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单边奇异积分与Lipschitz函数生成的Cohen型交换子的变差不等式

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摘 要

本文引入了单边奇异积分与Lipschitz函数生成的Cohen型交换子的变差算子. 利用单边权的外推法，建立了上述算子从加权单边Triebel-Lizorkin空间到加权Lebesgue空间的有界性.

关键词

ρ-变差，Cohen型交换子，单边权，加权单边Triebel-Lizorkin空间

Variation Inequalities for Cohen Type Commutator of One-Sided Singular Integral with Lipschitz Function

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Abstract

In this paper, we introduce ρ-variation operator of Cohen type commutator of one-sided singular integral. By the extrapolation of one-sided weights, we establish the boundedness of the above operator from weighted Lebesgue spaces to weighed one-sided Triebel-Lizorkin spaces.

Keywords

ρ-Variation, Cohen Type Commutator, One-Sided Weights, Weighed One-Sided Triebel-Lizorkin Spaces

1. 引言与主要结果

设$\mathcal{T} = \{T_\varepsilon\}_\varepsilon$是一列算子并满足$\lim_{\varepsilon\to 0} T_\varepsilon f(x) = Tf(x)$几乎处处成立. 经典的方法是通过研究$(\sum_{i=1}^{\infty} |T_{\varepsilon_i} f - T_{\varepsilon_i+1} f|^2)^{1/2}$类型的函数来测量算子族$\{T_\varepsilon\}$ 的收敛速度. 更一般而言, 定义下面的振荡算子：

$$\mathcal{O}(\mathcal{T}f)(x) = \Big(\sum_{i=1}^{\infty} \sup_{t_{i+1}\le\varepsilon_{i+1}<\varepsilon_i\le t_i} |T_{\varepsilon_i+1} f(x) - T_{\varepsilon_i} f(x)|^2\Big)^{1/2},$$

其中$\{t_i\}$是一列给定的单调趋于0的数列. 此外, 另一种方法是考虑下面的ρ变差算子:

$$\mathcal{V}_\rho(\mathcal{T}f)(x) = \sup_{\varepsilon_i\searrow 0} \Big(\sum_{i=1}^{\infty} |T_{\varepsilon_i+1} f(x) - T_{\varepsilon_i} f(x)|^\rho\Big)^{1/\rho},$$

其中$\rho > 2$, 上确界取遍所有的递减趋于0的序列$\{\varepsilon_i\}$.

变差算子的研究在概率论, 遍历理论及调和分析等领域有着重要的应用. Lépingle [1]首先得到了鞅序列的振荡与变差不等式. Bourgain [2]建立了二进系统遍历平均的变差不等式. 有关振荡与变差算子的研究及应用参见文献 [3] [4] [5]. 奇异积分算子交换子的研究也是调和分析中重要分支, 它在微分方程中有着极其重要的作用. Liu和Wu [6]给出了一个一维情况下具有标准核的Calderón-Zygmund奇异积分算子交换子的振荡与变差算子的加权L^p有界性的判定定理. 单边奇异积分算

子与BMO函数生成的交换子的变差算子的加权有界性在文献 [7]中被建立. Fu和Lin [8]得到了单边算子与Lipschitz函数生成的交换子的加权有界性. Zhang和Wu [9]建立了奇异积分与Lipschitz函数生成的交换子的振荡与变差不等式. Zhang 和Hou [10]给出了单边奇异积分与Cohen型交换子的Lipschitz估计. Cohen [11]引入了奇异积分的Cohen型交换子并研究了其有界性. Chen和Lu [12]建立了奇异积分的Cohen型交换子从Triebel-Lizorkin空间到Lebesgue的有界性. 受以上结果的启发, 本文将引入单边奇异积分与Lipschitz函数生成的Cohen型交换子的变差算子并研究其加权有界性.

2. 预备知识

在文献 [13]中, Aimar, Forzani与Martín-Reyes引入了单边Calderón-Zygmund奇异积分:

$$T^+f(x) = \lim_{\varepsilon\to 0^+} T^+_\varepsilon f(x) = \lim_{\varepsilon\to 0^+}\int_{x+\varepsilon}^{\infty} K(x-y)f(y)dy,$$

其中支在$\mathbb{R}^- = (-\infty, 0)$的核函数$K$称为单边Calderón-Zygmund核且满足

$$\Big|\int_{a<|x|<b} K(x)dx\Big| \le C, \quad 0<a<b,$$

$$|K(x)| \le C/|x|, \quad x\neq 0, \tag{2.1}$$

$$|K(x-y)-K(x)| \le C|y|/|x|^2, \quad |x|>2|y|>0. \tag{2.2}$$

这类核函数的一个例子为

$$K(x) = \frac{\sin(\log|x|)}{(x\log|x|)}\chi_{(-\infty,0)}(x).$$

1986年, Sawyer [14]首先引入了单边Muckenhoupt权A_p^+及A_p^-来处理下面的单边极大算子:

$$M^+f(x) := \sup_{h>0}\frac{1}{h}\int_x^{x+h}|f(y)|dy, \quad M^-f(x) := \sup_{h>0}\frac{1}{h}\int_{x-h}^{x}|f(y)|dy.$$

对于$1\le p<\infty$, 若ω满足

$$\sup_{a<b<c}\frac{1}{(c-a)^p}\int_a^b \omega(x)dx\Big(\int_b^c \omega(x)^{1-p'}dx\Big)^{p-1} < \infty,$$

或

$$\sup_{a<b<c}\frac{1}{(c-a)^p}\int_b^c \omega(x)dx\Big(\int_a^b \omega(x)^{1-p'}dx\Big)^{p-1} < \infty,$$

则称ω属于A_p^+或A_p^-. 当$p=1$时, 存在常数C使得ω满足

$$M^-\omega(x)\le C\omega(x), \quad M^+\omega(x)\le C\omega(x).$$

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则称ω属于A_1^+或A_1^-. 对于$1\le p<\infty$, 有$A_p \subsetneqq A_p^+$ 且$A_p \subsetneqq A_p^-$. 注意到$\omega(x)=e^x$属于A_p^+ 但不属于A_p. 类似的, 对于所有的$a<b<c\in\mathbb{R}$, $0<\alpha<1$, $1<p<q$ 且$1/p-1/q=\alpha$, ω满足

$$\frac{1}{(c-a)^{1-\alpha}}\Big(\int_a^b \omega^q\Big)^{1/q}\Big(\int_b^c \omega^{-p'}\Big)^{1/p'}<C,$$

或

$$\frac{1}{(c-a)^{1-\alpha}}\Big(\int_b^c \omega^q\Big)^{1/q}\Big(\int_a^b \omega^{-p'}\Big)^{1/p'}<C$$

则称ω属于$A^+(p,q)$或$A^-(p,q)$. 当$p=1$, $1-1/q=\alpha$时, 存在常数C使得ω满足

$$M^-\omega(x)^q\le C\omega(x)^q,\quad M^+\omega(x)^q\le C\omega(x)^q$$

则称ω属于$A^+(1,q)$或$A^-(1,q)$. 对于$0<\alpha<1$, 定义单边分数次积分:

$$I_\alpha^+(f)(x):=\int_x^\infty \frac{1}{(y-x)^{1-\alpha}}f(y)dy.$$

Andersen与Sawyer [15]得到了I_α^+是从$L^p(\omega^p)$到$L^q(\omega^q)$有界的, 其中$\omega\in A^+(p,q)$且$1/p-1/q=\alpha$.

令$A(x)$为$\mathbb{R}$上的局部可积函数. 对于$m\ge 1$, $R_m(A;x,y)$为函数$A(x)$在y处泰勒级数的m阶余项. 即

$$R_m(A;x,y):=A(x)-\sum_{|k|\le m-1}\frac{1}{k!}A^{(k)}(y)(x-y)^k.$$

定义如下的单边奇异积分Cohen型交换子:

$$T_{A,m}^{\varepsilon,+}f(x)=\int_{x+\varepsilon}^\infty \frac{K(x-y)}{(y-x)^{m-1}}R_m(A;x,y)f(y)dy. \tag{2.3}$$

令$\Theta=\{\beta:\beta=\{\varepsilon_i\}, i\in\mathbb{R}, \varepsilon\searrow 0\}$. 考虑集合$\mathbb{N}\times\Theta$且用$F_\rho$表示所有满足$\|g\|_{F_\rho}<\infty$的函数$g(i,\beta)$构成的混合范数空间, 其中

$$\|g\|_{F_\rho}=\sup_\beta\Big(\sum_i |g(i,\beta)|^\rho\Big)^{1/\rho}.$$

给定$L^p(\mathbb{R})$上的一列算子$\mathcal{T}=\{T_t^+\}_{t>0}$, 考虑$F_\rho$-值算子$V(\mathcal{T}):f\to V(\mathcal{T})f$ 为:

$$V(\mathcal{T})f(x)=\big\{T_{\varepsilon_i+1}f(x)-T_{\varepsilon_i}f(x)\big\}_{\beta=\{\varepsilon_i\}\in\Theta},$$

其中$\big\{T_{\varepsilon_i+1}f(x)-T_{\varepsilon_i}f(x)\big\}_{\beta=\{\varepsilon_i\}\in\Theta}$ 是F_ρ中元素, 且以下面的形式给出

$$(i,\beta)=(i,\{\varepsilon_i\})\to T_{[\varepsilon_i+1,\varepsilon_i]}f(x).$$

注意到

$$\mathcal{V}_\rho(\mathcal{T}f)(x)=\|V(\mathcal{T})f(x)\|_{F_\rho}.$$

定义 2.1 对于$0<\alpha<1$, 若函数f满足

$$\|f\|_{Lip_\alpha}=\sup_{x,h\in\mathbb{R},h\neq 0}\frac{f(x+h)-f(x)}{|h|^\alpha}<\infty,$$

则称$f\in Lip_\alpha$.

定义 2.2 对于$0<\alpha<1$, $1<p<\infty$及合适的权函数ω, 若函数f满足

$$\|f\|_{\dot{F}^{\alpha,\infty}_{p,+}}\approx\left\|\sup_{h>0}\frac{1}{h^{1+\alpha}}\int_x^{x+h}\left|f(y)-f_{[x,x+h]}\right|dy\right\|_{L^p(\omega)}<\infty,$$

则称f属于加权单边*Tribel-Lizorkin*空间$\dot{F}^{\alpha,\infty}_{p,+}$.

在本文中, 我们将建立如下结果:

定理 2.1 令$m\geq 1$, $\rho>2$, $0<\alpha<1$且$A^{(k)}\in Lip_\alpha$, $k=0,1,\ldots,m-1$. 设$\mathcal{T}^+_{A,m}=\{T^{\varepsilon,+}_{A,m}\}_{\varepsilon>0}$. 若$1<p<q<\infty$, $1/p-1/q=\alpha$, 则对$\omega\in A^+(p,q)$, 有

$$\|\mathcal{V}_\rho(\mathcal{T}^+_{A,m})f\|_{L^q(w^q)}\lesssim\|A^{(m-1)}\|_{Lip_\alpha}\|f\|_{L^p(w^p)}.$$

定理 2.2 令$m\geq 1$, $0<\alpha<1$且$A^{(k)}\in Lip_\alpha$, $k=0,1,\ldots,m-1$. 设$\mathcal{T}^+_{A,m}=\{T^{\varepsilon,+}_{A,m}\}_{\varepsilon>0}$. 若$1<p<\infty$, $\rho>\max\{2,1/(1-\alpha)\}$, 则对$\omega\in A^+_p$, 有

$$\|\mathcal{V}_\rho(\mathcal{T}^+_{A,m})f\|_{\dot{F}^{\alpha,\infty}_{p,+}}\lesssim\|f\|_{L^p(w)}.$$

为方便起见, 在后文中我们用字母C表示一个与本质变量无关的正常数, 但在不同的位置其值可以不同. 若$f\leq Cg$和$f\lesssim g\lesssim f$, 分别记为$f\lesssim g$, $f\sim g$.

3. 定理 2.1 和定理 2.2 的证明

在证明定理2.1之前, 我们先给出下面的引理.

引理 3.1 *[16]* 若$A^{(m-1)}\in Lip_\alpha$ $(0<\alpha\leq 1)$, 存在常数C使得

$$|R_m(A;x,y)|\leq C\|A^{(m-1)}\|_{Lip_\alpha}|x-y|^{m-1+\alpha};\tag{3.1}$$

$$\begin{aligned}&|R_m(A;x,y)-R_m(A;z,y)|\\ &\le C\|A^{(m-1)}\|_{Lip_\alpha}\Big(\sum_{l=1}^{m-1}|x-z|^l|z-y|^{m-1-l+\alpha}+|x-z|^{m-1+\alpha}\Big).\end{aligned}\tag{3.2}$$

引理 3.2 *[17] 令T 为定义在$C_c^\infty(\mathbb{R})$ 上的次线性算子并满足*

$$\|\omega Tf\|_\infty\le C\|f\omega\|_\infty,$$

其中$\omega^{-1}\in A_1^-$; 那么, 对于$1<p<\infty$, $\omega\in A_p^+$ 有

$$\|Tf\|_{L^p(\omega)}\le C\|f\|_{L^p(\omega)}.$$

引理 3.3 *[18] 设$\omega\in A_1^-$, 存在$\varepsilon_1>0$使得, 对任意的$1<r\le 1+\varepsilon_1$, 有$\omega^r\in A_1^-$.*

证 [定理2.1的证明] 观察到

$$\big\|\{\chi_{(y+\varepsilon_{i+1},y+\varepsilon_i]}(z)\}_{i\in\mathbb{N},\gamma=\{\varepsilon_i\}\in\Theta}\big\|_{F_\rho}\le 1,\quad \forall y\in\mathbb{R}.\tag{3.3}$$

利用(3.1), 可知

$$\begin{aligned}|\mathcal{V}_\rho(\mathcal{T}_{A,m}^+)f(x)|&\le\big\|V(\mathcal{T}_{A,m}^+)f(x)\big\|_{F_\rho}\\ &\le\int_x^\infty\big\|\{\chi_{(x+\varepsilon_{i+1},x+\varepsilon_i]}(y)\}_{i\in\mathbb{N},\gamma=\{\varepsilon_i\}\in\Theta}\big\|_{F_\rho}\frac{|K(x-y)|}{|y-x|^{m-1}}|R_m(A;x,y)f(y)|dy\\ &\lesssim\|A^{(m-1)}\|_{Lip_\alpha}\int_x^\infty\frac{|f(y)|}{|x-y|^{1-\alpha}}dy\\ &=\|A^{(m-1)}\|_{Lip_\alpha}I_\alpha^+(|f|)(x).\end{aligned}$$

利用I_α^+的$L^p(\omega^p)$到$L^q(\omega^q)$有界性, 可得

$$\|\mathcal{V}_\rho(\mathcal{T}_{A,m}^+)f\|_{L^q(w^q)}\lesssim\|A^{(m-1)}\|_{Lip_\alpha}\|I_\alpha^+(|f|)\|_{L^q(w^q)}\lesssim\|A^{(m-1)}\|_{Lip_\alpha}\|f\|_{L^p(w^p)}.$$

这就完成了定理2.1的证明. ∎

证 [定理2.2的证明] 令$x\in\mathbb{R},h>0$ 且$I=[x,x+8h]$. 记$f=f_1+f_2$, 其中$f_1(x):=f\chi_I(x)$. 注意到

$$\begin{aligned}&\frac{1}{h^{1+\alpha}}\int_x^{x+h}\big|\mathcal{V}_\rho(\mathcal{T}_{A,m}^+)f(y)-\big(\mathcal{V}_\rho(\mathcal{T}_{A,m}^+)f\big)_{[x,x+h]}\big|dy\\ &\le\frac{2}{h^{1+\alpha}}\int_x^{x+2h}\big|\mathcal{V}_\rho(\mathcal{T}_{A,m}^+)f(y)-\mathcal{V}_\rho(\mathcal{T}_{A,m}^+)f_2(x)\big|dy\\ &\le\frac{2}{h^{1+\alpha}}\int_x^{x+2h}\big|\mathcal{V}_\rho(\mathcal{T}_{A,m}^+)f_1(y)\big|dy+\frac{2}{h^{1+\alpha}}\int_x^{x+2h}\big|\mathcal{V}_\rho(\mathcal{T}_{A,m}^+)f_2(y)dy-\mathcal{V}_\rho(\mathcal{T}_{A,m}^+)f_2(x)\big|dy\\ &=:I_1(x)+I_2(x).\end{aligned}$$

对于$I_2(x)$, 有

$$\begin{aligned}&\left|V(\mathcal{T}^+_{A,m})f_2(y)-V(\mathcal{T}^+_{A,m})f_2(x)\right|\\ &\le\int_{\mathbb{R}}\frac{|k(y-z)|}{|y-z|^{m-1}}|R_m(A;x,z)-R_m(A;y,z)||f_2(z)|\chi_{(y+\varepsilon_{i+1},y+\varepsilon_i)}(z)dz\\ &+\int_{\mathbb{R}}\left|\frac{k(y-z)}{(z-y)^{m-1}}-\frac{k(x-z)}{(z-x)^{m-1}}\right||R_m(A;x,z)f_2(z)|\chi_{(y+\varepsilon_{i+1},y+\varepsilon_i)}(z)dz\\ &+\int_{\mathbb{R}}\frac{|k(x-z)|}{|x-z|^{m-1}}\left(\chi_{(y+\varepsilon_{i+1},y+\varepsilon_i)}(z)-\chi_{(x+\varepsilon_{i+1},x+\varepsilon_i)}(z)\right)|R_m(A;x,z)f_2(z)|dz\\ &=:J_1(x,y)+J_2(x,y)+J_3(x,y).\end{aligned}$$

由此可得

$$\begin{aligned}\left|\mathcal{V}_\rho(\mathcal{T}^+_{A,m})f_2(y)-\mathcal{V}_\rho(\mathcal{T}^+_{A,m})f_2(x)\right|\le&\left\|V(\mathcal{T}^+_{A,m})f_2(y)dy-V(\mathcal{T}^+_{A,m})f_2(x)\right\|_{F_\rho}\\ \le&\left\|J_1(x,y)\right\|_{F_\rho}+\left\|J_2(x,y)\right\|_{F_\rho}+\left\|J_3(x,y)\right\|_{F_\rho}.\end{aligned}$$

对于$\left\|J_1(x,y)\right\|_{F_\rho}$, 注意到$z\in(x+8h,\infty)$, $y\in(x,x+2h)$. 可得$|x-z|\sim|z-y|$, $|x-y|\le|z-y|$. 由(3.2), 可知

$$\begin{aligned}|R_m(A;x,z)-R_m(A;y,z)|&\lesssim\|A^{(m-1)}\|_{Lip_\alpha}\left(\sum_{l=1}^{m-1}|x-y|^l|z-y|^{m-1-l+\alpha}+|x-y|^{m-1+\alpha}\right)\\ &\lesssim\|A^{(m-1)}\|_{Lip_\alpha}|x-y||z-y|^{m-2+\alpha}.\end{aligned}$$

再结合(2.1)和(3.3), 可得

$$\begin{aligned}\|J_1(x,y)\|_{F_\rho}\le&\int_{\mathbb{R}}\left\|\left\{\chi_{(y+\varepsilon_{i+1},y+\varepsilon_i]}(z)\right\}_{i\in\mathbb{N},\gamma=\{\varepsilon_i\}\in\Theta}\right\|_{F_\rho}\frac{|k(y-z)|}{|y-z|^{m-1}}\times\\ &|R_m(A;x,z)-R_m(A;y,z)||f_2(z)|dz\\ \lesssim&\int_{\mathbb{R}}\frac{|x-y|}{|x-z|^{2-\alpha}}|f_2(z)|dz\\ \le&\,h\int_{\mathbb{R}}\frac{1}{|x-z|^{2-\alpha}}|f_2(z)|dz.\end{aligned}$$

对于$J_2(x,y)$, 注意到

$$\begin{aligned}J_2(x,y)=&\int_{\mathbb{R}}\left|\frac{k(y-z)}{(z-y)^{m-1}}-\frac{k(x-z)}{(z-x)^{m-1}}\right||R_m(A;x,z)f_2(z)|\chi_{(y+\varepsilon_{i+1},y+\varepsilon_i)}(z)dz\\ \le&\int_{\mathbb{R}}\frac{|k(y-z)-k(x-z)|}{|z-y|^{m-1}}|R_m(A;x,z)f_2(z)|\chi_{(y+\varepsilon_{i+1},y+\varepsilon_i)}(z)dz\\ &+\int_{\mathbb{R}}\left|\frac{1}{(z-y)^{m-1}}-\frac{1}{(z-x)^{m-1}}\right||k(y-z)R_m(A;x,z)f_2(z)|\chi_{(y+\varepsilon_{i+1},y+\varepsilon_i)}(z)dz\\ =:&\,J_{21}(x,y)+J_{22}(x,y).\end{aligned}$$

对于$z\in(x+8h,\infty)$, $y\in(x,x+2h)$, 易证$|y-z|\geq 2|x-y|$. 由(2.2), (3.3)和(3.1), 可得

$$\begin{aligned}\|J_{21}(x,y)\|_{F_\rho}&\leq\int_{\mathbb{R}}\left\|\left\{\chi_{(y+\varepsilon_{i+1},y+\varepsilon_i]}(z)\right\}_{i\in\mathbb{N},\gamma=\{\varepsilon_i\}\in\Theta}\right\|_{F_\rho}\times\\&\qquad\frac{|k(y-z)-k(x-z)|}{|z-y|^{m-1}}|R_m(A;x,z)f_2(z)|dz\\&\lesssim\int_{\mathbb{R}}\frac{|x-y|}{|x-z|^{2-\alpha}}|f_2(z)|dz\\&\leq h\int_{\mathbb{R}}\frac{1}{|x-z|^{2-\alpha}}|f_2(z)|dz.\end{aligned}$$

利用类似于$\|J_{21}(x,y)\|_{F_\rho}$的处理, 可得

$$\|J_{22}(x,y)\|_{F_\rho}\lesssim h\int_{\mathbb{R}}\frac{1}{|x-z|^{2-\alpha}}|f_2(z)|dz.$$

综上可得

$$\|J_2(x,y)\|_{F_\rho}\lesssim h\int_{\mathbb{R}}\frac{1}{|x-z|^{2-\alpha}}|f_2(z)|dz.$$

对于$\{\varepsilon_i\}\in\Theta$, 令$\mathbb{N}_1=\{i\in\mathbb{Z}:\varepsilon_i-\varepsilon_{i+1}\geq y-x\}$, $\mathbb{N}_2=\{i\in\mathbb{Z}:\varepsilon_i-\varepsilon_{i+1}<y-x\}$. 则

$$\begin{aligned}\|J_3(x,y)\|_{F_\rho}&\leq\Big\|\Big\{\int_{\mathbb{R}}\big(\chi_{(y+\varepsilon_{i+1},y+\varepsilon_i)}(z)-\chi_{(x+\varepsilon_{i+1},x+\varepsilon_i)}(z)\big)\times\\&\qquad\frac{k(x-z)}{(z-x)^{m-1}}R_m(A;x,z)f_2(z)dz\Big\}_{i\in\mathbb{N}_1,\beta=\{\varepsilon_i\}\in\Theta}\Big\|_{F_\rho}\\&\quad+\Big\|\Big\{\int_{\mathbb{R}}\big(\chi_{(y+\varepsilon_{i+1},y+\varepsilon_i)}(z)-\chi_{(x+\varepsilon_{i+1},x+\varepsilon_i)}(z)\big)\times\\&\qquad\frac{k(x-z)}{(z-x)^{m-1}}R_m(A;x,z)f_2(z)dz\Big\}_{i\in\mathbb{N}_2,\beta=\{\varepsilon_i\}\in\Theta}\Big\|_{F_\rho}\\&=:\|J_{31}(x,y)\|_{F_\rho}+\|J_{32}(x,y)\|_{F_\rho}.\end{aligned}$$

对于$i\in\mathbb{N}_1$, 易证

$$\begin{aligned}\|J_{31}(x,y)\|_{F_\rho}&\leq\Big\|\Big\{\int_{\mathbb{R}}\chi_{(x+\varepsilon_{i+1},y+\varepsilon_{i+1})}(z)\frac{k(x-z)}{(z-x)^{m-1}}R_m(A;x,z)f_2(z)dz\Big\}_{i\in\mathbb{N}_1,\beta=\{\varepsilon_i\}\in\Theta}\Big\|_{F_\rho}\\&\quad+\Big\|\Big\{\int_{\mathbb{R}}\chi_{(x+\varepsilon_i,y+\varepsilon_i)}(z)\frac{k(x-z)}{(z-x)^{m-1}}R_m(A;x,z)f_2(z)dz\Big\}_{i\in\mathbb{N}_1,\beta=\{\varepsilon_i\}\in\Theta}\Big\|_{F_\rho}\\&=:\|L_1(x,y)\|_{F_\rho}+\|L_2(x,y)\|_{F_\rho}.\end{aligned}$$

设$1/(1-\alpha)<r<\rho$, 由Hölder不等式及(2.1), (3.1), 可得

$$\begin{aligned}\|L_1(x,y)\|_{F_\rho}&\lesssim\Big\|\Big\{\int_{\mathbb{R}}\chi_{(x+\varepsilon_{i+1},y+\varepsilon_{i+1})}(z)\frac{|f_2(z)|}{|z-x|^{1-\alpha}}dz\Big\}_{i\in\mathbb{N}_1,\beta=\{\varepsilon_i\}\in\Theta}\Big\|_{F_\rho}\\&\le h^{1/r'}\Big(\sup_\beta\sum_{i\in\mathbb{N}_1}\Big(\int_{\mathbb{R}}\chi_{(x+\varepsilon_{i+1},x+\varepsilon_i)}(z)\frac{|f_2(z)|^r}{|z-x|^{(1-\alpha)r}}dz\Big)^{\rho/r}\Big)^{1/\rho}\\&\le h^{1/r'}\Big(\int_{\mathbb{R}}\frac{|f_2(z)|^r}{|z-x|^{(1-\alpha)r}}dz\Big)^{1/r}.\end{aligned}$$

类似可得

$$\|L_2(x,y)\|_{F_\rho}\lesssim h^{1/r'}\Big(\int_{\mathbb{R}}\frac{|f_2(z)|^r}{|z-x|^{(1-\alpha)r}}dz\Big)^{1/r}.$$

对于$i\in\mathbb{N}_2$, 注意到$\varepsilon_i-\varepsilon_{i+1}\ge y-x<2h$. 由Hölder不等式及(2.1), (3.1), 通过类似对于$\|L_1(x,y)\|_{F_\rho}$的处理, 有

$$\begin{aligned}\|J_{32}(x,y)\|_{F_\rho}\le&\Big\|\Big\{\int_{\mathbb{R}}\chi_{(x+\varepsilon_{i+1},x+\varepsilon_i)}(z)\frac{k(x-z)}{(z-x)^{m-1}}R_m(A;x,z)f_2(z)dz\Big\}_{i\in\mathbb{N}_2,\beta=\{\varepsilon_i\}\in\Theta}\Big\|_{F_\rho}\\&+\Big\|\Big\{\int_{\mathbb{R}}\chi_{(y+\varepsilon_{i+1},y+\varepsilon_i)}(z)\frac{k(x-z)}{(z-x)^{m-1}}R_m(A;x,z)f_2(z)dz\Big\}_{i\in\mathbb{N}_2,\beta=\{\varepsilon_i\}\in\Theta}\Big\|_{F_\rho}\\&\lesssim h^{1/r'}\Big(\sup_\beta\sum_{i\in\mathbb{N}_2}\Big(\int_{\mathbb{R}}\chi_{(x+\varepsilon_{i+1},x+\varepsilon_i)}(z)\frac{|f_2(z)|^r}{|z-x|^{(1-\alpha)r}}dz\Big)^{\rho/r}\Big)^{1/\rho}\\&\lesssim h^{1/r'}\Big(\int_{\mathbb{R}}\frac{|f_2(z)|^r}{|z-x|^{(1-\alpha)r}}dz\Big)^{1/r}.\end{aligned}$$

综上可知,

$$\|J_3(x,y)\|_{F_\rho}\lesssim h^{1/r'}\Big(\int_{\mathbb{R}}\frac{|f_2(z)|^r}{|z-x|^{(1-\alpha)r}}dz\Big)^{1/r}.$$

考虑定义在$C_c^\infty(\mathbb{R})$上的三个次线性算子:

$$M_1^+f(x)=\sup_{h>0}\frac{1}{h^{1+\alpha}}\int_x^{x+2h}\big|\mathcal{V}_\rho(\mathcal{T}_{A,m}^+)f_2(y)\big|dy,$$

$$M_2^+f(x)=\sup_{h>0}\frac{1}{h^{1+\alpha}}\int_x^{x+2h}\int_{\mathbb{R}}\frac{h}{|x-z|^{2-\alpha}}|f_2(z)|dzdy,$$

$$M_3^+f(x)=\sup_{h>0}\frac{1}{h^{1+\alpha}}\int_x^{x+2h}h^{1/r'}\Big(\int_{\mathbb{R}}\frac{|f_2(z)|^r}{|z-x|^{(1-\alpha)r}}dz\Big)^{1/r}dy.$$

对于$\omega^{-1} \in A_1^-$, 利用引理3.3, 存在$s > 1$使得$\omega^{-s} \in A_1^-$. 对于$I(x)$, 由Hölder不等式和定理2.1, 可得

$$\begin{aligned}\frac{1}{h^{1+\alpha}} \int_x^{x+2h} \left|\mathcal{V}_\rho(\mathcal{T}_{A,m}^+) f_2(y)\right| dy &\leq \frac{1}{h^{\alpha+1/t}} \Big(\int_x^{x+2h} \left|\mathcal{V}_\rho(\mathcal{T}_{A,m}^+) f_1(y)\right|^t dy \Big)^{1/t} \\ &\lesssim \frac{1}{h^{\alpha+1/t}} \Big(\int_x^{x+2h} |f_1(y)|^s dy \Big)^{1/s} \\ &\lesssim \frac{h^{1/s}}{h^{\alpha+1/t}} \Big(\frac{1}{8h} \int_x^{x+8h} |f(y)|^s \omega(y)^s \omega(y)^{-s} dy \Big)^{1/s} \\ &\lesssim \|f\omega\|_\infty \omega(x)^{-1},\end{aligned}$$

其中$1/s - 1/t = \alpha$, $\omega^{-s} \in A_1^-$ 对于$\omega^{-1} \in A_1^-$. 那么,

$$\|\omega M_1^+ f\|_\infty \lesssim \|f\omega\|_\infty.$$

再结合引理3.2, 可知

$$\|M_1^+ f\|_{L^p(\omega)} \lesssim \|f\|_{L^p(\omega)}. \tag{3.4}$$

对于$M_2^+ f$, 通过Hölder不等式可得

$$\begin{aligned}&\frac{1}{h^{1+\alpha}} \int_x^{x+2h} \int_{\mathbb{R}} \frac{h}{|x-z|^{2-\alpha}} |f_2(z)| dz dy \\ &\leq \frac{1}{h^\alpha} \int_x^{x+2h} \sum_{k=3}^\infty \frac{1}{(2^k h)^{2-\alpha}} \int_{x+2^k h}^{x+2^{k+1}h} |f(z)| dz dy \\ &\leq \sum_{k=3}^\infty \frac{1}{2^{k(1-\alpha)}} \Big(\frac{1}{2^{k+1}h} \int_x^{x+2^{k+1}h} |f(z)|^s \omega(z)^s \omega(z)^{-s} dz \Big)^{1/s} \\ &\leq \|f\omega\|_\infty \sum_{k=3}^\infty \frac{1}{2^{k(1-\alpha)}} \Big(\frac{1}{2^{k+1}h} \int_x^{x+2^{k+1}h} \omega(z)^{-s} dz \Big)^{1/s} \\ &\lesssim \|f\omega\|_\infty \omega(x)^{-1},\end{aligned}$$

其中$0 < \alpha < 1$, $\omega^{-s} \in A_1^-$ 对于$\omega^{-1} \in A_1^-$. 则

$$\|\omega M_2^+ f\|_\infty \lesssim \|f\omega\|_\infty.$$

再结合引理3.2推得

$$\|M_2^+ f\|_{L^p(\omega)} \lesssim \|f\|_{L^p(\omega)}. \tag{3.5}$$

对于$M_3^+ f$, 有

$$
\begin{aligned}
&\frac{1}{h^{1+\alpha}}\int_x^{x+2h} h^{1/r'}\Big(\int_{\mathbb{R}}\frac{|f_2(z)|^r}{|z-x|^{(1-\alpha)r}}dz\Big)^{1/r}dy\\
&\le \frac{1}{h^{\alpha+1/r}}\int_x^{x+2h}\Big(\sum_{k=3}^{\infty}\frac{1}{(2^k h)^{(1-\alpha)r}}\int_{x+2^k h}^{x+2^{k+1}h}|f(z)|dz\Big)^{1/r}dy\\
&\lesssim \sum_{k=3}^{\infty}\frac{1}{2^{k(1-\alpha-1/r)}}\Big(\frac{1}{2^{k+1}h}\int_x^{x+2^{k+1}h}|f(z)|^r dz\Big)^{1/r}\\
&\le \sum_{k=3}^{\infty}\frac{1}{2^{k(1-\alpha-1/r)}}\Big(\frac{1}{2^{k+1}h}\int_x^{x+2^{k+1}h}|f(z)|^r\omega(z)^r\omega(z)^{-r}dz\Big)^{1/r}\\
&\le \|f\omega\|_\infty\sum_{k=3}^{\infty}\frac{1}{2^{k(1-\alpha-1/r)}}\Big(\frac{1}{2^{k+1}h}\int_x^{x+2^{k+1}h}\omega(z)^{-r}dz\Big)^{1/r}\\
&\lesssim \|f\omega\|_\infty\omega(x)^{-1},
\end{aligned}
$$

其中$0<\alpha<1$, $\omega^{-r}\in A_1^-$ 对于$\omega^{-1}\in A_1^-$. 因此,

$$\|\omega M_3^+ f\|_\infty \lesssim \|f\omega\|_\infty.$$

利用引理3.2, 得到

$$\|M_3^+ f\|_{L^p(\omega)} \lesssim \|f\|_{L^p(\omega)}.$$

再结合(3.4)及(3.5), 完成了定理2.2的证明. ∎

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