Pure Mathematics 理论数学, 2021, 11(12), 1974-1986
Published Online December 2021 in Hans. http://www.hanspub.org/journal/pm
https://doi.org/10.12677/pm.2021.1112220

二维时间反向热传导问题的两种正则化方法及后验误差估计

侯佳琪

西北师范大学数学与统计学院，甘肃 兰州

收稿日期：2021年11月4日；录用日期：2021年12月6日；发布日期：2021年12月13日

摘　要

本文讨论了一类二维时间反向热传导问题，它从终值时刻$t = T\,(T > 0)$的温度分布来反演初始时刻的温度分布。该问题在图像处理方面有重要应用。这是一个严重不适定问题，它的解在一定条件下不连续依赖于数据。针对传统正则化方法的缺陷，本文采用拟逆正则化方法和分数次*Tikhonov*正则化方法，来恢复解对数据的依赖性。同时我们还给出了两种方法相应的后验参数选取规则及其正则解与精确解的误差估计。

关键词

不适定问题，时间反向热传导问题，分数次*Tikhonov*正则化方法，拟逆正则化方法，正则化后验参数选取，误差估计

Two Regularization Methods for the Two-Dimensional Time-Inverse Heat Conduction Problem and Its Posterior Error Estimation

Jiaqi Hou

College of Mathematics and Statistics, Northwest Normal University, Lanzhou Gansu

Received: Nov. 4th, 2021; accepted: Dec. 6th, 2021; published: Dec. 13th, 2021

Abstract

The time-inverse heat conduction problem was concerned in two-dimensional space, which re-

文章引用：侯佳琪. 二维时间反向热传导问题的两种正则化方法及后验误差估计[J]. 理论数学，2021，11(12): 1974-1986. DOI: 10.12677/pm.2021.1112220

trieves the temperature distribution from the temperature distribution at the final moment. This problem has important application in image processing. This is a serious ill-posed problem, *i.e.* its solution is not continuously dependent on the data under certain conditions. For the defects of traditional regularization methods, the quasi-reversibility regularization method and the fractional *Tikhonov* regularization method are proposed to restore the dependence of the solution on the data. Meanwhile, the errors between the approximate solutions and the exact solution for the ill-posed problem are estimated, and the posteriori regularization parameter selection rules are given.

Keywords

Ill-Posed Problem, Time-Inverse Heat Conduction Problem, Fractional *Tikhonov* Regularization Method, Quasi-Reversibility Regularization Method, Selection of Regularized Posterior Parameter, Error Estimation

1. 引言

随着科学技术的发展，人们发现实际问题的建模中，不适定问题占有重要地位，于是不适定问题的研究得以发展。对于热传导问题各类典型的不适定问题，学者们对于该问题提出了一些正则化方法，如 *Tikhonov* 正则化方法[1] [2]、拟逆正则化方法[3] [4]、数值微分正则化方法[5]、边界元方法[6] [7]等。同时也给出了相应近似解与精确解的误差估计。通过文献检索，我们发现对于一维空间中的时间反向热传导问题的研究相对较多，同时大多给出的误差估计为先验误差估计，而对二维空间的此类问题[8] [9] [10]研究相对较少。本文将对一类二维时间反向热传导问题进行讨论，并给出其后验误差估计。

本文考虑二维时间反向热传导问题[11]

$$\begin{cases} u_t(x,y,t) = \Delta u(x,y,t),(x,y)\in \mathbf{R}^2, 0\le t<T, \\ u(x,y,T) = g(x,y),(x,y)\in \mathbf{R}^2. \end{cases} \tag{1}$$

其中 $\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$， $u(x,y,\cdot)$ 和 $g(x,y)$ 是定义在 $L^2(\mathbf{R}^2)$ 上的函数。我们想由数据 $g(x,y)$ 来确定 $u(x,y,t)$ 在 $0\le t<T$ 上的温度分布。而这里的输入数据 $g(x,y)$ 往往是被测量出来的，记 $g^\delta(x,y)\in L^2(\mathbf{R}^2)$ 为带噪音的测量数据，且满足

$$\left\|g-g^\delta\right\|\le \delta, \tag{2}$$

δ 表示输入数据的噪音水平且 $\delta>0$。$\|\cdot\|$ 是 $L^2(\mathbf{R}^2)$ 空间的范数。进一步，给出如下先验假设

$$\left\|u(\cdot,\cdot,0)\right\|\le E, \tag{3}$$

其中 E 为大于 0 的有界常数。

2. 问题的解与不适定性分析

下面分析问题(1)在 $L^2(\mathbf{R}^2)$ 空间中的不适定性。我们定义 $f(x,y)\in L^2(\mathbf{R}^2)$ 的 *Fourier* 变换 $\hat{f}(\xi,\eta)$ 如

下：

$$\hat{f}(\xi,\eta):=\frac{1}{2\pi}\int_{\boldsymbol{R}^2}\mathrm{e}^{-i(\xi x+\eta y)}f(x,y)\mathrm{d}x\mathrm{d}y,$$

对问题(1)中的精确解 $u(x,y,t)$ 进行 *Fourier* 变换得到

$$\hat{u}(\xi,\eta,t)=\mathrm{e}^{(\xi^2+\eta^2)(T-t)}\hat{g}(\xi,\eta), \tag{4}$$

因此，作 *Fourier* 逆变换有

$$u(x,y,t)=\frac{1}{2\pi}\int_{\boldsymbol{R}^2}\mathrm{e}^{i(\xi x+\eta y)}\mathrm{e}^{(\xi^2+\eta^2)(T-t)}\hat{g}(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta, \tag{5}$$

当 $t=0$ 时，

$$\hat{u}(\xi,\eta,0)=\mathrm{e}^{(\xi^2+\eta^2)T}\hat{g}(\xi,\eta), \tag{6}$$

注意到，$\mathrm{e}^{(\xi^2+\eta^2)(T-t)}$ 为放大因子，即对固定的 $t\in(0,T)$，它关于 ξ,η 是无界的。若假设精确解 $\hat{u}(\xi,\eta,t)=\mathrm{e}^{(\xi^2+\eta^2)(T-t)}\hat{g}(\xi,\eta)$ 在 $L^2(\boldsymbol{R}^2)$ 中，则当 $\xi,\eta\to\infty$ 时，$\hat{g}(\xi,\eta)$ 必须为急降函数。否则当输入数据 $g(\xi,\eta)$ 有微小扰动时，放大因子 $\mathrm{e}^{(\xi^2+\eta^2)(T-t)}$ 将解无限放大，导致解的爆破。我们只有带噪音的数据 $\hat{g}^{\delta}(\xi,\eta)\in L^2(\boldsymbol{R}^2)$，而 $\hat{g}^{\delta}(\xi,\eta)$ 一般不是急降函数，故问题严重不适定。

针对传统的正则化方法，如 *Fourier* 正则化方法和经典 *Tihonov* 正则化方法给出的近似解过度光滑，因此本文将用拟逆正则化方法和分数次 *Tihonov* 正则化方法来解决该不适定问题，并给出后验参数选取及相应的误差估计。

3. 预备定理

引理 1. [12]若常数 $\alpha>0$，$0<p<q$，且 $s\geq 0$，则有下述不等式成立

$$\sup_{s\geq 0}\frac{\mathrm{e}^{sp}}{1+\alpha\mathrm{e}^{sq}}\leq\alpha^{-\frac{p}{q}}.$$

引理 2. [13]对于 $r\geq 0$，有下述不等式成立

$$1-\mathrm{e}^{-r}\leq r.$$

4. 分数次 *Tikhonov* 正则化方法后验参数选取及误差估计

由前述内容可知问题(1)的精确解为

$$\hat{u}(\xi,\eta,t)=\mathrm{e}^{(\xi^2+\eta^2)(T-t)}\hat{g}(\xi,\eta),$$

我们选取分数次 *Tihonov* 正则化解

$$\hat{u}_{\alpha,\delta}(\xi,\eta,t)=\frac{\mathrm{e}^{(\xi^2+\eta^2)(T-t)}}{1+\alpha\mathrm{e}^{\gamma(\xi^2+\eta^2)T}}\hat{g}^{\delta}(\xi,\eta). \tag{7}$$

这里 $\alpha>0$ 为正则化参数，$1\leq\gamma\leq 2$ 为分数次参数。当 $\gamma=1$ 时为拟边界方法；当 $\gamma=2$ 时为经典 *Tihonov* 方法。记

$$\hat{u}_{\alpha}(\xi,\eta,t)=\frac{\mathrm{e}^{(\xi^2+\eta^2)(T-t)}}{1+\alpha\mathrm{e}^{\gamma(\xi^2+\eta^2)T}}\hat{g}(\xi,\eta). \tag{8}$$

下面我们考虑分数次 *Tihonov* 正则化方法的后验参数选取。当噪音水平 δ 已知时，一般采用 *Morozov's* 偏差原理进行后验正则化参数选取，找到一个 α 满足方程

$$\rho(\alpha)=\left\|\frac{1}{1+\alpha e^{(\xi^2+\eta^2)T\gamma}}\hat{g}^{\delta}(\xi,\eta)-\hat{g}^{\delta}(\xi,\eta)\right\|=\tau\delta, \tag{9}$$

其中 $\tau>1$ 是一个常数， $\alpha>0$ 是正则化参数。

引理 3. 设对固定的 $\delta>0$， $\rho(\alpha)$ 满足如下几条性质：

1) $\rho(\alpha)$ 是连续函数；
2) $\lim\limits_{\alpha\to 0}\rho(\alpha)=0$；
3) $\lim\limits_{\alpha\to\infty}\rho(\alpha)=\left\|g^{\delta}\right\|$；
4) $\rho(\alpha)$ 是严格单调增函数。

证明：由 $\rho(\alpha)$ 的表达式

$$\begin{aligned}\rho(\alpha)&=\left\|\frac{1}{1+\alpha e^{(\xi^2+\eta^2)T\gamma}}\hat{g}^{\delta}(\xi,\eta)-\hat{g}^{\delta}(\xi,\eta)\right\|\\&=\left\|\frac{\alpha e^{(\xi^2+\eta^2)T\gamma}}{1+\alpha e^{(\xi^2+\eta^2)T\gamma}}\hat{g}^{\delta}(\xi,\eta)\right\|\end{aligned}$$

容易验证上述结论成立。若选择常数 τ 满足 $0<\tau\delta<\left\|g^{\delta}\right\|$ 时，根据引理内容知(9)的解是存在且唯一的。

引理 4. 假设噪音水平(2)和先验条件(3)成立，如果 α 是(9)的解，可得如下不等式

$$\left\|\frac{1}{1+\alpha e^{(\xi^2+\eta^2)T\gamma}}\hat{g}^{\delta}(\xi,\eta)-\hat{g}(\xi,\eta)\right\|\le(\tau+1)\delta. \tag{10}$$

证明：由三角不等式，噪音水平(2)及(9)有

$$\begin{aligned}&\left\|\frac{1}{1+\alpha e^{(\xi^2+\eta^2)T\gamma}}\hat{g}^{\delta}(\xi,\eta)-\hat{g}(\xi,\eta)\right\|\\&=\left\|\frac{1}{1+\alpha e^{(\xi^2+\eta^2)T\gamma}}\hat{g}^{\delta}(\xi,\eta)-\hat{g}^{\delta}(\xi,\eta)+\hat{g}^{\delta}(\xi,\eta)-\hat{g}(\xi,\eta)\right\|\\&\le\left\|\frac{1}{1+\alpha e^{(\xi^2+\eta^2)T\gamma}}\hat{g}^{\delta}(\xi,\eta)-\hat{g}^{\delta}(\xi,\eta)\right\|+\left\|g^{\delta}(\xi,\eta)-\hat{g}(\xi,\eta)\right\|\\&\le(\tau+1)\delta.\end{aligned}$$

引理 5. 假设噪音水平(2)和先验界(3)成立。如果 α 是(9)的解，有不等式

$$\alpha^{-\frac{1}{\gamma}}\le\frac{E}{(\tau-1)\delta}. \tag{11}$$

证明：由噪音水平(2)和(9)，可得

$$\begin{aligned}
\tau\delta &= \left\| \frac{1}{1+\alpha e^{(\xi^2+\eta^2)T\gamma}} \hat{g}^{\delta}(\xi,\eta) - \hat{g}^{\delta}(\xi,\eta) \right\| \\
&= \left\| \frac{\alpha e^{(\xi^2+\eta^2)T\gamma}}{1+\alpha e^{(\xi^2+\eta^2)T\gamma}} \hat{g}^{\delta}(\xi,\eta) \right\| \\
&= \left\| \frac{\alpha e^{(\xi^2+\eta^2)T\gamma}}{1+\alpha e^{(\xi^2+\eta^2)T\gamma}} \left(\hat{g}^{\delta}(\xi,\eta) - \hat{g}(\xi,\eta) + \hat{g}(\xi,\eta) \right) \right\| \\
&\le \left\| \frac{\alpha e^{(\xi^2+\eta^2)T\gamma}}{1+\alpha e^{(\xi^2+\eta^2)T\gamma}} \left(\hat{g}^{\delta}(\xi,\eta) - \hat{g}(\xi,\eta) \right) \right\| + \left\| \frac{\alpha e^{(\xi^2+\eta^2)T(\gamma-1)}}{1+\alpha e^{(\xi^2+\eta^2)T\gamma}} e^{(\xi^2+\eta^2)T} \hat{g}(\xi,\eta) \right\| \\
&\le \delta + \alpha E \sup_{\xi,\eta\in\mathbf{R}} \left(\frac{e^{(\xi^2+\eta^2)T(\gamma-1)}}{1+\alpha e^{(\xi^2+\eta^2)T\gamma}} \right) \\
&\le \delta + E\alpha^{\frac{1}{\gamma}},
\end{aligned}$$

由此，有

$$\alpha^{-\frac{1}{\gamma}} \le \frac{E}{(\tau-1)\delta}.$$

定理 1：假设噪音水平(2)及先验条件(3)成立。正则化参数α是由 *Morozov's* 偏差原理选取的，则有如下误差估计

$$\left\| u_{\alpha,\delta}(\cdot,\cdot,t) - u(\cdot,\cdot,t) \right\| \le \left(\frac{\tau}{\tau-1} \right)^{1-\frac{t}{T}} (\tau+1)^{\frac{t}{T}} E^{1-\frac{t}{T}} \delta^{\frac{t}{T}}. \tag{12}$$

证明：设 $J = \left\| u(\cdot,\cdot,t) - u_{\alpha,\delta}(\cdot,\cdot,t) \right\|$。由 *Parseval's* 等式，(4)及(7)有

$$\begin{aligned}
J^2 &= \left\| u(\cdot,\cdot,t) - u_{\alpha,\delta}(\cdot,\cdot,t) \right\|^2 \\
&= \left\| \hat{u}(\cdot,\cdot,t) - \hat{u}_{\alpha,\delta}(\cdot,\cdot,t) \right\|^2 \\
&= \left\| e^{(\xi^2+\eta^2)(T-t)} \hat{g}(\xi,\eta) - \frac{e^{(\xi^2+\eta^2)(T-t)}}{1+\alpha e^{(\xi^2+\eta^2)T\gamma}} \hat{g}^{\delta}(\xi,\eta) \right\|^2 \\
&= \left\| e^{(\xi^2+\eta^2)(T-t)} \left(\hat{g}(\xi,\eta) - \frac{1}{1+\alpha e^{(\xi^2+\eta^2)T\gamma}} \hat{g}^{\delta}(\xi,\eta) \right) \right\|^2,
\end{aligned}$$

由 *Hölder* 不等式，得

$$J^2=\int_{\boldsymbol{R}^2}\mathrm{e}^{2\left(\xi^2+\eta^2\right)(T-t)}\left|\hat{g}(\xi,\eta)-\frac{1}{1+\alpha\mathrm{e}^{\left(\xi^2+\eta^2\right)T\gamma}}\hat{g}^{\delta}(\xi,\eta)\right|^2\mathrm{d}\xi\mathrm{d}\eta$$

$$=\int_{\boldsymbol{R}^2}\mathrm{e}^{2\left(\xi^2+\eta^2\right)(T-t)}\left|\hat{g}(\xi,\eta)-\frac{1}{1+\alpha\mathrm{e}^{\left(\xi^2+\eta^2\right)T\gamma}}\hat{g}^{\delta}(\xi,\eta)\right|^{2\left(1-\frac{t}{T}\right)}$$

$$\cdot\left|\hat{g}(\xi,\eta)-\frac{1}{1+\alpha\mathrm{e}^{\left(\xi^2+\eta^2\right)T\gamma}}\hat{g}^{\delta}(\xi,\eta)\right|^{\frac{2t}{T}}\mathrm{d}\xi\mathrm{d}\eta$$

$$\leq\left(\int_{\boldsymbol{R}^2}\left(\mathrm{e}^{2\left(\xi^2+\eta^2\right)(T-t)}\left|\hat{g}(\xi,\eta)-\frac{1}{1+\alpha\mathrm{e}^{\left(\xi^2+\eta^2\right)T\gamma}}\hat{g}^{\delta}(\xi,\eta)\right|^{2\left(1-\frac{t}{T}\right)}\right)^{\frac{T}{T-t}}\mathrm{d}\xi\mathrm{d}\eta\right)^{1-\frac{t}{T}}$$

$$\cdot\left(\int_{\boldsymbol{R}^2}\left(\left|\hat{g}(\xi,\eta)-\frac{1}{1+\alpha\mathrm{e}^{\left(\xi^2+\eta^2\right)T\gamma}}\hat{g}^{\delta}(\xi,\eta)\right|^{\frac{2t}{T}}\right)^{\frac{T}{t}}\mathrm{d}\xi\mathrm{d}\eta\right)^{\frac{t}{T}}$$

$$\leq\left(\int_{\boldsymbol{R}^2}\mathrm{e}^{2\left(\xi^2+\eta^2\right)T}\left|\hat{g}(\xi,\eta)-\frac{1}{1+\alpha\mathrm{e}^{\left(\xi^2+\eta^2\right)T\gamma}}\hat{g}^{\delta}(\xi,\eta)\right|^2\mathrm{d}\xi\mathrm{d}\eta\right)^{1-\frac{t}{T}}$$

$$\cdot\left(\int_{\boldsymbol{R}^2}\left|\hat{g}(\xi,\eta)-\frac{1}{1+\alpha\mathrm{e}^{\left(\xi^2+\eta^2\right)T\gamma}}\hat{g}^{\delta}(\xi,\eta)\right|^2\mathrm{d}\xi\mathrm{d}\eta\right)^{\frac{t}{T}},$$

因此有

$$J^2\leq\left\|\mathrm{e}^{\left(\xi^2+\eta^2\right)T}\left(\hat{g}(\xi,\eta)-\frac{1}{1+\alpha\mathrm{e}^{\left(\xi^2+\eta^2\right)T\gamma}}\hat{g}^{\delta}(\xi,\eta)\right)\right\|^{2\left(1-\frac{t}{T}\right)}\left\|\hat{g}(\xi,\eta)-\frac{1}{1+\alpha\mathrm{e}^{\left(\xi^2+\eta^2\right)T\gamma}}\hat{g}^{\delta}(\xi,\eta)\right\|^{\frac{2t}{T}}$$

$$=\left\|\mathrm{e}^{\left(\xi^2+\eta^2\right)T}\hat{g}(\xi,\eta)-\frac{\mathrm{e}^{\left(\xi^2+\eta^2\right)T}}{1+\alpha\mathrm{e}^{\left(\xi^2+\eta^2\right)T\gamma}}\hat{g}^{\delta}(\xi,\eta)\right\|^{2\left(1-\frac{t}{T}\right)}\left\|\hat{g}(\xi,\eta)-\frac{1}{1+\alpha\mathrm{e}^{\left(\xi^2+\eta^2\right)T\gamma}}\hat{g}^{\delta}(\xi,\eta)\right\|^{\frac{2t}{T}}$$

$$\leq\left(\left\|\mathrm{e}^{\left(\xi^2+\eta^2\right)T}\hat{g}(\xi,\eta)\left(1-\frac{1}{1+\alpha\mathrm{e}^{\left(\xi^2+\eta^2\right)T\gamma}}\right)\right\|+\left\|\frac{\mathrm{e}^{\left(\xi^2+\eta^2\right)T}}{1+\alpha\mathrm{e}^{\left(\xi^2+\eta^2\right)T\gamma}}\left(\hat{g}(\xi,\eta)-\hat{g}^{\delta}(\xi,\eta)\right)\right\|\right)^{2\left(1-\frac{t}{T}\right)}$$

$$\cdot\left\|\hat{g}(\xi,\eta)-\frac{1}{1+\alpha\mathrm{e}^{\left(\xi^2+\eta^2\right)T\gamma}}\hat{g}^{\delta}(\xi,\eta)\right\|^{\frac{2t}{T}},$$

根据引理 1、引理 4、引理 5，易得

$$J^2 \le \left(E+\delta \sup_{\xi,\eta\in\boldsymbol{R}}\left(\frac{e^{\left(\xi^2+\eta^2\right)T}}{1+\alpha e^{\left(\xi^2+\eta^2\right)T\gamma}}\right)\right)^{2\left(1-\frac{t}{T}\right)}\left(\left(\tau+1\right)\delta\right)^{\frac{2t}{T}}$$
$$\le E^{2\left(1-\frac{t}{T}\right)}\delta^{\frac{2t}{T}}\left(\frac{\tau}{\tau-1}\right)^{2\left(1-\frac{t}{T}\right)}\left(\tau+1\right)^{\frac{2t}{T}},$$

即

$$\left\|u_{\alpha,\delta}\left(\cdot,\cdot,t\right)-u\left(\cdot,\cdot,t\right)\right\|^2 \le \left(\frac{\tau}{\tau-1}\right)^{2\left(1-\frac{t}{T}\right)}\left(\tau+1\right)^{\frac{2t}{T}}E^{2\left(1-\frac{t}{T}\right)}\delta^{\frac{2t}{T}},$$

故

$$\left\|u_{\alpha,\delta}\left(\cdot,\cdot,t\right)-u\left(\cdot,\cdot,t\right)\right\| \le \left(\frac{\tau}{\tau-1}\right)^{1-\frac{t}{T}}\left(\tau+1\right)^{\frac{t}{T}}E^{1-\frac{t}{T}}\delta^{\frac{t}{T}}.$$

5. 拟逆正则化方法后验参数选取及误差估计

对于问题(1)给出拟逆正则化方法近似如下：

$$\begin{cases} u_t\left(x,y,t\right)-\alpha\left(\Delta u\right)_t\left(x,y,t\right)-\Delta u\left(x,y,t\right)=0,\left(x,y\right)\in\boldsymbol{R}^2,t\in\left(0,T\right), \\ u\left(x,y,T\right)=g^\delta\left(x,y\right),\left(x,y\right)\in\boldsymbol{R}^2, \end{cases} \tag{13}$$

这里的 $\alpha>0$ 为正则化参数。对(13)作 *Fourier* 变换有

$$\begin{cases} \hat{u}_t\left(\xi,\eta,t\right)+\alpha\left(\xi^2+\eta^2\right)\hat{u}_t\left(\xi,\eta,t\right)+\left(\xi^2+\eta^2\right)\hat{u}\left(\xi,\eta,t\right)=0,\left(\xi,\eta\right)\in\boldsymbol{R}^2,t\in\left(0,T\right), \\ \hat{u}\left(\xi,\eta,T\right)=\hat{g}^\delta\left(\xi,\eta\right),\left(\xi,\eta\right)\in\boldsymbol{R}^2. \end{cases} \tag{14}$$

其频域上相应的正则解为

$$\hat{u}_{\alpha,\delta}\left(\xi,\eta,t\right)=e^{\frac{\left(\xi^2+\eta^2\right)\left(T-t\right)}{1+\alpha\left(\xi^2+\eta^2\right)}}\hat{g}^\delta\left(\xi,\eta\right).$$

记

$$\hat{u}_\alpha\left(\xi,\eta,t\right)=e^{\frac{\left(\xi^2+\eta^2\right)\left(T-t\right)}{1+\alpha\left(\xi^2+\eta^2\right)}}\hat{g}\left(\xi,\eta\right). \tag{15}$$

下面我们考虑拟逆正则化方法的后验参数选取以及相应的误差估计。当噪音水平 δ 已知时，一般采用 *Morozov's* 偏差原理进行后验正则化参数选取，找到一个 α 满足方程

$$\begin{aligned}\rho\left(\alpha\right)&=\left\|e^{-\frac{\alpha\left(\xi^2+\eta^2\right)^2\left(T-t\right)}{1+\alpha\left(\xi^2+\eta^2\right)}}\hat{g}^\delta\left(\xi,\eta\right)-\hat{g}^\delta\left(\xi,\eta\right)\right\| \\ &=\delta+\tau\left(\log\log\left(\frac{1}{\delta}\right)\right)^{-1}.\end{aligned} \tag{16}$$

其中$\tau>0$是一个常数，$\alpha>0$是正则化参数。

引理 6. 设对固定的$\delta>0$，$\rho(\alpha)$满足以下几条性质：

1) $\rho(\alpha)$是连续函数；

2) $\lim_{\alpha\to 0}\rho(\alpha)=0$；

3) $\lim_{\alpha\to\infty}\rho(\alpha)=\|g^\delta\|$；

4) $\rho(\alpha)$是严格单调增函数。

证明：由$\rho(\alpha)$的表达式

$$\rho(\alpha)=\left\| e^{-\frac{\alpha(\xi^2+\eta^2)^2(T-t)}{1+\alpha(\xi^2+\eta^2)}}\hat{g}^\delta(\xi,\eta)-\hat{g}^\delta(\xi,\eta)\right\|$$

容易验证上述结论成立。若选择常数τ满足$0<\delta+\tau\left(\log\log\left(\frac{1}{\delta}\right)\right)^{-1}<\|\hat{g}^\delta\|$时，根据引理内容知(16)的解是存在且唯一的。

引理 7. 如果α是(16)的解，则有下述不等式成立

$$\left\|\hat{g}(\xi,\eta)-e^{-\frac{\alpha(\xi^2+\eta^2)^2(T-t)}{1+\alpha(\xi^2+\eta^2)}}\hat{g}^\delta(\xi,\eta)\right\|\le 2\delta+\tau\left(\log\log\left(\frac{1}{\delta}\right)\right)^{-1}. \tag{17}$$

证明：由三角不等式和噪音水平(2)可得

$$\begin{aligned}
&\left\|\hat{g}(\xi,\eta)-e^{-\frac{\alpha(\xi^2+\eta^2)^2(T-t)}{1+\alpha(\xi^2+\eta^2)}}\hat{g}^\delta(\xi,\eta)\right\|\\
&=\left\|e^{-\frac{\alpha(\xi^2+\eta^2)^2(T-t)}{1+\alpha(\xi^2+\eta^2)}}\hat{g}^\delta(\xi,\eta)-\hat{g}^\delta(\xi,\eta)+\hat{g}^\delta(\xi,\eta)-\hat{g}(\xi,\eta)\right\|\\
&\le\left\|e^{-\frac{\alpha(\xi^2+\eta^2)^2(T-t)}{1+\alpha(\xi^2+\eta^2)}}\hat{g}^\delta(\xi,\eta)-\hat{g}^\delta(\xi,\eta)\right\|+\left\|\hat{g}^\delta(\xi,\eta)-\hat{g}(\xi,\eta)\right\|\\
&\le 2\delta+\tau\left(\log\log\left(\frac{1}{\delta}\right)\right)^{-1}.
\end{aligned}$$

引理 8. 假设噪音水平(2)成立，正则化参数α是由*Morozov's*偏差原理(16)选取，则有下述不等式成立：

$$\frac{1}{\alpha}\le\frac{4(T-t)E}{\tau e^2T^2}\log\log\left(\frac{1}{\delta}\right). \tag{18}$$

证明：由噪音水平(2)和先验界(3)以及(16)，得

$$
\begin{aligned}
\delta+\tau\left(\log\log\left(\frac{1}{\delta}\right)\right)^{-1} &= \left\| e^{-\frac{\alpha(\xi^2+\eta^2)^2(T-t)}{1+\alpha(\xi^2+\eta^2)}}\hat{g}^{\delta}(\xi,\eta)-\hat{g}^{\delta}(\xi,\eta) \right\| \\
&= \left\| \left(1-e^{-\frac{\alpha(\xi^2+\eta^2)^2(T-t)}{1+\alpha(\xi^2+\eta^2)}}\right)\hat{g}^{\delta}(\xi,\eta) \right\| \\
&\le \left\| \left(1-e^{-\frac{\alpha(\xi^2+\eta^2)^2(T-t)}{1+\alpha(\xi^2+\eta^2)}}\right)\left(\hat{g}^{\delta}(\xi,\eta)-\hat{g}(\xi,\eta)\right) \right\| + \left\| \left(1-e^{-\frac{\alpha(\xi^2+\eta^2)^2(T-t)}{1+\alpha(\xi^2+\eta^2)}}\right)\hat{g}(\xi,\eta) \right\| \\
&\le \delta+\left\| \left(1-e^{-\frac{\alpha(\xi^2+\eta^2)^2(T-t)}{1+\alpha(\xi^2+\eta^2)}}\right) e^{-(\xi^2+\eta^2)T} e^{(\xi^2+\eta^2)T}\hat{g}(\xi,\eta) \right\| \\
&\le \delta+E\sup\left(\left(1-e^{-\frac{\alpha(\xi^2+\eta^2)^2(T-t)}{1+\alpha(\xi^2+\eta^2)}}\right) e^{-(\xi^2+\eta^2)T}\right) \\
&\le \delta+EG(\xi,\eta),
\end{aligned}
$$

令

$$
G(\xi,\eta):=\sup_{\xi,\eta\in\boldsymbol{R}}\left(\left(1-e^{-\frac{\alpha(\xi^2+\eta^2)^2(T-t)}{1+\alpha(\xi^2+\eta^2)}}\right) e^{-(\xi^2+\eta^2)T}\right),
$$

由于

$$
1-e^{-\frac{\alpha(\xi^2+\eta^2)^2(T-t)}{1+\alpha(\xi^2+\eta^2)}} \le \frac{\alpha(\xi^2+\eta^2)^2(T-t)}{1+\alpha(\xi^2+\eta^2)} \le \alpha(\xi^2+\eta^2)^2(T-t),
$$

故有

$$
G(\xi,\eta)\le\sup_{\xi,\eta\in\boldsymbol{R}}\left(\alpha(\xi^2+\eta^2)^2(T-t)e^{-(\xi^2+\eta^2)T}\right),
$$

令 $\xi^2+\eta^2=m$，则对 m 求导，得到极大值点 $m=\frac{2}{T}$，则

$$
\sup_{\xi,\eta\in\boldsymbol{R}}G(\xi,\eta)\le\sup_{\xi,\eta\in\boldsymbol{R}}\left(\alpha(\xi^2+\eta^2)^2(T-t)e^{-(\xi^2+\eta^2)T}\right)\le\frac{4\alpha(T-t)}{e^2T^2},
$$

于是有

$$\tau\left(\log\log\left(\frac{1}{\delta}\right)\right)^{-1}\le\frac{4\alpha(T-t)}{\mathrm{e}^2T^2}E,$$

从而

$$\frac{1}{\alpha}\le\frac{4(T-t)E}{\tau\mathrm{e}^2T^2}\log\log\left(\frac{1}{\delta}\right).$$

定理 2. 假设噪音水平(2)和先验条件(3)成立，并存在$\tau>0$，使得$0<\delta+\tau\left(\log\log\left(\frac{1}{\delta}\right)\right)^{-1}<\left\|g^\delta\right\|$，对于$0\le t<T$，正则化参数$\alpha$是由后验正则化参数选取得到的，则有如下估计：

$$\left\|\hat{u}_{\alpha,\delta}(\cdot,\cdot,t)-\hat{u}(\cdot,\cdot,t)\right\|\le\delta\cdot\left(\frac{4(T-t)E}{\tau e^2T^2}\log\log\left(\frac{1}{\delta}\right)\right)^{T-t}+E^{1-\frac{t}{T}}\left(2\delta+\tau\left(\log\log\left(\frac{1}{\delta}\right)\right)^{-1}\right)^{\frac{t}{T}}.$$

证明：令$I=\left\|u_{\alpha,\delta}(\cdot,\cdot,t)-u(\cdot,\cdot,t)\right\|$，由 Parseval’s 等式及三角不等式得

$$\begin{aligned}I&=\left\|u_{\alpha,\delta}(\cdot,\cdot,t)-u(\cdot,\cdot,t)\right\|\\&=\left\|\hat{u}_{\alpha,\delta}(\cdot,\cdot,t)-\hat{u}(\cdot,\cdot,t)\right\|\\&\le\left\|\hat{u}_{\alpha,\delta}(\cdot,\cdot,t)-\hat{u}_\alpha(\cdot,\cdot,t)\right\|+\left\|\hat{u}_\alpha(\cdot,\cdot,t)-\hat{u}(\cdot,\cdot,t)\right\|\\&=A_1+A_2.\end{aligned}$$

则先对A_1做估计，根据引理 1 有

$$\begin{aligned}A_1&=\left\|u_{\alpha,\delta}(\cdot,\cdot,t)-u_\alpha(\cdot,\cdot,t)\right\|\\&=\left\|\hat{u}_{\alpha,\delta}(\cdot,\cdot,t)-\hat{u}_\alpha(\cdot,\cdot,t)\right\|\\&=\left\|\mathrm{e}^{\frac{(\xi^2+\eta^2)(T-t)}{1+\alpha(\xi^2+\eta^2)}}\hat{g}^\delta(\xi,\eta)-\mathrm{e}^{\frac{(\xi^2+\eta^2)(T-t)}{1+\alpha(\xi^2+\eta^2)}}\hat{g}(\xi,\eta)\right\|\\&=\left\|\mathrm{e}^{\frac{(\xi^2+\eta^2)(T-t)}{1+\alpha(\xi^2+\eta^2)}}\left(\hat{g}^\delta(\xi,\eta)-\hat{g}(\xi,\eta)\right)\right\|\\&\le\delta\cdot\mathrm{e}^{\frac{T-t}{\alpha}}.\end{aligned}$$

由引理 8，则

$$A_1\le\delta\cdot\mathrm{e}^{\frac{4(T-t)^2E}{\tau\mathrm{e}^2T^2}\log\log\left(\frac{1}{\delta}\right)}.$$

再对A_2做估计，由 *Hölder* 不等式有

$$
\begin{aligned}
A_2^2 &= \left\| u_\alpha(\cdot,\cdot,t) - u(\cdot,\cdot,t) \right\|^2 \\
&= \left\| \hat{u}_\alpha(\cdot,\cdot,t) - \hat{u}(\cdot,\cdot,t) \right\|^2 \\
&= \left\| \mathrm{e}^{\frac{(\xi^2+\eta^2)(T-t)}{1+\alpha(\xi^2+\eta^2)}} \hat{g}(\xi,\eta) - \mathrm{e}^{(\xi^2+\eta^2)(T-t)} \hat{g}(\xi,\eta) \right\| \\
&= \left\| \mathrm{e}^{(\xi^2+\eta^2)(T-t)} \left(\hat{g}(\xi,\eta) - \mathrm{e}^{-\frac{\alpha(\xi^2+\eta^2)^2(T-t)}{1+\alpha(\xi^2+\eta^2)}} \hat{g}(\xi,\eta) \right) \right\|^2 \\
&= \iint_{\boldsymbol{R}^2} \mathrm{e}^{2(\xi^2+\eta^2)(T-t)} \left| \hat{g}(\xi,\eta) - \mathrm{e}^{-\frac{\alpha(\xi^2+\eta^2)^2(T-t)}{1+\alpha(\xi^2+\eta^2)}} \hat{g}(\xi,\eta) \right|^{2\left(1-\frac{t}{T}\right)} \\
&\quad \cdot \left| \hat{g}(\xi,\eta) - \mathrm{e}^{-\frac{\alpha(\xi^2+\eta^2)^2(T-t)}{1+\alpha(\xi^2+\eta^2)}} \hat{g}(\xi,\eta) \right|^{\frac{2t}{T}} \mathrm{d}\xi \mathrm{d}\eta \\
&\le \left(\iint_{\boldsymbol{R}^2} \left(\mathrm{e}^{2(\xi^2+\eta^2)(T-t)} \left| \hat{g}(\xi,\eta) - \mathrm{e}^{-\frac{\alpha(\xi^2+\eta^2)^2(T-t)}{1+\alpha(\xi^2+\eta^2)}} \hat{g}(\xi,\eta) \right|^{2\left(1-\frac{t}{T}\right)} \right)^{\frac{T}{T-t}} \mathrm{d}\xi \mathrm{d}\eta \right)^{\frac{T-t}{T}} \\
&\quad \cdot \left(\iint_{\boldsymbol{R}^2} \left(\left| \hat{g}(\xi,\eta) - \mathrm{e}^{-\frac{\alpha(\xi^2+\eta^2)^2(T-t)}{1+\alpha(\xi^2+\eta^2)}} \hat{g}(\xi,\eta) \right|^{\frac{2t}{T}} \right)^{\frac{T}{t}} \mathrm{d}\xi \mathrm{d}\eta \right)^{\frac{t}{T}} \\
&\le \left(\iint_{\boldsymbol{R}^2} \mathrm{e}^{2(\xi^2+\eta^2)(T-t)} \left| \hat{g}(\xi,\eta) - \mathrm{e}^{-\frac{\alpha(\xi^2+\eta^2)^2(T-t)}{1+\alpha(\xi^2+\eta^2)}} \hat{g}(\xi,\eta) \right|^2 \mathrm{d}\xi \mathrm{d}\eta \right)^{\frac{T-t}{T}} \\
&\quad \cdot \left(\iint_{\boldsymbol{R}^2} \left| \hat{g}(\xi,\eta) - \mathrm{e}^{-\frac{\alpha(\xi^2+\eta^2)^2(T-t)}{1+\alpha(\xi^2+\eta^2)}} \hat{g}(\xi,\eta) \right|^2 \mathrm{d}\xi \mathrm{d}\eta \right)^{\frac{t}{T}},
\end{aligned}
$$

则由(16)式有

$$
\begin{aligned}
A_2^2 &= \left\| u_\alpha(\cdot,\cdot,t) - u(\cdot,\cdot,t) \right\|^2 \\
&= \left\| e^{(\xi^2+\eta^2)(T-t)} \left(\hat{g}(\xi,\eta) - e^{-\frac{\alpha(\xi^2+\eta^2)^2(T-t)}{1+\alpha(\xi^2+\eta^2)}} \hat{g}(\xi,\eta) \right) \right\|^{2\left(1-\frac{t}{T}\right)} \left\| \hat{g}(\xi,\eta) - e^{-\frac{\alpha(\xi^2+\eta^2)^2(T-t)}{1+\alpha(\xi^2+\eta^2)}} \hat{g}(\xi,\eta) \right\|^{\frac{2t}{T}} \\
&= \left\| e^{(\xi^2+\eta^2)(T-t)} \hat{g}(\xi,\eta) \left(1 - e^{-\frac{\alpha(\xi^2+\eta^2)^2(T-t)}{1+\alpha(\xi^2+\eta^2)}} \right) \right\|^{2\left(1-\frac{t}{T}\right)} \left\| \hat{g}(\xi,\eta) \left(1 - e^{-\frac{\alpha(\xi^2+\eta^2)^2(T-t)}{1+\alpha(\xi^2+\eta^2)}} \right) \right\|^{\frac{2t}{T}} \\
&\le E^{2\left(1-\frac{t}{T}\right)} \cdot \left\| \left(\hat{g}(\xi,\eta) - \hat{g}^\delta(\xi,\eta) + \hat{g}^\delta(\xi,\eta) \right) \left(1 - e^{-\frac{\alpha(\xi^2+\eta^2)^2(T-t)}{1+\alpha(\xi^2+\eta^2)}} \right) \right\|^{\frac{2t}{T}} \\
&\le E^{2\left(1-\frac{t}{T}\right)} \cdot \left(\left\| \left(\hat{g}(\xi,\eta) - \hat{g}^\delta(\xi,\eta) \right) \left(1 - e^{-\frac{\alpha(\xi^2+\eta^2)^2(T-t)}{1+\alpha(\xi^2+\eta^2)}} \right) \right\| + \left\| \hat{g}^\delta(\xi,\eta) \left(1 - e^{-\frac{\alpha(\xi^2+\eta^2)^2(T-t)}{1+\alpha(\xi^2+\eta^2)}} \right) \right\| \right)^{\frac{2t}{T}} \\
&\le E^{2\left(1-\frac{t}{T}\right)} \cdot \left(\left\| \hat{g}(\xi,\eta) - \hat{g}^\delta(\xi,\eta) \right\| + \left\| e^{-\frac{\alpha(\xi^2+\eta^2)^2(T-t)}{1+\alpha(\xi^2+\eta^2)}} \hat{g}^\delta(\xi,\eta) - \hat{g}^\delta(\xi,\eta) \right\| \right)^{\frac{2t}{T}} \\
&\le E^{2\left(1-\frac{t}{T}\right)} \cdot \left(2\delta + \tau \log\log\left(\frac{1}{\delta}\right) \right)^{\frac{2t}{T}}.
\end{aligned}
$$

于是

$$
A_2 \le E^{1-\frac{t}{T}} \cdot \left(2\delta + \tau \log\log\left(\frac{1}{\delta}\right) \right)^{\frac{t}{T}},
$$

从而

$$
\begin{aligned}
&\left\| u_{\alpha,\delta}(\cdot,\cdot,t) - u(\cdot,\cdot,t) \right\| \\
&= \left\| \hat{u}_{\alpha,\delta}(\cdot,\cdot,t) - \hat{u}(\cdot,\cdot,t) \right\| \\
&\le \left\| \hat{u}_{\alpha,\delta}(\cdot,\cdot,t) - \hat{u}_\alpha(\cdot,\cdot,t) \right\| + \left\| \hat{u}_\alpha(\cdot,\cdot,t) - \hat{u}(\cdot,\cdot,t) \right\| \\
&\le \delta \cdot e^{\frac{4(T-t)^2 E}{\tau e^2 T^2} \log\log\left(\frac{1}{\delta}\right)} + E^{1-\frac{t}{T}} \cdot \left(2\delta + \tau \log\log\left(\frac{1}{\delta}\right) \right)^{\frac{t}{T}}.
\end{aligned}
$$

6. 结束语

本文针对二维空间的时间反向热传导问题采用分数次 *Tihonov* 正则化方法和拟逆正则化方法进行了正则化，并得到了正则解。同时给出了后验参数选取规则以及正则解与精确解之间稳定的误差估计。

参考文献

[1] Cheng, W. and Fu, C.L. (2010) A Modified *Tikhonov* Regularization Method for an Axisymmetric Backward Heat Equation. *Acta Mathematica Sinica, English Series*, **26**, 2157-2164. https://doi.org/10.1007/s10114-010-8509-5

[2] Hochstenbach, M.E. and Reichel, L. (2011) Fractional *Tikhonov* Regularization for Linear Discrete Ill-Posed Problems. *BIT Numerical Mathematics*, **51**, 197-215. https://doi.org/10.1007/s10543-011-0313-9

[3] Lattes, R. and Lions, J.L. (1969) The Method of Quasi-Reversibility: Applications to Partial Differential Equations. American Elsevier, New York.

[4] Koba, H. and Matsuoka, H. (2015) Generalized Quasi-Reversibility Method for a Backward Heat Equation with a Fractional Laplacian. *Analysis*, **35**, 47-57. https://doi.org/10.1515/anly-2014-1262

[5] Shukla, A. and Mehra, M. (2020) Compact Filtering as a Regularization Technique for a Backward Heat Conduction Problem. *Applied Numerical Mathematics*, **153**, 82-97. https://doi.org/10.1016/j.apnum.2020.02.003

[6] Mera, N.S., Elliott, L., Ingham, D.B. and Lesnic, D. (2001) An Iterative Boundary Element Method for Solving the One Dimensional Backward Heat Conduction Problem. *International Journal of Heat and Mass Transfer*, **44**, 1973-1946. https://doi.org/10.1016/S0017-9310(00)00235-0

[7] Lesnic, D., Elliott, L. and Ingham, D.B. (1998) An Iterative Boundary Element Method for Solving the Backward Heat Conduction Problem Using an Elliptic Approximation. *Inverse Problems in Engineering*, **6**, 255-279. https://doi.org/10.1080/174159798088027679

[8] Li, M., and Xiong, X.T. (2012) On a Fractional Backward Heat Conduction Problem: Application to Deblurring. *Computers and Mathematics with Applications*, **64**, 2594-2602. https://doi.org/10.1016/j.camwa.2012.07.003

[9] Reinhardt, H.J., Hào, D.N., Frohne, J. and Suttmeier, F.T. (2007) Numerical Solution of Inverse Heat Conduction Problems in Two Spatial Dimensions. *Journal of Inverse and Ill-Posed Problems*, **15**, 181-198. https://doi.org/10.1515/JIIP.2007.010

[10] Xiong, X.T., Wang, J.X. and Li, M. (2012) An Optimal Method for Fractional Heat Conduction Problem Backward in Time. *Applicable Analysis*, **91**, 823-840. https://doi.org/10.1080/00036811.2011.601455

[11] Carasso, A.S., Sanderson, J.G. and Hyman, J.M. (1978) Digital Removal of Random Media Image Degradations by Solving the Diffusion Equation Backwards in Time. *SIAM Journal on Numerical Analysis*, **15**, 344-367. https://doi.org/10.1137/0715023

[12] Xiong, X.T. (2010) A Regularization Method for a Cauchy Problem of the Helmholtz Equation. *Journal of Computational and Applied Mathematics*, **233**, 1723-1732. https://doi.org/10.1016/j.cam.2009.09.001

[13] Qian, A.L., Xiong, X.T. and Wu, Y.J. (2010) On a Quasi-Reversibility Regularization Method for a Cauchy Problem of the Helmholtz Equation. *Journal of Computational and Applied Mathematics*, **233**, 1969-1979. https://doi.org/10.1016/j.cam.2009.09.031