优再保险策略的影响

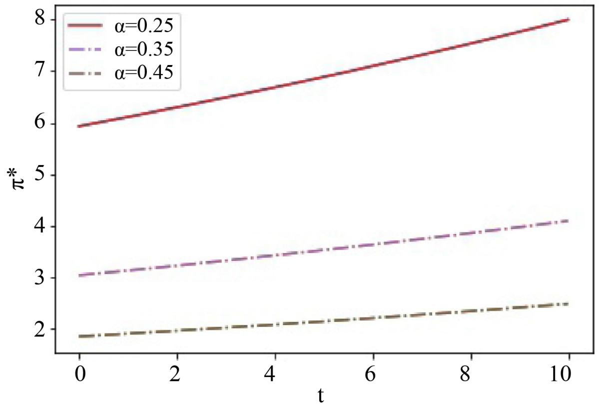

Figure 2. The effect of γ on optimal reinsurance strategy
图 2. γ对最优再保险策略的影响

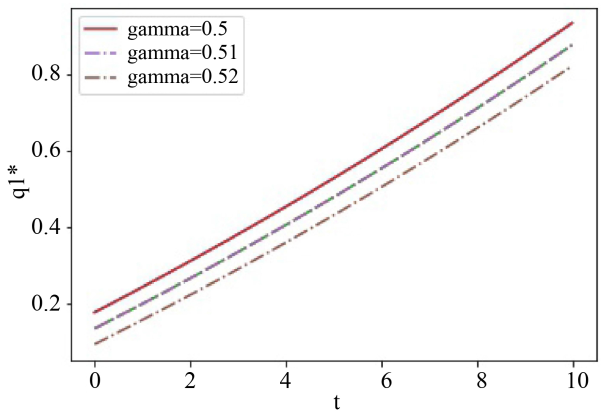

Figure 3. The effect of γ on optimal investment strategy
图 3. γ对最优投资策略的影响.

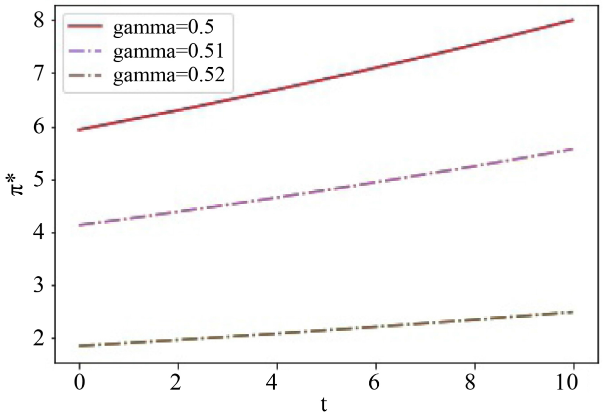

Figure 4. The effect of σ on optimal investment strategy
图 4. σ对最优投资策略的影响

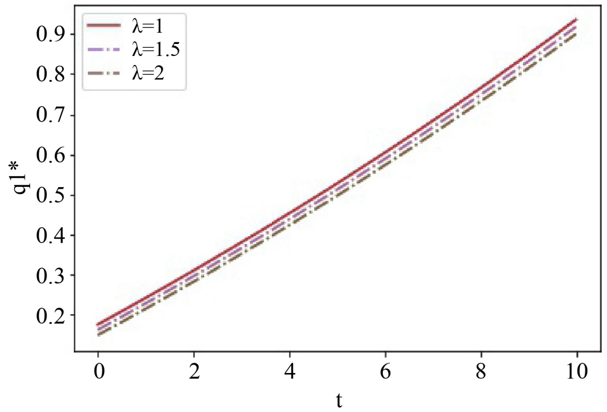

Figure 5. The effect of λ on optimal reinsurance strategy
图 5. λ对最优再保险策略的影响

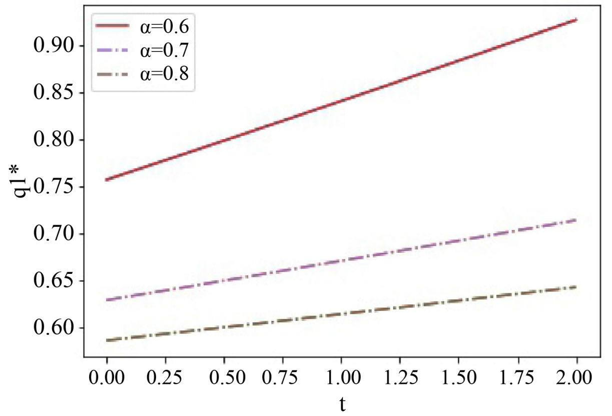

Figure 6. The effect of α on optimal investment strategy
图 6. α对最优投资策略的影响

6. 总结

我们在均值-方差准则下研究了保险公司和再保险公司联合收益的最优投资再保险问题，其中两类风险业务之间具有相依性，再保险公司采用期望保费原理收取保费，保险公司和再保险公司都可以投资无风险资产，同时保险公司还可以投资风险资产，通过求解扩展的HJB方程组，得到了最优策略的表达式以及最优值函数的表达式，并给出了数值例子分析参数对最优策略的影响。

基金项目

国家自然科学基金(62073071)。

参考文献

[1] Hipp, C. and Plum, M. (2000) Optimal Investment for Insurers. *Insurance Mathematics and Economics*, **27**, 215-228. https://doi.org/10.1016/S0167-6687(00)00049-4

[2] Schmidli, H. (2002) On Minimizing the Ruin Probability by Investment and Reinsurance. *Annals of Applied Probability*, **12**, 890-907. https://doi.org/10.1214/aoap/1031863173

[3] Promislow, S.D. and Young, V.R. (2005) Minimizing the Probability of Ruin When Claims Follow Brownian Motion with Drift. *North American Actuarial Journal*, **9**, 110-128. https://doi.org/10.1080/10920277.2005.10596214

[4] Chen, S., Li, Z. and Li, K. (2010) Optimal Investment-Reinsurance Policy for an Insurance Company with VaR Constraint. *Insurance Mathematics and Economics*, **47**, 144-153. https://doi.org/10.1016/j.insmatheco.2010.06.002

[5] Browne, S. (1995) Optimal Investment Policies for a Firm with a Random Risk Process: Exponential Utility and Minimizing the Probability of Ruin. *Mathematics of Operations Research*, **20**, 937-958. https://doi.org/10.1287/moor.20.4.937

[6] Bai, L. and Guo, J. (2008) Optimal Proportional Reinsurance and Investment with Multiple Risky Assets and No-Shorting Constraint. *Insurance Mathematics Economics*, **42**, 968-975. https://doi.org/10.1016/j.insmatheco.2007.11.002

[7] Gu, A., Guo, X., Li, Z., *et al.* (2012) Optimal Control of Excess-of-Loss Reinsurance and Investment for Insurers under a CEV Model. *Insurance: Mathematics and Economics*, **51**, 674-684. https://doi.org/10.1016/j.insmatheco.2012.09.003

[8] Markowitz, H.M. (1952) Portfolio Selection. *The Journal of Finance*, **7**, 77-91. https://doi.org/10.2307/2975974

[9] Zeng, Y. and Li, Z. (2011) Optimal Time-Consistent Investment and Reinsurance Policies for Mean-Variance Insurers. *Insurance Mathematics Economics*, **49**, 145-154. https://doi.org/10.1016/j.insmatheco.2011.01.001

[10] Bi, J. and Guo, J. (2013) Optimal Mean-Variance Problem with Constrained Controls in a Jump-Diffusion Financial Market for an Insurer. *Journal of Optimization Theory Applications*, **157**, 252-275. https://doi.org/10.1007/s10957-012-0138-y

[11] Borch, K. (1969) The Optimal Reinsurance Treaty. *ASTIN Bulletin*, **5**, 293-297. https://doi.org/10.1017/S051503610000814X

[12] Cai, J., Fang, Y., Li, Z., *et al.* (2013) Optimal Reciprocal Reinsurance Treaties under the Joint Survival Probability and the Joint Profitable Probability. *Journal of Risk and Insurance*, **80**, 145-168. https://doi.org/10.1111/j.1539-6975.2012.01462.x

[13] Fang, Y. and Qu, Z. (2014) Optimal Combination of Quota-Share and Stop-Loss Reinsurance Treaties under the Joint Survival Probability. *IMA Journal of Management Mathematics*, **25**, 89-103. https://doi.org/10.1093/imaman/dps029

[14] Li, D., Rong, X. and Zhao, H. (2014) Optimal Reinsurance-Investment Problem for Maximizing the Product of the Insurer's and the Reinsurer's Utilities under a CEV Model. *Journal of Computational and Applied Mathematics*, **255**, 671-683. https://doi.org/10.1016/j.cam.2013.06.033

[15] Li, D., Rong, X. and Zhao, H. (2015) Time-Consistent Reinsurance-Investment Strategy for an Insurer and a Reinsurer with Mean-Variance Criterion under the CEV Model. *Journal of Computational Applied Mathematics*, **283**, 142-162. https://doi.org/10.1016/j.cam.2015.01.038

[16] Li, D., Rong, X. and Zhao, H. (2016) Optimal Reinsurance and Investment Problem for an Insurer and a Reinsurer with Jump-Diffusion Risk Process under the Heston Model. *Computational and Applied Mathematics*, **35**, 533-557. https://doi.org/10.1007/s40314-014-0204-1

[17] Hu, H., Yin, Z. and Gao, X. (2018) Optimal Reinsurance-Investment Problem for an Insurer and a Reinsurer with Jump-Diffusion Process. *Discrete Dynamics in Nature and Society*, **2018**, Article ID: 9424908. https://doi.org/10.1155/2018/9424908

[18] Chen, L. and Shen, Y. (2019) Stochastic Stackelberg Differential Reinsurance Games under Time-Inconsistent Mean-Variance Framework. *Insurance: Mathematics and Economics*, **88**, 120-137. https://doi.org/10.1016/j.insmatheco.2019.06.006

[19] Bai, Y., Zhou, Z., Gao, R., *et al.* (2020) Nash Equilibrium Investment-Reinsurance Strategies for an Insurer and a Reinsurer with Intertemporal Restrictions and Common Interests. *Mathematics*, **8**, 139. https://doi.org/10.3390/math8010139

[20] Liang, Z. and Yuen, K.C. (2016) Optimal Dynamic Reinsurance with Dependent Risks: Variance Premium Principle. *Scandinavian Actuarial Journal*, **2016**, 18-36. https://doi.org/10.1080/03461238.2014.892899

[21] Liang, Z., Bi, J. and Yuen, K.C., *et al.* (2016) Optimal Mean-Variance Reinsurance and Investment in a Jump-Diffusion Financial Market with Common Shock Dependence. *Mathematical Methods of Operations Research*, **84**, 155-181. https://doi.org/10.1007/s00186-016-0538-0

[22] Bjork, T. and Murgoci, A. (2010) A General Theory of Markovian Time Inconsistent Stochastic Control Problems. *SSRN Electronic Journal*.

附录

证明定理（4.3）：

（1）将$q_1^* = \overline{q}_1, q_2^* = \overline{q}_2, \pi^*$代入(4.5)和(4.2)，有

$$(\dot{A}(t) + rA(t))x + \dot{B}(t) + A(t)(\alpha c - (1+\eta_1)\alpha a_1 - (1+\eta_2)\alpha a_2) + \frac{A(t)^2}{2\gamma a(t)^2}\xi = 0, \tag{6.1}$$

$$(\dot{a}(t) + ra(t))x + \dot{b}(t) + a(t)(\alpha c - (1+\eta_1)\alpha a_1 - (1+\eta_2)\alpha a_2) + \frac{A(t)^2}{\gamma a(t)^2}\xi = 0, \tag{6.2}$$

其中

$$\xi = a_1^2\eta_1^2 b_2^2 + a_2^2\eta_2^2 b_2^1 - 2a_1 a_2 \eta_1 \eta_2 \lambda \mu_{11}\mu_{21} + \frac{(\beta - r)^2}{\sigma^2}, \tag{6.3}$$

要想(4.11)和(4.12)成立，则

$$\dot{A}(t) + rA(t) = 0, A(T) = 1,$$

$$\dot{B}(t) + A(t)(\alpha c - (1+\eta_1)\alpha a_1 - (1+\eta_2)\alpha a_2) + \frac{A(t)^2}{2\gamma a(t)^2}\xi(t) = 0,$$

$$\dot{a}(t) + ra(t) = 0, a(T) = 1,$$

$$\dot{b}(t) + a(t)(\alpha c - (1+\eta_1)\alpha a_1 - (1+\eta_2)\alpha a_2) + \frac{A(t)^2}{\gamma a(t)^2}\xi(t) = 0.$$

解上述微分方程，有

$$A(t) = e^{r(T-t)} \tag{6.4}$$

$$B(t) = (\alpha c - (1+\eta_1)\alpha a_1 - (1+\eta_2)\alpha a_2)\frac{1}{r}(e^{r(T-t)} - 1) + \frac{1}{2\gamma}\xi(T-t) \tag{6.5}$$

$$a(t) = e^{r(T-t)} \tag{6.6}$$

$$b(t) = (\alpha c - (1+\eta_1)\alpha a_1 - (1+\eta_2)\alpha a_2)\frac{1}{r}(e^{r(T-t)} - 1) + \frac{1}{\gamma}\xi(T-t) \tag{6.7}$$

于是，最优值函数为

$$V(t,x) = U(t,x) = e^{r(T-t)}x + (\alpha c - (1+\eta_1)\alpha a_1 - (1+\eta_2)\alpha a_2)\frac{1}{r}(e^{r(T-t)} - 1) + \frac{1}{2\gamma}\xi(T-t) \tag{6.8}$$

此外，

$$E_{t,x}[R_T^{Q^*}] = g(t,x) = e^{r(T-t)}x + (\alpha c - (1+\eta_1)\alpha a_1 - (1+\eta_2)\alpha a_2)\frac{1}{r}(e^{r(T-t)} - 1) + \frac{1}{\gamma}\xi(T-t) \tag{6.9}$$

$$Var_{t,x}[R_T^{Q^*}] = \frac{2}{\gamma}\{E_{t,x}[R_T^{Q^*}] - V(t,x)\} = \frac{1}{\gamma^2}\xi(T-t) \tag{6.10}$$

因此$\frac{1}{\gamma} = \sqrt{Var_{t,x}[R_T^{Q^*}]/\xi(T-t)}$，代入（6.9）得：

$$E_{t,x}[R_T^{Q^*}] = e^{r(T-t)}x + (\alpha c - (1+\eta_1)\alpha a_1 - (1+\eta_2)\alpha a_2)\frac{1}{r}(e^{r(T-t)} - 1) + \sqrt{Var_{t,x}[R_T^{Q^*}]\xi(T-t)} \tag{6.11}$$

上式中的关系也称为现代投资组合理论中问题（3.1）在初始状态(t,x) 的有效边界。

（2）将$q_1^* = 0, q_2^* = 0, \pi^*$代入(4.5)和(4.2)，有

$$\dot{B}(t) + A(t)(\alpha c + (1-2\alpha)(1+\eta_1)a_1 + (1-2\alpha)(1+\eta_2)a_2 - (1-\alpha)a_1 - (1-\alpha)a_2)$$
$$+ (\dot{A}(t) + rA(t))x - \frac{\gamma}{2}a(t)^2(1-\alpha)^2(b_1^2 + b_2^2 + 2\lambda\mu_{11}\mu_{21}) + \frac{A(t)^2}{2\gamma a(t)^2}\frac{(\beta-r)^2}{\sigma^2} = 0, \tag{6.12}$$

$$\dot{b}(t) + a(t)(\alpha c + (1-2\alpha)(1+\eta_1)a_1 + (1-2\alpha)(1+\eta_2)a_2 - (1-\alpha)a_1 - (1-\alpha)a_2)$$
$$+ (\dot{a}(t) + ra(t))x + \frac{A(t)}{\gamma a(t)}\frac{(\beta-r)^2}{\sigma^2} = 0, \tag{6.13}$$

类似（1）中解法，我们可以得到最优值函数

$$V(t,x) = (\alpha c + (1-2\alpha)(1+\eta_1)a_1 + (1-2\alpha)(1+\eta_2)a_2 - (1-\alpha)a_2)\frac{1}{r}(e^{r(T-t)} - 1)$$
$$- (1-\alpha)a_1\frac{1}{r}(e^{r(T-t)} - 1) + e^{r(T-t)}x - \frac{\gamma}{4r}(e^{2r(T-t)} - 1)(1-\alpha)^2(b_1^2 + b_2^2 + 2\lambda\mu_{11}\mu_{21})$$
$$+ \frac{1}{2\gamma}\frac{(\beta-r)^2}{\sigma^2}(T-t) \tag{6.14}$$

（3）将$q_1^* = 0, q_2^* = \widehat{q}_2, \pi^*$代入(4.5)和(4.2)，有

$$(\dot{A}(t) + rA(t))x + \dot{B}(t) + A(t)(\alpha c - (1+\eta_2)\alpha a_2 + (1-2\alpha)(1+\eta_1)a_1 - \frac{(1-\alpha)\lambda\mu_{11}\mu_{21}\eta_2 a_2}{b_2^2})$$
$$- A(t)(1-\alpha)a_1 - \frac{\gamma}{2}a(t)^2(1-\alpha)^2[b_1^2 - \frac{\lambda^2\mu_{11}^2\mu_{21}^2}{b_2^2}] + \frac{A(t)^2}{2\gamma a(t)^2}[\frac{(\beta-r)^2}{\sigma^2} + \frac{\eta_2^2 a_2^2}{b_2^2}] = 0,$$

$$(\dot{a}(t) + ra(t))x + \dot{b}(t) + a(t)(\alpha c - (1+\eta_2)\alpha a_2 + (1-2\alpha)(1+\eta_1)a_1 - \frac{(1-\alpha)\lambda\mu_{11}\mu_{21}\eta_2 a_2}{b_2^2})$$
$$- a(t)(1-\alpha)a_1 + \frac{A(t)}{\gamma a(t)}[\frac{(\beta-r)^2}{\sigma^2} + \frac{\eta_2^2 a_2^2}{b_2^2}] = 0, \tag{6.15}$$

类似（1）中解法，我们可以得到最优值函数

$$V(t,x)=(\alpha c-(1+\eta_2)\alpha a_2+(1-2\alpha)(1+\eta_1)a_1-\frac{(1-\alpha)\lambda\mu_{11}\mu_{21}\eta_2 a_2}{b_2^2})\frac{1}{r}(e^{r(T-t)}-1)$$
$$-(1-\alpha)a_1\frac{1}{r}(e^{r(T-t)}-1)+e^{r(T-t)}x-\frac{\gamma}{4r}(e^{2r(T-t)}-1))(1-\alpha)^2[b_1^2-\frac{\lambda^2\mu_{11}^2\mu_{21}^2}{b_2^2}]$$
$$+\frac{1}{2\gamma}[\frac{(\beta-r)^2}{\sigma^2}+\frac{\eta_2^2 a_2^2}{b_2^2}](T-t) \tag{6.16}$$

（4）将$q_1^*=\widehat{q}_1,q_2^*=0,\pi^*$代入(4.5)和(4.2)，有

$$(\dot{B}(t)+A(t)(\alpha c-(1+\eta_1)\alpha a_1+(1-2\alpha)(1+\eta_2)a_2-(1-\alpha)a_2-\frac{(1-\alpha)\lambda\mu_{11}\mu_{21}\eta_1 a_1}{b_1^2})$$
$$+\dot{A}(t)+rA(t))x-\frac{\gamma}{2}a(t)^2(1-\alpha)^2[b_2^2-\frac{\lambda^2\mu_{11}^2\mu_{21}^2}{b_1^2}]+\frac{A(t)^2}{2\gamma a(t)^2}[\frac{(\beta-r)^2}{\sigma^2}+\frac{\eta_1^2 a_1^2}{b_1^2}]=0, \tag{6.17}$$

$$\dot{b}(t)+a(t)(\alpha c-(1+\eta_1)\alpha a_1+(1-2\alpha)(1+\eta_2)a_2-(1-\alpha)a_2-\frac{(1-\alpha)\lambda\mu_{11}\mu_{21}\eta_2 a_2}{b_1^2})$$
$$+(\dot{a}(t)+ra(t))x+\frac{A(t)}{\gamma a(t)}[\frac{(\beta-r)^2}{\sigma^2}+\frac{\eta_1^2 a_1^2}{b_1^2}]=0, \tag{6.18}$$

类似（1）中解法，我们可以得到最优值函数

$$V(t,x)=(\alpha c-(1+\eta_1)\alpha a_1+(1-2\alpha)(1+\eta_2)a_2-\frac{(1-\alpha)\lambda\mu_{11}\mu_{21}\eta_1 a_1}{b_1^2})\frac{1}{r}(e^{r(T-t)}-1)$$
$$-(1-\alpha)a_2\frac{1}{r}(e^{r(T-t)}-1)+e^{r(T-t)}x-\frac{\gamma}{4r}(e^{2r(T-t)}-1))(1-\alpha)^2[b_2^2-\frac{\lambda^2\mu_{11}^2\mu_{21}^2}{b_1^2}]$$
$$+\frac{1}{2\gamma}[\frac{(\beta-r)^2}{\sigma^2}+\frac{\eta_1^2 a_1^2}{b_1^2}](T-t) \tag{6.19}$$

（5）将$q_1^*=\widetilde{q}_1,q_2^*=1,\pi^*$代入(4.5)和(4.2)，有

$$(\dot{A}(t)+rA(t))x+\dot{B}(t)+A(t)(\alpha c-(1+\eta_1)\alpha a_1-\alpha a_2-\frac{\alpha\lambda\mu_{11}\mu_{21}\eta_1 a_1}{b_1^2})$$
$$-\frac{\gamma}{2}a(t)^2\alpha^2[b_2^2-\frac{\lambda^2\mu_{11}^2\mu_{21}^2}{b_1^2}]+\frac{A(t)^2}{2\gamma a(t)^2}[\frac{(\beta-r)^2}{\sigma^2}+\frac{\eta_1^2 a_1^2}{b_1^2}]=0, \tag{6.20}$$

$$(\dot{a}(t)+ra(t))x+\dot{b}(t)+a(t)(\alpha c-(1+\eta_1)\alpha a_1-\alpha a_2-\frac{\alpha\lambda\mu_{11}\mu_{21}\eta_1 a_1}{b_1^2})$$
$$+\frac{A(t)}{\gamma a(t)}[\frac{(\beta-r)^2}{\sigma^2}+\frac{\eta_1^2 a_1^2}{b_1^2}]=0, \tag{6.21}$$

类似（1）中解法，我们可以得到最优值函数

$$V(t,x)=e^{r(T-t)}x+(\alpha c-(1+\eta_1)\alpha a_1-\alpha a_2-\frac{\alpha\lambda\mu_{11}\mu_{21}\eta_1 a_1}{b_1^2})\frac{1}{r}(e^{r(T-t)}-1)$$
$$-\frac{\gamma}{4r}(e^{2r(T-t)}-1))\alpha^2[b_2^2-\frac{\lambda^2\mu_{11}^2\mu_{21}^2}{b_1^2}]+\frac{1}{2\gamma}[\frac{(\beta-r)^2}{\sigma^2}+\frac{\eta_1^2 a_1^2}{b_1^2}](T-t) \tag{6.22}$$

（6）将$q_1^*=1,q_2^*=\widetilde{q}_2,\pi^*$代入(4.5)和(4.2)，有

$$(\dot{A}(t)+rA(t))x+\dot{B}(t)+A(t)(\alpha c-(1+\eta_2)\alpha a_2-\alpha a_1-\frac{\alpha\lambda\mu_{11}\mu_{21}\eta_2 a_2}{b_2^2})$$
$$-\frac{\gamma}{2}a(t)^2\alpha^2[b_1^2-\frac{\lambda^2\mu_{11}^2\mu_{21}^2}{b_2^2}]+\frac{A(t)^2}{2\gamma a(t)^2}[\frac{(\beta-r)^2}{\sigma^2}+\frac{\eta_2^2 a_2^2}{b_2^2}]=0, \tag{6.23}$$

$$(\dot{a}(t)+ra(t))x+\dot{b}(t)+a(t)(\alpha c-(1+\eta_2)\alpha a_2-\alpha a_1-\frac{\alpha\lambda\mu_{11}\mu_{21}\eta_2 a_2}{b_2^2})$$
$$+\frac{A(t)}{\gamma a(t)}[\frac{(\beta-r)^2}{\sigma^2}+\frac{\eta_2^2 a_2^2}{b_2^2}]=0, \tag{6.24}$$

类似（1）中解法，我们可以得到最优值函数

$$V(t,x)=e^{r(T-t)}x+(\alpha c-(1+\eta_2)\alpha a_2-\alpha a_1-\frac{\alpha\lambda\mu_{11}\mu_{21}\eta_2 a_2}{b_2^2})\frac{1}{r}(e^{r(T-t)}-1)$$
$$-\frac{\gamma}{4r}(e^{2r(T-t)}-1))\alpha^2[b_1^2-\frac{\lambda^2\mu_{11}^2\mu_{21}^2}{b_2^2}]+\frac{1}{2\gamma}[\frac{(\beta-r)^2}{\sigma^2}+\frac{\eta_2^2 a_2^2}{b_2^2}](T-t) \tag{6.25}$$

（7）将$q_1^*=1,q_2^*=1,\pi^*$代入(4.5)和(4.2)，有

$$(\dot{A}(t)+rA(t))x+\dot{B}(t)+A(t)(\alpha c-(1+\eta_2)\alpha a_2-\alpha a_1-\frac{\alpha\lambda\mu_{11}\mu_{21}\eta_2 a_2}{b_2^2})$$
$$-\frac{\gamma}{2}a(t)^2\alpha^2(b_1^2+b_2^2+2\lambda\mu_{11}\mu_{21})+\frac{A(t)^2}{2\gamma a(t)^2}\frac{(\beta-r)^2}{\sigma^2}=0, \tag{6.26}$$

$$(\dot{a}(t)+ra(t))x+\dot{b}(t)+a(t)(\alpha c-(1+\eta_2)\alpha a_2-\alpha a_1-\frac{\alpha\lambda\mu_{11}\mu_{21}\eta_2 a_2}{b_2^2})$$
$$+\frac{A(t)}{\gamma a(t)}\frac{(\beta-r)^2}{\sigma^2}=0, \tag{6.27}$$

类似（1）中解法，我们可以得到最优值函数

$$V(t,x)=e^{r(T-t)}x+(\alpha c-\alpha a_1-\alpha a_2)\frac{1}{r}(e^{r(T-t)}-1)+\frac{1}{2\gamma}\frac{(\beta-r)^2}{\sigma^2}(T-t)$$
$$-\frac{\gamma}{4r}(e^{2r(T-t)}-1)\alpha^2(b_1^2+b_2^2+2\lambda\mu_{11}\mu_{21}) \tag{6.28}$$

该定理证明完毕。