**Pure Mathematics 理论数学, 2023, 13(7), 2155-2168**
Published Online July 2023 in Hans. https://www.hanspub.org/journal/pm
https://doi.org/10.12677/pm.2023.137223

# 内蕴平方函数在 Morrey-Adams 空间上的有界性

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收稿日期：2023年6月18日；录用日期：2023年7月24日；发布日期：2023年7月31日

## 摘 要

**借助 Lebesgue 空间上的有界性, 利用函数分解方法和实变技巧, 证明了内蕴平方函数在 Morrey-Adams 空间上的有界性, 同时也给出了** BMO **交换子的相应结果。**

## 关键词

**内蕴平方函数，Morrey-Adams 空间，交换子**

# Boundedness of Intrinsic Square Functions on Morrey-Adams Spaces

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Received: Jun. $18^{th}$, 2023; accepted: Jul. $24^{th}$, 2023; published: Jul. $31^{st}$, 2023

## Abstract

**With the help of the boundedness of the Lebesgue spaces, by applying the decom-**

文章引用: 朱小杰. 内蕴平方函数在 Morrey-Adams 空间上的有界性[J]. 理论数学, 2023, 13(7): 2155-2168.
DOI: 10.12677/pm.2023.137223

position of function and real variable techniques, the boundedness of intrinsic square functions is obtained on Morrey-Adams spaces. Meanwhile, the corresponding result of its commutator with BMO functions is also given.

## Keywords

**Intrinsic Square Functions, Morrey-Adams Space, Commutator**



## 1. 引言及主要结果

记 $\mathbb{R}^{n+1}_{+}=\mathbb{R}\times(0,\infty)$, $\mu(x,t)=P_t*f(x)$, 其中 $P_t(x)=c_n\frac{t}{(t^2+|x|^2)^{\frac{n+1}{2}}}$ 表示 $\mathbb{R}^{n+1}_{+}$ 上的 Poisson 核.经典平方函数(Lusin面积积分)定义为

$$\mathcal{S}_\beta(f)(x)=\left(\int\int_{\Gamma_\beta(x)}|\bigtriangledown\mu(y,t)|^2\,t^{1-n}\mathrm{d}y\mathrm{d}t\right)^{\frac{1}{2}},$$

其中: $|\bigtriangledown\mu(y,t)|=|\frac{\partial\mu}{\partial t}|^2+\sum_{j=1}^{n}|\frac{\partial\mu}{\partial y_j}|^2$; $\Gamma_\beta(x)=\left\{(y,t)\in\mathbb{R}^{n+1}_{+}:|x-y|<\beta t\right\}$, $\beta>0$. 当 $\beta=1$, 时,记 $\mathcal{S}_\beta(f)$ 和 $\Gamma_\beta(x)$ 分别为 $\mathcal{S}(f)$ 和 $\Gamma(x)$. 相应地, Littlewood-Paley $g$ 函数和 $g^*_\lambda$ 函数分别定义为

$$g(f)(x)=\left(\int_0^\infty|\bigtriangledown\mu(y,t)|^2\,t\mathrm{d}t\right)^{\frac{1}{2}},$$

$$g^*_\gamma(f)(x)=\left[\int\int_{\mathbb{R}^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{\gamma n}|\bigtriangledown\mu(y,t)|^2\,t^{1-n}\mathrm{d}y\mathrm{d}t\right]^{\frac{1}{2}}.$$

经典平方函数 [1,2]在调和分析中具有重要作用,内蕴平方函数的广义函数 [1]是对经典平方函数的推广.设 $0<\alpha\le 1$, 函数 $\varphi:\mathbb{R}^n\to\mathbb{R}$ 满足 $\sup p\varphi\subset\{x\in\mathbb{R}^n:|x|\le 1\}$, $\int_{\mathbb{R}^n}\varphi(x)\mathrm{d}x=0$, 并对任意的 $x_1,x_2\in\mathbb{R}^n$,有 $|\varphi(x_1)-\varphi(x_2)|\le|x_1-x_2|^\alpha$. 满足上述条件的 $\varphi$ 构成的函数族用 $\mathcal{C}_\alpha$ 表示.对于 $(y,t)\in\mathbb{R}^{n+1}_{+}$, $f\in L^1_{loc}(\mathbb{R}^n)$, 记

$$A_\alpha(f)(y,t)=\sup_{\varphi\in\mathcal{C}_\alpha}|f*\varphi_t(y)|=\sup_{\varphi\in\mathcal{C}_\alpha}\left|\int_{\mathbb{R}^n}\varphi_t(y-z)f(z)\mathrm{d}z\right|,$$

其中 $\varphi_t(x)=t^{-n}\varphi(\frac{x}{t})$. $f$ 的内蕴平方函数定义为

$$\mathcal{S}_\alpha(f)(x)=\left(\int\int_{\Gamma(x)}(A_\alpha(f)(y,t))^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\right)^{\frac{1}{2}}. \tag{1}$$

相应地

$$\mathcal{S}_{\alpha,\beta}(f)(x)=\left(\int\int_{\Gamma_\beta(x)}(A_\alpha(f)(y,t))^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\right)^{\frac{1}{2}}.$$

内蕴 Littlewood-Paley $g$ 和 $g_\lambda^*$ 函数分别

$$g_\alpha(f)(x)=\left(\int_0^\infty(A_\alpha(f)(y,t))^2\frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}}, \tag{2}$$

$$g_{\gamma,\alpha}^*(f)(x)=\left[\int\int_{\mathbb{R}^{n+1}}\left(\frac{t}{t+\mid x-y\mid}\right)^{\gamma n}(A_\alpha(f)(y,t))^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\right]^{\frac{1}{2}}. \tag{3}$$

设 $b$ 是 $\mathbb{R}^n$ 上局部可积函数,内蕴平方交换子分别定义为

$$[b,\mathcal{S}_\alpha](f)(x)=\left(\int\int_{\Gamma(x)}\sup_{\varphi\in\mathcal{C}_\alpha}\left|\int_{\mathbb{R}^n}[b(x)-b(z)]\varphi_t(y-z)f_2(z)\mathrm{d}z\right|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\right)^{\frac{1}{2}}, \tag{4}$$

$$[b,g_\alpha](f)(x)=\left\{\int_0^\infty\sup_{\varphi\in\mathcal{C}_\alpha}\left|\int_{\mathbb{R}^n}[b(y)-b(z)]\varphi_t(y-z)f_2(z)\mathrm{d}z\right|^2\frac{\mathrm{d}t}{t}\right\}^{\frac{1}{2}}, \tag{5}$$

$$[b,g_{\gamma,\alpha}^*](f)(x)=\left\{\int\int_{\mathbb{R}^{n+1}}\left(\frac{t}{t+\mid x-y\mid}\right)^{\gamma n}\sup_{\varphi\in\mathcal{C}_\alpha}\left|\int_{\mathbb{R}^n}[b(x)-b(z)]\varphi_t(y-z)f_2(z)\mathrm{d}z\right|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\right\}^{\frac{1}{2}}. \tag{6}$$

**定义 1** [3] 设 $0\le\lambda\le n, 1\le p<\infty$, Morrey 空间定义为

$$L^{p,\lambda}(\mathbb{R}^n)=\left\{f(x)\in L_{loc}^p:\parallel f\parallel_{L^{p,\lambda}}=\sup_{x\in\mathbb{R}^n,r>0}r^{-\frac{\lambda}{p}}\parallel f\parallel_{L^p(B(x,r))}<\infty\right\},$$

其中, $B(x,r)=\{y\in\mathbb{R}^n:\ |x-y|<r\}$.

注意到, 定义 1 中的 Morrey 范数 $\|\cdot\|_{L^{p,\lambda}}$ 可写成如下形式

$$\parallel f\parallel_{L^{p,\lambda}}=\sup_{x\in\mathbb{R}^n}\parallel|\cdot|^{-\frac{\lambda}{p}}\parallel f\parallel_{L^p(B(x,\cdot))}\|_{L^\infty(0,\infty)}.$$

1938 年 Morrey 在研究二阶椭圆偏微分方程解的局部特征性质时首次引进了经典的 Morrey 空间 [3], 这类函数空间不仅可视为经典 Lebesgue 空间的推广, 而且在偏微分方程等领域有着重要应用. 1981 年, Adams 在文献 [4]中用 $L^\theta$ 范数 $\|\cdot\|_{L^\theta(0,\infty)}$ 代替 Morrey 范数中的 $L^\infty$ 范数 $\|\cdot\|_{L^\infty(0,\infty)}$, 得到了一种范围更广的 Morrey 空间 $L_\theta^{p,\lambda}(\mathbb{R}^n)$. 近些年, Morrey 型空间上的算子有界性被国内外许多学者广泛关注, 见文献 [5–8]. 最近, Salim 和 Budhi 在文献 [9]中证明了粗糙核分数次积

分算子在 Morrey-Adams 空间上的有界性. 受上述文献研究结果的启发, 本文主要讨论了内蕴平方函数及其与 BMO 函数生成的交换子在 Morrey-Adams 空间上的有界性. 为此, 我们首先回顾 Morrey-Adams 空间的定义.

**定义 2** [9] 设 $1 \le p < \infty, \lambda \in \mathbb{R}, 1 \le \theta < \infty$ , Morrey-Adams 空间定义为

$$L_\theta^{p,\lambda}(\mathbb{R}^n) = \left\{ f(x) \in L_{loc}^p : \| f \|_{L_\theta^{p,\lambda}} = \sup_{x\in\mathbb{R}^n} \left( \int_0^\infty r^{-\frac{\lambda\theta}{p}} \| f \|_{L^p(B(x,r))}^\theta \,\mathrm{d}r \right)^{\frac{1}{\theta}} < \infty \right\}.$$

如果 $\frac{p}{\theta} < \lambda < n + \frac{p}{\theta}$, 那么 $L_\theta^{p,\lambda}$ 空间是非平凡的. 例如, 取 $f(x) = \chi_{B(0,r_0)}$, 则有 $\| \chi_{B(0,r_0)} \|_{L_\theta^{p,\lambda}} = c_n r_0^{\frac{n-\lambda}{p}+\frac{1}{\theta}}$, 其中 $c_n$ 是正常数, 从而 $f \in L_\theta^{p,\lambda}$, 并且在 $L_\theta^{p,\lambda}$ 空间中 $\lambda$ 可以大于函数值域的维数.

**定义 3** [10] BMO 空间定义为

$$\mathrm{BMO}(\mathbb{R}^n) = \{ f \in L_{loc}^1(\mathbb{R}^n) : \| f \|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{| B(x,r) |} \int_{B(x,r)} | f(y) - f_{B(x,r)} | \,\mathrm{d}y < \infty \}.$$

其中 $f_B = \frac{1}{|B|} \int_B f(y)\mathrm{d}y$.

**本文的主要结果如下.**

**定理 1** 设 $0 < \alpha \le 1$且 $1 < p < \infty$, $S_\alpha$ 由(1)式所定义. 如果 $\lambda < n + \frac{p}{\theta}$, 那么存在不依赖于 $f$ 的常数 $C > 0$, 使得

$$\| S_\alpha(f) \|_{L_\theta^{p,\lambda}} \le C \|f\|_{L_\theta^{p,\lambda}}.$$

**定理 2** 设 $0 < \alpha \le 1, 1 < p < \infty$ 且 $b \in \mathrm{BMO}(\mathbb{R}^n)$, $[b, \mathcal{S}_\alpha]$ 由(4)式所定义.如果 $\lambda < n + \frac{p}{\theta}$, 那么存在不依赖于 $f$ 的常数 $C > 0$, 使得

$$\| [b, \mathcal{S}_\alpha](f) \|_{L_\theta^{p,\lambda}} \le C \| f \|_{L_\theta^{p,\lambda}} .$$

**定理 3** 设 $0 < \alpha \le 1$且 $1 < p < \infty$, $g_{\gamma,\alpha}^*$ 由(3)式所定义. 如果 $\lambda < n + \frac{p}{\theta}, \gamma > 3 > \frac{2}{p}$, 那么存在不依赖于 $f$ 的常数 $C > 0$, 使得

$$\| g_{\gamma,\alpha}^*(f) \|_{L_\theta^{p,\lambda}} \le C \|f\|_{L_\theta^{p,\lambda}}.$$

**定理 4** 设 $0 < \alpha \le 1, 1 < p < \infty$ 且 $b \in \mathrm{BMO}(\mathbb{R}^n)$, $[b, g_{\gamma,\alpha}^*]$ 由(6)式所定义.如果 $\lambda < n + \frac{p}{\theta}$, $\gamma > 3$, 那么存在不依赖于 $f$ 的常数 $C > 0$, 使得

$$\| [b, g_{\gamma,\alpha}^*](f) \|_{L_\theta^{p,\lambda}} \le C \|f\|_{L_\theta^{p,\lambda}}.$$

在文献{1中, Wilson 证得对任意 $0 < \alpha \le 1$ 时, $g_\alpha(f)$ 可由 $S_\alpha(f)$ 逐点控制,因此由定理 1 和定理 2 可得下面的推论.

**推论 5** 设 $g_\alpha$ 由(2)式所定义,在定理 1 的条件下,存在不依赖于 $f$ 的常数 $C > 0$, 使得

$$\| g_\alpha(f) \|_{L_\theta^{p,\lambda}} \le C \|f\|_{L_\theta^{p,\lambda}}.$$

**推论 6** 设 $[b, g_\alpha]$ 由(5)式所定义,在定理 2 的条件下,存在不依赖于 $f$ 的常数 $C > 0$, 使得

$$\| [b, g_\alpha](f) \|_{L_\theta^{p,\lambda}} \leq C \| f \|_{L_\theta^{p,\lambda}} .$$

## 2. 定理的证明

为证明定理, 我们需要下面引理.

**引理 1** [1,2] 设 $0 < \alpha \leq 1, 1 < p < \infty$, 则存在不依赖于 $f$ 的常数 $C > 0$, 使得

$$\| \mathcal{S}_\alpha(f) \|_{L^p} \leq C \| f \|_{L^p} .$$

**引理 2** [11] 设 $0 < \alpha \leq 1$, 且 $2 < p < \infty$, 则对任意 $i \in \mathbb{Z}_+$, 存在不依赖于 $f$ 的常数 $C > 0$, 使得

$$\| \mathcal{S}_{\alpha,2^i}(f) \|_{L^p} \leq C \cdot 2^{\frac{in}{2}} \| \mathcal{S}_\alpha(f) \|_{L^p} .$$

**引理 3** [11] 设 $0 < \alpha \leq 1$, 且 $1 \leq p < 2$, 则对任意 $i \in \mathbb{Z}_+$, 存在不依赖于 $f$ 的常数 $C > 0$, 使得

$$\| \mathcal{S}_{\alpha,2^i}(f) \|_{L^p} \leq C \cdot 2^{\frac{in}{p}} \| \mathcal{S}_\alpha(f) \|_{L^p} .$$

**引理 4** [11] 设 $0 < \alpha \leq 1, 1 < p < \infty$, 则当 $\gamma > 3, b \in \mathrm{BMO}(\mathbb{R}^n)$, 时,交换子 $[b, g^*_{\gamma,\alpha}]$ 和 $[b, g^*_{\gamma,\alpha}]$ 是
$L^p(\mathbb{R}^n)$ 上的有界算子.

**引理 5** [12,13] 设 $1 \leq p < \infty$, 则当 $b \in \mathrm{BMO}(\mathbb{R}^n)$ 时,有

$$\sup_{x\in\mathbb{R}^n, r>0} \left( \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(x) - b_{B(x,r)}|^p \mathrm{d}x \right)^{\frac{1}{p}} \leq C \| b \|_* .$$

**定理 1的证明** 设 $B = B(x_0, r)$ 是 $\mathbb{R}^n$ 中的一个以 $x_0$ 为中心,以 $r$ 为半径的球, 把 $f$ 分解为 $f = f_1 + f_2$, 其中 $f_1 = f\chi_{2B}$, $f$ 的分解依赖于 $r$. 于是

$$\begin{aligned}
& \left( \int_0^\infty r^{-\frac{\lambda\theta}{p}} \| S_\alpha(f) \|^\theta_{L^p(B(x_0,r))} \mathrm{d}r \right)^{\frac{1}{\theta}} \\
\leq & \left( \int_0^\infty r^{-\frac{\lambda\theta}{p}} \| S_\alpha(f_1) \|^\theta_{L^p(B(x_0,r))} \mathrm{d}r \right)^{\frac{1}{\theta}} + \left( \int_0^\infty r^{-\frac{\lambda\theta}{p}} \| S_\alpha(f_2) \|^\theta_{L^p(B(x_0,r))} \mathrm{d}r \right)^{\frac{1}{\theta}} \\
:= & I_1 + I_2.
\end{aligned}$$

对 $I_1$, 由引理 1, 得

$$\begin{aligned}
I_1 &\leq C \left( \int_0^\infty r^{-\frac{\lambda\theta}{p}} \left( \int_B |f(x)|^p \mathrm{d}x \right)^{\frac{\theta}{p}} \mathrm{d}r \right)^{\frac{1}{\theta}} \\
&\leq C \| f \|_{L_\theta^{p,\lambda}} .
\end{aligned}$$

对 $I_2$, 由任意的 $\varphi \in \mathcal{C}_\alpha, 0 < \alpha \le 1$, 且 $(y,t) \in \Gamma(x)$, 有

$$\begin{aligned} |\ f_2 * \varphi_t(y)\ | &= |\int_{(2B)^c} \varphi_t(y-z) f(z) \mathrm{d}z\ | \\ &\le C \cdot t^{-n} \int_{(2B)^c \bigcap \{z:|y-z|\le t\}} |\ f(z)\ | \,\mathrm{d}z \\ &\le C \cdot t^{-n} \sum_{j=1}^{\infty} \int_{(2^{j+1}B \backslash 2^j B) \bigcap \{z:|y-z|\le t\}} |\ f(z)\ | \,\mathrm{d}z. \end{aligned}$$

注意到当 $x \in B$, $(y,t) \in \Gamma(x)$, $z \in (2^{j+1} \backslash 2^j B) \bigcap B(y,t)$ 时,有

$$2^{j-1} r \le |\ z - x_0\ | - |\ x - x_0\ | \le |\ x - z\ | \le |\ x - y\ | + |\ y - z\ | \le 2t.$$

因此,利用上式和 Minkowski 不等式,得

$$\begin{aligned} |\ S_\alpha(f_2)(x)\ | &\le C \left( \int_{2^{j-2} r_B}^{\infty} \int_{|x-y|<t} \left| t^{-n} \sum_{j=1}^{\infty} \int_{2^{j+1}B \backslash 2^j B} |\ f(z)\ | \,\mathrm{d}z \right|^2 \frac{\mathrm{d}y \mathrm{d}t}{t^{n+1}} \right)^{\frac{1}{2}} \\ &\le C \left( \sum_{j=1}^{\infty} \int_{2^{j+1}B \backslash 2^j B} |\ f(z)\ | \,\mathrm{d}z \right) \left( \int_{2^{j-2} r}^{\infty} \frac{\mathrm{d}t}{t^{2n+1}} \right)^{\frac{1}{2}} \\ &\le C \sum_{j=1}^{\infty} \frac{1}{|\ 2^{j+1} B\ |} \int_{2^{j+1}B} |\ f(z)\ | \,\mathrm{d}z. \end{aligned}$$

利用 Hölder 不等式,得

$$\begin{aligned} \frac{1}{|\ 2^{j+1} B\ |} \int_{2^{j+1}B} |\ f(z)\ | \,\mathrm{d}z &\le \frac{1}{|\ 2^{j+1} B\ |} \left( \int_{2^{j+1}B} |\ f(z)\ |^p \,\mathrm{d}z \right)^{\frac{1}{p}} \left( \int_{2^{j+1}B} \mathrm{d}z \right)^{\frac{1}{p'}} \\ &\le \frac{1}{|\ 2^{j+1} B\ |} \left( \int_{2^{j+1}B} |\ f(z)\ |^p \,\omega(z) \mathrm{d}z \right)^{\frac{1}{p}} |\ 2^{j+1} B\ |^{\frac{1}{p'}} \\ &\le C (2^j r)^{-\frac{n}{p}} \left( \int_{2^{j+1}B} |\ f(z)\ |^p \,\mathrm{d}z \right)^{\frac{1}{p}}. \end{aligned}$$

由 Minkowski 不等式,且令 $t = 2^{j+1} r$ ,得

$$\begin{aligned} I_2 &\le \sum_{j=1}^{\infty} 2^{-\frac{jn}{p}} \left( r^{-\frac{\lambda\theta}{p}} \parallel f \parallel_{L^p(B(x_0, 2^{j+1} r))}^{\theta} \mathrm{d}r \right)^{\frac{1}{\theta}} \\ &\le C \sum_{j=1}^{\infty} 2^{\frac{j(\lambda-n)}{p}} \left( \int_0^{\infty} t^{-\frac{\lambda\theta}{p}} \parallel f \parallel_{L^p(B(x_0,t))}^{\theta} \frac{\mathrm{d}t}{2^j} \right)^{\frac{1}{\theta}} \\ &\le C \parallel f \parallel_{L_\theta^{p,\lambda}} \sum_{j=1}^{\infty} 2^{\frac{j(\lambda-n)}{p} - \frac{j}{\theta}}. \end{aligned}$$

注意到当 $\lambda < n + \frac{p}{\theta}$, 级数 $\sum_{j=1}^{\infty} 2^{\frac{j(\lambda-n)}{p} - \frac{j}{\theta}}$ 收敛. 定理1 证毕.

DOI: 10.12677/pm.2023.137223

**定理 2的证明** 设 $B=B(x_0,r)$ 是 $\mathbb{R}^n$ 中的一个以 $x_0$ 为中心,以 $r$ 为半径的球, 把 $f$ 分解为 $f=f_1+f_2$, 其中 $f_1=f\chi_{2B}$, $f$ 的分解依赖于 $r$. 于是

$$\begin{aligned}&\left(\int_0^\infty r^{-\frac{\lambda\theta}{p}}\parallel S_{b,\alpha}(f)\parallel_{L^p(B(x_0,r))}^\theta \mathrm{d}r\right)^{\frac{1}{\theta}}\\ \leq\ &\left(\int_0^\infty r^{-\frac{\lambda\theta}{p}}\parallel S_{b,\alpha}(f_1)\parallel_{L^p(B(x_0,r))}^\theta \mathrm{d}r\right)^{\frac{1}{\theta}}\\ &+\left(\int_0^\infty r^{-\frac{\lambda\theta}{p}}\parallel S_{b,\alpha}(f_2)\parallel_{L^p(B(x_0,r))}^\theta \mathrm{d}r\right)^{\frac{1}{\theta}}\\ :=\ &T_1+T_2.\end{aligned}$$

对 $T_1$, 由引理 4,得

$$T_1\leq C\left(\int_0^\infty r^{-\frac{\lambda\theta}{p}}\left(\int_B\mid f_1(x)\mid^p \mathrm{d}x\right)^{\frac{\theta}{p}}\mathrm{d}r\right)^{\frac{1}{\theta}}\leq C\parallel f\parallel_{L_\theta^{p,\lambda}}.$$

对 $T_2$ ,注意到对任意的 $x\in B$, $(y,t)\in\Gamma(x)$, 有

$$\begin{aligned}&\sup_{\varphi\in\mathcal{C}_\alpha}\left|\int_{\mathbb{R}^n}[b(x)-b(z)]\varphi_t(y-z)f_2(z)\mathrm{d}z\right|\\ \leq\ &\mid b(x)-b_B\mid\sup_{\varphi\in\mathcal{C}_\alpha}\left|\int_{\mathbb{R}^n}\varphi_t(y-z)f_2(z)\mathrm{d}z\right|\\ &+\sup_{\varphi\in\mathcal{C}_\alpha}\left|\int_{\mathbb{R}^n}[b(z)-b_B]\varphi_t(y-z)f_2(z)\mathrm{d}z\right|.\end{aligned}$$

从而,

$$\begin{aligned}&\mid[b,\mathcal{S}_\alpha(f_2)(x)\mid\\ \leq\ &\mid b(x)-b_B\mid\mid\mathcal{S}_\alpha(f_2)(x)\mid\\ &+\left(\int\int_{\Gamma(x)}\sup_{\varphi\in\mathcal{C}_\alpha}\left|\int_{\mathbb{R}^n}[b(z)-b_B]\varphi_t(y-z)f_2(z)\mathrm{d}z\right|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\right)^{\frac{1}{2}}\\ =:\ &V_1+V_2.\end{aligned}$$

利用 Hölder 不等式,有

$$\begin{aligned}V_1\ &\leq\ C\mid b(x)-b_B\mid\sum_{j=1}^\infty\frac{1}{\mid 2^{j+1}B\mid}\int_{2^{j+1}B}\mid f(z)\mid \mathrm{d}z\\ &\leq\ C\mid b(x)-b_B\mid\frac{1}{\mid 2^{j+1}B\mid}\left(\int_{2^{j+1}B}\mid f(z)\mid^p\omega(z)\mathrm{d}z\right)^{\frac{1}{p}}\mid 2^{j+1}B\mid^{\frac{1}{p'}}\\ &\leq\ C\mid b(x)-b_B\mid(2^jr)^{-\frac{n}{p}}\left(\int_{2^{j+1}B}\mid f(z)\mid^p \mathrm{d}z\right)^{\frac{1}{p}}.\end{aligned}$$

令 $t=2^{j+1}r$, 由引理 5, 得

$$
\begin{aligned}
&\left(\int_0^\infty r^{-\frac{\lambda\theta}{p}}\parallel V_1\parallel^\theta_{L^p(B(x_0,r))}\,\mathrm{d}r\right)^{\frac{1}{\theta}}\\
\leq\ & C\left(\int_0^\infty r^{-\frac{\lambda\theta}{p}}\left(\sum_{j=1}^\infty(2^jr)^{-n}\parallel f\parallel^p_{L^p(B(x_0,2^{j+1}r))}\int_{B(x_0,r)}|b(x)-b_{B(x,r)}|^p\mathrm{d}x\right)^{\frac{\theta}{p}}\mathrm{d}r\right)^{\frac{1}{\theta}}\\
\leq\ & C\sum_{j=1}^\infty 2^{-\frac{jn}{p}}\left(r^{-\frac{\lambda\theta}{p}}\parallel f\parallel^\theta_{L^p(B(x_0,2^{j+1}r))}\left(\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)}|b(x)-b_{B(x,r)}|^p\mathrm{d}x\right)^{\frac{\theta}{p}}\mathrm{d}r\right)^{\frac{1}{\theta}}\\
\leq\ & C\parallel b\parallel_*\sum_{j=1}^\infty 2^{\frac{j(\lambda-n)}{p}}\left(\int_0^\infty t^{-\frac{\lambda\theta}{p}}\parallel f\parallel^\theta_{L^p(B(x_0,t))}\frac{\mathrm{d}t}{2^j}\right)^{\frac{1}{\theta}}\\
\leq\ & C\parallel b\parallel_*\parallel f\parallel_{L^{p,\lambda}_\theta}\sum_{j=1}^\infty 2^{\frac{j(\lambda-n)}{p}-\frac{j}{\theta}}.
\end{aligned}
$$

下面估计 $V_2$

$$
\begin{aligned}
V_2\ \leq\ & C\left(\int\int_{\Gamma(x)}\left|t^{-n}\sum_{j=1}^\infty\int_{(2^{j+1}B\backslash 2^jB)\bigcap\{z:|y-z|\leq t\}}|b(z)-b_B||f(z)|\,\mathrm{d}z\right|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\right)^{\frac12}\\
\leq\ & C\left(\int\int_{\Gamma(x)}\left|t^{-n}\sum_{j=1}^\infty\int_{(2^{j+1}B\backslash 2^jB)\bigcap\{z:|y-z|\leq t\}}|b(z)-b_{2^{j+1}B}||f(z)|\,\mathrm{d}z\right|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\right)^{\frac12}\\
&+C\left(\int\int_{\Gamma(x)}\left|t^{-n}\sum_{j=1}^\infty\int_{(2^{j+1}B\backslash 2^jB)\bigcap\{z:|y-z|\leq t\}}|b_{2^{j+1}B}-b_B||f(z)|\,\mathrm{d}z\right|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\right)^{\frac12}\\
=:\ & V_3+V_4.
\end{aligned}
$$

由定理 1 的证明有 $t>2^{j-2}r$, 再利用 Hölder 不等式, Minkowski 不等式以及引理 5,得

$$
\begin{aligned}
V_3\ \leq\ & C\left(\int_{2^{j-2}r}^\infty\int_{|x-y|<t}\left|t^{-n}\sum_{j=1}^\infty\int_{2^{j+1}B\backslash 2^jB}|b(z)-b_{2^{j+1}B}||f(z)|\,\mathrm{d}z\right|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\right)^{\frac12}\\
\leq\ & C\left(\sum_{j=1}^\infty\int_{2^{j+1}B\backslash 2^jB}|b(z)-b_{2^{j+1}B}||f(z)|\,\mathrm{d}z\right)\left(\int_{2^{j-2}r}^\infty\frac{\mathrm{d}t}{t^{2n+1}}\right)^{\frac12}\\
\leq\ & C\sum_{j=1}^\infty\left(\int_{2^{j+1}B}|b(z)-b_{2^{j+1}B}|^{p'}\,\mathrm{d}z\right)^{\frac{1}{p'}}\left(\int_{2^{j+1}B}|f(z)|^p\,\mathrm{d}z\right)^{\frac1p}\left(\int_{2^{j-2}r}^\infty\frac{\mathrm{d}t}{t^{2n+1}}\right)^{\frac12}\\
\leq\ & C\parallel b\parallel_*\sum_{j=1}^\infty|2^{j+1}B|^{\frac{1}{p'}}|2^{j+1}B|^{-1}\left(\int_{2^{j+1}B}|f(z)|^p\,\mathrm{d}z\right)^{\frac1p}\\
=\ & C\parallel b\parallel_*\sum_{j=1}^\infty(2^jr)^{-\frac{n}{p}}\left(\int_{2^{j+1}B}|f(z)|^p\,\mathrm{d}z\right)^{\frac1p}.
\end{aligned}
$$

对 $V_4$, 注意到当 $b \in \mathrm{BMO}(\mathbb{R}^n)$ 时, 有

$$|\, b_{2^{j+1}B} - b_B \,| \leq C(j+1) \parallel b \parallel_* .$$

利用 Hölder 不等式和 Minkowski 不等式,得

$$\begin{aligned}
V_4 &\leq C\left(\int_{2^{j-2}r}^{\infty}\int_{|x-y|<t}\left|t^{-n}\sum_{j=1}^{\infty}\int_{2^{j+1}B\backslash 2^jB}|\, b_{2^{j+1}B}-b_B \,||\, f(z)\,|\,\mathrm{d}z\right|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\right)^{\frac{1}{2}}\\
&\leq C\parallel b\parallel_*\sum_{j=1}^{\infty}(j+1)\left(\int_{2^{j+1}B\backslash 2^jB}|\, f(z)\,|\,\mathrm{d}z\right)\left(\int_{2^{j-2}r}^{\infty}\frac{\mathrm{d}t}{t^{2n+1}}\right)^{\frac{1}{2}}\\
&\leq C\parallel b\parallel_*\sum_{j=1}^{\infty}(j+1)\frac{1}{|\,2^{j+1}B\,|}\int_{2^{j+1}B}|\, f(z)\,|\,\mathrm{d}z\\
&\leq C\parallel b\parallel_*\sum_{j=1}^{\infty}(j+1)\,|\,2^{j+1}B\,|^{-1}|\,2^{j+1}B\,|^{\frac{1}{p'}}\left(\int_{2^{j+1}B}|\, f(z)\,|^p\,\mathrm{d}z\right)^{\frac{1}{p}}\\
&\leq C\parallel b\parallel_*\sum_{j=1}^{\infty}(j+1)(2^jr)^{-\frac{n}{p}}\left(\int_{2^{j+1}B}|\, f(z)\,|^p\,\mathrm{d}z\right)^{\frac{1}{p}}.
\end{aligned}$$

令 $t = 2^{j+1}r$, 由 Markowski 不等式, 得

$$\begin{aligned}
&\left(\int_0^{\infty} r^{-\frac{\lambda\theta}{p}}\parallel J_2\parallel^{\theta}_{L^p(B(z,r))}\,\mathrm{d}r\right)^{\frac{1}{\theta}}\\
\leq\;& C\parallel b\parallel_*\sum_{j=1}^{\infty}(j+1)2^{-\frac{jn}{p}}\left(r^{-\frac{\lambda\theta}{p}}\|f\|^{\theta}_{L^p(B(z,2^{j+1}r))}\mathrm{d}r\right)^{\frac{1}{\theta}}\\
\leq\;& \parallel b\parallel_*\sum_{j=1}^{\infty}(j+1)2^{\frac{j(\lambda-n)}{p}}\left(\int_0^{\infty}t^{-\frac{\lambda\theta}{p}}\|f\|^{\theta}_{L^p(B(z,t))}\frac{\mathrm{d}t}{2^j}\right)^{\frac{1}{\theta}}\\
\leq\;& \parallel b\parallel_*\parallel f\parallel_{L^{p,\lambda}_{\theta}}\sum_{j=1}^{\infty}(j+1)2^{\frac{j(\lambda-n)}{p}-\frac{j}{\theta}}.
\end{aligned}$$

注意到当 $\lambda < n + \frac{p}{\theta}$, 级数 $\sum_{j=1}^{\infty} 2^{\frac{j(\lambda-n)}{p}-\frac{j}{\theta}}$ 收敛, 定理 2 得证.

**定理 3的证明** 由 $g^*_{\gamma,\alpha}$ 的定义,得

$$\begin{aligned}
g^*_{\gamma,\alpha}(f)(x)^2 &= \int_0^{\infty}\int_{|x-y|<t}\left(\frac{t}{t+|\,x-y\,|}\right)^{\gamma n}(A_\alpha(f)(y,t))^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\\
&\quad+\sum_{i=1}^{\infty}\int_0^{\infty}\int_{2^{i-1}t\leq|x-y|<2^it}\left(\frac{t}{t+|\,x-y\,|}\right)^{\gamma n}(A_\alpha(f)(y,t))^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\\
&\leq C\left[\mathcal{S}_\alpha(f)(x)^2+\sum_{i=1}^{\infty}2^{-i\gamma n}\mathcal{S}_{\alpha,2^i}(f)(x)^2\right].
\end{aligned}$$

设 $B = B(x_0, r)$, 是 $\mathbb{R}^n$ 中的任意一个球,我们有

$$
\begin{aligned}
& \left(\int_0^\infty r^{-\frac{\lambda\theta}{p}} \left(\int_B \mid g^*_{\gamma,\alpha}(f)(x) \mid^p \mathrm{d}x\right)^{\frac{\theta}{p}} \mathrm{d}r\right)^{\frac{1}{\theta}} \\
\leq & \left(\int_0^\infty r^{-\frac{\lambda\theta}{p}} \left(\int_B \mid \mathcal{S}_\alpha(f)(x) \mid^p \mathrm{d}x\right)^{\frac{\theta}{p}} \mathrm{d}r\right)^{\frac{1}{\theta}} \\
& + \sum_{i=1}^\infty 2^{-\frac{i\gamma n}{2}} \left(\int_0^\infty r^{-\frac{\lambda\theta}{p}} \left(\int_B \mid \mathcal{S}_{\alpha,2^i}(f)(x) \mid^p \mathrm{d}x\right)^{\frac{\theta}{p}} \mathrm{d}r\right)^{\frac{1}{\theta}} \\
=: & \; M_0 + \sum_{i=1}^\infty 2^{-\frac{i\gamma n}{2}} M_i.
\end{aligned}
$$

由定理 1 ,$M_0 \leq C \parallel f \parallel_{L^{p,\lambda}_\theta}$, 下面估计 $M_i(i = 1, 2, \ldots)$. 设 $f = f_1 + f_2$, 其中 $f_1 = f\chi_{2B}$, 则有

$$
\begin{aligned}
M_i \leq & \left(\int_0^\infty r^{-\frac{\lambda\theta}{p}} \left(\int_B \mid \mathcal{S}_{\alpha,2^i}(f_1)(x) \mid^p \mathrm{d}x\right)^{\frac{\theta}{p}} \mathrm{d}r\right)^{\frac{1}{\theta}} \\
& + \left(\int_0^\infty r^{-\frac{\lambda\theta}{p}} \left(\int_B \mid \mathcal{S}_{\alpha,2^i}(f_2)(x) \mid^p \mathrm{d}x\right)^{\frac{\theta}{p}} \mathrm{d}r\right)^{\frac{1}{\theta}} \\
=: & \; M_i^{(1)} + M_i^{(2)}.
\end{aligned}
$$

由引理 2 和引理 3,得

$$
\begin{aligned}
M_i^{(1)} & \leq C(2^{\frac{in}{2}} + 2^{\frac{in}{p}}) \parallel \mathcal{S}_\alpha(f_1)(x) \parallel_{L^{p,\lambda}_\theta} \\
& \leq C(2^{\frac{in}{2}} + 2^{\frac{in}{p}}) \parallel f \parallel_{L^{p,\lambda}_\theta} .
\end{aligned}
$$

最后估计 $M_i^{(2)}$. 注意到当 $x \in B$, $(y,t) \in \Gamma_{2^i}(x)$, $z \in (2^{j+1}B \backslash 2^j B) \bigcap B(y,t)$时,有

$$
2^{j-1} r \leq \mid z - x_0 \mid - \mid x - x_0 \mid \leq \mid x - z \mid \leq \mid x - y \mid + \mid y - z \mid \leq t + 2^i t.
$$

因此,利用式和 Markowski 不等式, 得

$$
\begin{aligned}
\mid \mathcal{S}_{\alpha,2^i}(f)(x) \mid & \leq C \left(\int_{2^{j-2-i}r}^\infty \int_{|x-y|<2^i t} \left| t^{-n} \sum_{j=1}^\infty \int_{2^{j+1}B \backslash 2^j B} \mid f(z) \mid \mathrm{d}z \right|^2 \frac{\mathrm{d}y \mathrm{d}t}{t^{n+1}}\right)^{\frac{1}{2}} \\
& \leq C \left(\sum_{j=1}^\infty \int_{2^{j+1}B \backslash 2^j B} \mid f(z) \mid \mathrm{d}z\right) \left(\int_{2^{j-2-i}r}^\infty 2^{in} \frac{\mathrm{d}t}{t^{2n+1}}\right)^{\frac{1}{2}} \\
& \leq C \cdot 2^{\frac{3in}{2}} \sum_{j=1}^\infty \frac{1}{\mid 2^{j+1} B \mid} \int_{2^{j+1}B} \mid f(z) \mid \mathrm{d}z.
\end{aligned}
$$

与定理 1 的证明过程相似,可得

$$M_i^{(2)} \leq C \cdot 2^{\frac{3in}{2}} \parallel f \parallel_{L_\theta^{p,\lambda}} \sum_{j=1}^{\infty} 2^{\frac{j(\lambda-n)}{p}-\frac{j}{\theta}}.$$

注意到 $\gamma > 3 > \frac{2}{p}$, 且 $p > 1$, 因此

$$\begin{aligned}
&\left(\int_0^\infty r^{-\frac{\lambda\theta}{p}}\left(\int_B \mid g_{\gamma,\alpha}^*(f)(x) \mid^p \mathrm{d}x\right)^{\frac{\theta}{p}} \mathrm{d}r\right)^{\frac{1}{\theta}} \\
\leq\ & C \parallel f \parallel_{L_\theta^{p,\lambda}} \left(1+\sum_{i=1}^{\infty} 2^{-\frac{i\gamma n}{2}} 2^{\frac{3in}{2}} + \sum_{i=1}^{\infty} 2^{-\frac{i\gamma n}{2}} 2^{\frac{in}{p}}\right) \\
\leq\ & C \parallel f \parallel_{L_\theta^{p,\lambda}}.
\end{aligned}$$

对所有的 $B = B(x_0, r)$ 取上确界,定理 3 证毕.

**定理 4的证明** 设 $B = B(x_0, r)$ 是 $\mathbb{R}^n$ 中的任意一个球,记 $f = f_1 + f_2$, 其中 $f_1 = f\chi_{2B}$, 则有

$$\begin{aligned}
&\left(\int_0^\infty r^{-\frac{\lambda\theta}{p}}\left(\int_B \mid [b, g_{\gamma,\alpha}^*](f)(x) \mid^p \mathrm{d}x\right)^{\frac{\theta}{p}} \mathrm{d}r\right)^{\frac{1}{\theta}} \\
\leq\ & \left(\int_0^\infty r^{-\frac{\lambda\theta}{p}}\left(\int_B \mid [b, g_{\gamma,\alpha}^*](f_1)(x) \mid^p \mathrm{d}x\right)^{\frac{\theta}{p}} \mathrm{d}r\right)^{\frac{1}{\theta}} \\
& + \left(\int_0^\infty r^{-\frac{\lambda\theta}{p}}\left(\int_B \mid [b, g_{\gamma,\alpha}^*](f_2)(x) \mid^p \mathrm{d}x\right)^{\frac{\theta}{p}} \mathrm{d}r\right)^{\frac{1}{\theta}} \\
=:\ & F_1 + F_2.
\end{aligned}$$

由引理 1 ,有 $F_1 \leq C \parallel f \parallel_{L_\theta^{p,\lambda}}$. 下面估计 $F_2$. 由 $[b, g_{\gamma,\alpha}^*]$的定义可知,

$$\begin{aligned}
& [b, g_{\gamma,\alpha}^*]^2 \\
=\ & \int_0^\infty \int_{|x-y|<t} \left(\frac{t}{t + \mid x-y \mid}\right)^{\gamma n} \sup_{\varphi\in\mathcal{C}_\alpha} \left|\int_{\mathbb{R}^n} [b(x)-b(z)]\varphi_t(y-z) f_2(z) \mathrm{d}z\right|^2 \frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}} \\
& + \sum_{i=1}^{\infty} \int_0^\infty \int_{2^{i-1}t \leq |x-y| < 2^i t} \left(\frac{t}{t + \mid x-y \mid}\right)^{\gamma n} \sup_{\varphi\in\mathcal{C}_\alpha} \left|\int_{\mathbb{R}^n} [b(x)-b(z)]\varphi_t(y-z) f_2(z) \mathrm{d}z\right|^2 \frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}} \\
\leq\ & C([b, \mathcal{S}_\alpha]^2(f_2)(x) + \sum_{i=1}^{\infty} 2^{-i\gamma n} [b, \mathcal{S}_{\alpha,2^i}]^2 (f_2)(x)).
\end{aligned}$$

因此,

$$
\begin{aligned}
F_2 \quad \le \quad & \left(\int_0^\infty r^{-\frac{\lambda\theta}{p}}\left(\int_B |\,[b,\mathcal{S}_\alpha]^2(f_2)(x)\,|^p\,\mathrm{d}x\right)^{\frac{\theta}{p}}\mathrm{d}r\right)^{\frac{1}{\theta}}\\
& +\sum_{i=1}^{\infty}2^{-\frac{i\gamma n}{2}}\left(\int_0^\infty r^{-\frac{\lambda\theta}{p}}\left(\int_B |\,[b,\mathcal{S}_{\alpha,2^i}]^2(f_2)(x)\,|^p\,\mathrm{d}x\right)^{\frac{\theta}{p}}\mathrm{d}r\right)^{\frac{1}{\theta}}\\
=: \quad & W_0+\sum_{i=1}^{\infty}2^{-\frac{i\gamma n}{2}}W_i.
\end{aligned}
$$

由定理 2 可知 $W_0 \le C \parallel f \parallel_{L_\theta^{p,\lambda}}$. 下面估计 $W_i$. 对任意给定的 $x \in B$, $(y,t) \in \Gamma_{2^i}(x)$, $z \in (2^{j+1}B\backslash 2^jB)\bigcap B(y,t)$, 有

$$
\begin{aligned}
& \sup_{\varphi\in\mathcal{C}_\alpha}\left|\int_{\mathbb{R}^n}[b(x)-b(z)]\varphi_t(y-z)f_2(z)\mathrm{d}z\right|\\
\le \quad & |\,b(x)-b_B\,|\sup_{\varphi\in\mathcal{C}_\alpha}\left|\int_{\mathbb{R}^n}\varphi_t(y-z)f_2(z)\mathrm{d}z\right|+\sup_{\varphi\in\mathcal{C}_\alpha}\left|\int_{\mathbb{R}^n}[b(z)-b_B]\varphi_t(y-z)f_2(z)\mathrm{d}z\right|.
\end{aligned}
$$

因此,

$$
\begin{aligned}
& |\,[b,\mathcal{S}_{\alpha,2^i}](f_2)(x)\,|\\
\le \quad & |\,b(x)-b_B\,||\,\mathcal{S}_{\alpha,2^i}(f_2)(x)\,|+\left(\int\int_{\Gamma 2^i(x)}\sup_{\varphi\in\mathcal{C}_\alpha}\left|\int_{\mathbb{R}^n}[b(z)-b_B]\varphi_t(y-z)f_2(z)\mathrm{d}z\right|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\right)^{\frac{1}{2}}\\
=: \quad & W_i^{(1)}+W_i^{(2)}.
\end{aligned}
$$

由定理 2 和定理 3 的证明可得

$$
\left(\int_0^\infty r^{-\frac{\lambda\theta}{p}}\parallel W_i^{(1)}\parallel_{L^p(B(x_0,r))}^{\theta}\mathrm{d}r\right)^{\frac{1}{\theta}}\le C\parallel b\parallel_*\parallel f\parallel_{L_\theta^{p,\lambda}}\cdot 2^{\frac{3in}{2}}.
$$

另一方面,有

$$
\begin{aligned}
W_i^{(2)} \quad \le \quad & C\left(\int\int_{\Gamma 2^i(x)}\left|t^{-n}\sum_{j=1}^{\infty}\int_{(2^{j+1}B\backslash 2^jB)\bigcap\{z:|y-z|\le t\}}|\,b(z)-b_B\,||\,f(z)\,|\,\mathrm{d}z\right|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\right)^{\frac{1}{2}}\\
\le \quad & C\left(\int\int_{\Gamma 2^i(x)}\left|t^{-n}\sum_{j=1}^{\infty}\int_{(2^{j+1}B\backslash 2^jB)\bigcap\{z:|y-z|\le t\}}|\,b(z)-b_{2^{j+1}B}\,||\,f(z)\,|\,\mathrm{d}z\right|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\right)^{\frac{1}{2}}\\
& +C\left(\int\int_{\Gamma 2^i(x)}\left|t^{-n}\sum_{j=1}^{\infty}\int_{(2^{j+1}B\backslash 2^jB)\bigcap\{z:|y-z|\le t\}}|\,b_{2^{j+1}B}-b_B\,||\,f(z)\,|\,\mathrm{d}z\right|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\right)^{\frac{1}{2}}\\
=: \quad & W_i^{(21)}+W_i^{(22)}
\end{aligned}
$$

类似定理 2 和定理 3 的证明,有

$$W_i^{(21)} \le C \parallel b \parallel_* \cdot 2^{\frac{3in}{2}} \sum_{j=1}^{\infty} (2^j r)^{-\frac{n}{p}} \left( \int_{2^{j+1}B} \mid f(z) \mid^p \mathrm{d}z \right)^{\frac{1}{p}}.$$

从而,有

$$\left( \int_0^{\infty} r^{-\frac{\lambda\theta}{p}} \parallel W_i^{(21)} \parallel_{L^p(B(x_0,r))}^{\theta} \mathrm{d}r \right)^{\frac{1}{\theta}} \le C \parallel b \parallel_* \parallel f \parallel_{L_\theta^{p,\lambda}} \cdot 2^{\frac{3in}{2}}.$$

最后估计 $W_i^{(22)}$, 有

$$\begin{aligned} W_i^{(22)} &\le C \left( \int_{2^{j-2-i}r}^{\infty} \int_{|x-y|<2^i t} \left| t^{-n} \sum_{j=1}^{\infty} \int_{2^{j+1}B \backslash 2^j B} \mid b_{2^{j+1}B} - b_B \mid\mid f(z) \mid \mathrm{d}z \right|^2 \frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}} \right)^{\frac{1}{2}} \\ &\le C \parallel b \parallel_* \cdot 2^{\frac{3in}{2}} \sum_{j=1}^{\infty} (j+1)(2^j r)^{-\frac{n}{p}} \left( \int_{2^{j+1}B} \mid f(z) \mid^p \mathrm{d}z \right)^{\frac{1}{p}}. \end{aligned}$$

从而,有

$$\left( \int_0^{\infty} r^{-\frac{\lambda\theta}{p}} \parallel W_i^{(22)} \parallel_{L^p(B(x_0,r))}^{\theta} \mathrm{d}r \right)^{\frac{1}{\theta}} \le C \parallel b \parallel_* \parallel f \parallel_{L_\theta^{p,\lambda}} \cdot 2^{\frac{3in}{2}}.$$

由于 $\gamma > 3$, 所以

$$\begin{aligned} &\left( \int_0^{\infty} r^{-\frac{\lambda\theta}{p}} \left( \int_B \mid [b, g_{\gamma,\alpha}^*](f)(x) \mid^p \mathrm{d}x \right)^{\frac{\theta}{p}} \mathrm{d}r \right)^{\frac{1}{\theta}} \\ \le\ & C \parallel b \parallel_* \parallel f \parallel_{L_\theta^{p,\lambda}} \left( 1 + \sum_{i=1}^{\infty} 2^{-\frac{i\gamma n}{2}} 2^{\frac{3in}{2}} + \sum_{i=1}^{\infty} 2^{-\frac{i\gamma n}{2}} 2^{\frac{3in}{2}} + \sum_{i=1}^{\infty} 2^{-\frac{i\gamma n}{2}} 2^{\frac{3in}{2}} \right) \\ \le\ & C \parallel b \parallel_* \parallel f \parallel_{L_\theta^{p,\lambda}}. \end{aligned}$$

定理 4 证毕.

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