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拟线性微分方程无限区间上正解的存在性

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摘 要

运用锥上的不动点定理讨论了无限区间上拟线性问题

$$\begin{cases} (r^{N-1}w')' + r^{N-1}K(r)f(w,w') = 0, & r \in (R,\infty), \\ w(R) = 0, \ \ \lim_{r\to\infty} w(r) = 0 \end{cases}$$

正解的存在性，其中 $K:(0,\infty)\to(0,\infty)$ 和 $f:[0,\infty)\times\mathbb{R}\to[0,\infty)$ 连续，$N>2$ 是整数，$\mathbf{R}$ 是一个正参数。

关键词

不动点定理，拟线性问题，无限区间，正解

The Existence of Positive Solutions to Quasilinear Differential Equation on Infinite Intervals

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Abstract

In this paper, by using the fixed point theorem, we discuss the following quasilinear problems on infinite intervals

$$\begin{cases} (r^{N-1}w')' + r^{N-1}K(r)f(w,w') = 0, & r \in (R,\infty), \\ w(R) = 0, \ \lim_{r\to\infty} w(r) = 0, \end{cases}$$

where $K : (R,\infty) \to (0,\infty)$ **and** $f : [0,\infty) \times \mathbb{R} \to [0,\infty)$ **are continuous,** $N > 2$ **is an integer,** R **is a positive parameter.**

Keywords

Fixed Point Theorem, Quasilinear Problems, Infinite Intervals, Positive Solution

1. 引言

无限区间上拟线性微分方程问题运用于研究非线性椭圆方程的径向对称解 [1]. 这类问题在物理, 基础工程, 流体力学等邻域广泛的应用 [2–5]. 近年来, 越来越多的学者运用锥上不动点定理, 上下解方法, 打靶法和变分法研究了无限区间上拟线性问题并取得了一系列的成果 [6–14].

文 [11], Iaia 讨论了如下问题

$$\begin{cases} \triangle u + K(|x|)f(u) = 0, & |x| \in (R,\infty), \\ u(x) = 0, |x| = R, \ \lim_{|x|\to\infty} u(x) = 0 \end{cases}$$

的径向解. 其中, $u : \mathbb{R}^N \to \mathbb{R}$, $N \geq 2$是整数, $R > 0$是正常数, f 是奇的且是局部 Lipschitz 连续函数, 当存在 $\beta > 0$ 时, $f'(0) < 0$, 当 $(0,\beta)$ 时, $f(u) < 0$, 当 (β,∞) 时, $f(u) > 0$.

文 [15], Jeong 等人运用不动点定理讨论了 p-Laplacian 问题

$$\begin{cases} \frac{1}{r^{N-1}}(r^{N-1}\varphi_p(u'))' + K(r)f(u) = 0, & r \in (R,\infty), \\ u(R) = 0, & \lim_{r\to\infty} u(r) = 0 \end{cases}$$

正解的存在性与非平凡解的不存在性. 其中, N 是整数, $1 < p < N$, $\varphi_p(s) := |s|^{p-2}s$, $s \in \mathbb{R}\backslash\{0\}$, $\varphi_p(0) = 0$, $K \in C^1(\mathbb{R}_+, \mathbb{R}_+)$, $\mathbb{R}_+ = (0,\infty)$, f 是奇的且是局部 Lipschitz 连续函数, R 是一个正参数.

文 [16], 利用不动点指数的性质在 $C^1[0,1]$ 空间上建立了一个新的不动点理论, 并证明了二阶常微分三点边值问题

$$\begin{cases} u''(t) + f(t,u(t),u'(t)) = 0, & t \in (0,1), \\ u(0) = 0, & u(1) = \alpha u(\eta) \end{cases}$$

正解的存在性, 其中 $\eta \in (0,1)$, $0 < \alpha\eta < 1$, $f : [0,1] \times [0,\infty) \times \mathbb{R} \to [0,\infty)$ 连续.

受上述文献的启发, 本文应用文 [16] 的锥上不动点定理证明如下拟线性问题

$$\begin{cases} (r^{N-1}w')' + r^{N-1}K(r)f(w,w') = 0, & r \in (R,\infty), \\ w(R) = 0, & \lim_{r\to\infty} w(r) = 0 \end{cases} \tag{1}$$

至少存在一个正解, 但关于正解的唯一性和多重性并未进行讨论, 因此这也是一个未来值得研究和讨论的课题.

2. 预备知识

本文假设

(H1) 存在正常数 q_1, q_2, C_1 和 C_2, $2 < q_2 \le q_1$, $q_2 > N$, 当 $r \in (0,R_0)$ 时, $C_2 r^{-q_2} \le K(r) \le C_1 r^{-q_1}$. 其中, $q_1 > q_2$, $R_0 = (\frac{C_1}{C_2})^{\frac{1}{q_1-q_2}} \in (0,\infty)$; $q_1 = q_2$, $R_0 = \infty$, $C_2 \le C_1$.

(H2) $f : [0,\infty) \times \mathbb{R} \to [0,\infty)$ 连续.

本文使用的空间是 $E = C^1[0,1]$, 在范数 $\|x\| = \max\{\|x\|_0, \|x'\|_0\}$ 下构成 Banach 空间, 其中 $\|x\|_0 = \max_{t\in[0,1]} |x(t)|$, 记 $P = \{x \in E \mid x(t) \ge 0\}$ 是 E 上的一个锥.

本文所使用的工具

引理 1 ([16]) 假设 $A : P \to P$ 是一个全连续算子且存在常数 $b, c > 0$, $0 \le t_1 < t_2 \le 1$, $\gamma \in (0,1)$, 使得 $b < \gamma c$, 有

(1) $\|Ax\| \le c$, 对任意的 $x \in \bar{P}_c = \{x \in P : \|x\| \le c\}$;

(2) $\min_{t\in[t_1,t_2]} Ax(t) > b$, 对任意的 $x \in P$, $b \le x(t) \le \frac{b}{\gamma}$, $t \in [t_1,t_2]$;

(3) $\min_{t\in[t_1,t_2]} Ax(t) > \gamma\|Ax\|$, 对任意的 $x \in P$, $\frac{b}{\gamma} < \|x\| \le c$, 且 $x(t) \ge b$, $t \in [t_1,t_2]$.

则 A 至少有一个不动点 $x^* \in \bar{P}_c$, 其中 $\min_{t\in[t_1,t_2]} x^*(t) > b$.

3. 正解的存在性

对问题 (1) 做如下变换. 令 $u(t) = w(r)$, $t = (\frac{r}{R})^{2-N}$, $\frac{dt}{dr} = (2-N)R^{N-2}r^{1-N}$, $r = t^{\frac{1}{2-N}}R$, $r^{1-N} = t^{\frac{1-N}{2-N}}R^{1-N}$,

$$
\begin{aligned}
(r^{N-1}w')' &= ((2-N)R^{N-2}u')' \\
&= (2-N)R^{N-2}\frac{dt}{dr}u'' \\
&= (2-N)R^{N-2}u''(2-N)R^{N-2}r^{1-N} \\
&= (2-N)R^{N-2}u''(2-N)R^{N-2}t^{\frac{1-N}{2-N}}R^{1-N}u'' \\
&= (2-N)^2R^{N-3}t^{\frac{1-N}{2-N}}u'', \\
r^{N-1}K(r)f(w,w') &= t^{\frac{N-1}{2-N}}R^{N-1}K(t^{\frac{1}{2-N}}R)f(u,(2-N)R^{-1}t^{\frac{1-N}{2-N}}u'), \\
u'' &+ h_R(t)f(u,(2-N)R^{-1}t^{\frac{1-N}{2-N}}u') = 0, \\
w(R) &= u(1) = 0, \\
\lim_{r\to\infty} w(r) &= \lim_{r\to\infty}(u(\frac{r}{R})^{2-N}) = u(0) = 0,
\end{aligned}
$$

从而 (1) 等价于

$$
\begin{cases}
u'' + h_R(t)f(u,(2-N)R^{-1}t^{\frac{1-N}{2-N}}u') = 0, & t \in (0,1), \\
u(0) = 0, \quad u(1) = 0,
\end{cases} \tag{2}
$$

其中

$$
h_R(t) = \frac{1}{(2-N)^2}R^2t^{\frac{2(N-1)}{2-N}}K(t^{\frac{1}{2-N}}R).
$$

显然 $h_R : (0,1] \to (0,\infty)$, $f : [0,\infty) \times \mathbb{R} \to [0,\infty)$ 连续, 这意味着问题 (1) 的解等价于问题 (2) 的解.

注 1　由 (H1) 可得,

$$
C_2R^{2-q_2}\frac{1}{(2-N)^2}t^{\frac{2(N-1)-q_1}{2-N}} \le h_R(t) \le C_1R^{2-q_1}\frac{1}{(2-N)^2}t^{\frac{2(N-1)-q_2}{2-N}}, t \in (0,1),\ R \in (0,R_0). \tag{3}
$$

因为 $2 < q_2 \le q_1$, $q_2 > N$, $\frac{2(N-1)-q_1}{2-N} \ge \frac{2(N-1)-q_2}{2-N} > -2$, 所以 $h_R \in \{h \in C((0,1],(0,\infty)) : \int_0^1 th(t)dt < \infty\}$.

记

$$
m = \frac{(N-q_2)(2-N)}{6C_1R^{2-q_1}}, \quad l = \frac{20(2-q_1)(4-q_1-N)}{C_2R^{2-q_2}}.
$$

定理 1 假定 (H2) 成立. 设 $R \in (0, R_0)$, b, c, γ为正常数, 且 $b < \gamma c$, $0 < \gamma < \frac{1}{4}$, 使得 f 满足

(H3) $f(u, v) \le mc$, 对 $0 \le u \le c$, $0 < v < \frac{(N-2)c}{R}$;

(H4) $f(u, v) \ge \min\{lb, l\gamma c\}$, 对 $b \le u \le \frac{b}{\gamma}$, $0 < v < \frac{(N-2)c}{R}$;

(H5) $f(u, v) \ge l\gamma c$, 对 $\frac{b}{\gamma} < u \le c$, $0 < v < \frac{(N-2)c}{R}$.

则拟线性问题 (2) 至少有一个正解.

证明 显然 $u \in P$ 是问题 (2) 的解当且仅当 u 满足积分方程

$$u(t) = \int_0^1 G(t,\tau) h_R(\tau) f(u, (2-N)R^{-1}\tau^{\frac{1-N}{2-N}} u') d\tau,$$

其中

$$G(t,\tau) = \begin{cases} t(1-\tau), & t \le \tau, \\ \tau(1-t), & \tau \le t. \end{cases}$$

定义一个算子 $T: E \to E$,

$$Tu(t) = \int_0^1 G(t,\tau) h_R(\tau) f(u, (2-N)R^{-1}\tau^{\frac{1-N}{2-N}} u') d\tau.$$

易证 $T: P \to P$ 是一个全连续算子.

对任意的 $u \in \bar{P}_c$ 可推知, $0 \le t \le 1$, $0 \le u \le c$, $0 < \frac{2-N}{R}\tau^{\frac{1-N}{2-N}} u' < \frac{(N-2)c}{R}$, 从而根据 (H3), 有

$$\begin{aligned}
\|Tu\|_0 &= \max_{t\in[0,1]} |Tu(t)| \\
&= \max_{t\in[0,1]} Tu(t) \\
&= \max_{t\in[0,1]} \int_0^1 G(t,\tau) h_R(\tau) f(u, (2-N)R^{-1}\tau^{\frac{1-N}{2-N}} u') d\tau \\
&\le \max_{t\in[0,1]} mc \int_0^1 G(\tau,\tau) C_1 R^{2-q_1} \frac{1}{(2-N)^2} \tau^{\frac{2(N-1)-q_2}{2-N}} d\tau \\
&= C_1 R^{2-q_1} \frac{mc}{(2-N)^2} \int_0^1 \tau^{\frac{N-q_2}{2-N}} - \tau^{\frac{2-q_2}{2-N}} d\tau \\
&= C_1 R^{2-q_1} \frac{mc}{(2-N)^2} (\frac{2-N}{2-q_2} - \frac{2-N}{4-q_2-N}) \\
&\le \frac{C_1 R^{2-q_1} mc}{(2-q_2)(2-N)} \\
&= \frac{(N-q_2)c}{6(2-q_2)},
\end{aligned}$$

以及

$$\begin{aligned}
\|Tu'\|_0 &= \max_{t\in[0,1]}|Tu'| \\
&= \max_{t\in[0,1]}|\int_0^t -\tau h_R(\tau)f(u,(2-N)R^{-1}\tau^{\frac{1-N}{2-N}}u')d\tau \\
&\quad +\int_t^1(1-\tau)h_R(\tau)f(u,(2-N)R^{-1}\tau^{\frac{1-N}{2-N}}u')d\tau| \\
&\le \max_{t\in[0,1]}(\int_0^t \tau C_1R^{2-q_1}\frac{mc}{(2-N)^2}\tau^{\frac{2(N-1)-q_2}{2-N}}d\tau \\
&\quad +\int_t^1(1-\tau)C_1R^{2-q_1}\frac{mc}{(2-N)^2}\tau^{\frac{2(N-1)-q_2}{2-N}}d\tau) \\
&= \max_{t\in[0,1]}\frac{C_1R^{2-q_1}mc}{(2-N)^2}(\int_0^t\tau^{\frac{N-q_2}{2-N}}d\tau+\int_t^1\tau^{\frac{2(N-1)-q_2}{2-N}}-\tau^{\frac{N-q_2}{2-N}}d\tau) \\
&= \max_{t\in[0,1]}\frac{C_1R^{2-q_1}mc}{(2-N)^2}[\frac{2-N}{2-q_2}t^{\frac{2-q_2}{2-N}}+\frac{2-N}{N-q_2}(1-t^{\frac{N-q_2}{2-N}})-\frac{2-N}{2-q_2}(1-t^{\frac{2-q_2}{2-N}})] \\
&\le \max_{t\in[0,1]}\frac{C_1R^{2-q_1}mc}{(2-N)^2}\frac{2-N}{N-q_2}(2t^{\frac{2-q_2}{2-N}}+1) \\
&= \frac{3mcC_1R^{2-q_1}}{(N-q_2)(2-N)} \\
&= \frac{c}{2}.
\end{aligned}$$

从而可得, $\|Tu\|=\max\{\|Tu\|_0,\|Tu'\|_0\}\le c$. 因此,引理 1 的条件 (1) 满足.

对任意的 $u\in P$, $b\le u(t)\le\frac{b}{\gamma}$, 有 $0<\frac{2-N}{R}\tau^{\frac{1-N}{2-N}}u'<\frac{(N-2)c}{R}$, 从而根据 (H4) 可得

$$\begin{aligned}
Tu(\frac{1}{4}) &= \int_0^1 G(\frac{1}{4},\tau)h_R(\tau)f(u,(2-N)R^{-1}\tau^{\frac{1-N}{2-N}}u')d\tau \\
&\ge \int_0^1\frac{1}{4}G(\tau,\tau)h_R(\tau)f(u,(2-N)R^{-1}\tau^{\frac{1-N}{2-N}}u')d\tau \\
&\ge lb\int_0^1\frac{1}{4}G(\tau,\tau)C_2R^{2-q_2}\frac{1}{(2-N)^2}\tau^{\frac{2(N-1)-q_1}{2-N}}d\tau \\
&= \frac{lb}{4}C_2R^{2-q_2}\frac{1}{(2-N)^2}\int_0^1\tau^{\frac{N-q_1}{2-N}}-\tau^{\frac{2-q_1}{2-N}}d\tau \\
&= \frac{lb}{4}C_2R^{2-q_2}\frac{1}{(2-N)^2}\frac{(N-2)^2}{(2-q_1)(4-q_1-N)} \\
&= \frac{lbC_2R^{2-q_2}}{4(2-q_1)(4-q_1-N)} \\
&= 5b.
\end{aligned}$$

则 $\min_{t\in[\frac{1}{4},\frac{3}{4}]}Tu(t)\ge\frac{1}{4}\|Tu\|_0\ge\frac{1}{4}Tu(\frac{1}{4})>b$, 对 $t\in[\frac{1}{4},\frac{3}{4}]$. 因此,引理 1 的条件 (2) 满足.

对任意的 $u\in P$, $\frac{b}{\gamma}<\|u\|\le c$, $u(t)\ge b$, 由 (H4) 和 (H5) 可得, $f(u,(2-N)R^{-1}\tau^{\frac{1-N}{2-N}}u')\ge l\gamma c$,

$b \le u \le \; c, 0 < \frac{2-N}{R}\tau^{\frac{1-N}{2-N}}u' < \frac{(N-2)c}{R}$, 从而

$$\begin{aligned}
Tu(\frac{1}{4}) &= \int_0^1 G(\frac{1}{4},\tau)h_R(\tau)f(u,(2-N)R^{-1}\tau^{\frac{1-N}{2-N}}u')d\tau \\
&\ge \int_0^1 \frac{1}{4}G(\tau,\tau)h_R(\tau)f(u,(2-N)R^{-1}\tau^{\frac{1-N}{2-N}}u')d\tau \\
&\ge l\gamma c\int_0^1 \frac{1}{4}G(\tau,\tau)C_2R^{2-q_2}\frac{1}{(2-N)^2}\tau^{\frac{2(N-1)-q_1}{2-N}}d\tau \\
&= \frac{l\gamma c}{4}C_2R^{2-q_2}\frac{1}{(2-N)^2}\int_0^1 \tau^{\frac{N-q_1}{2-N}}-\tau^{\frac{2-q_1}{2-N}}d\tau \\
&= \frac{l\gamma c}{4}C_2R^{2-q_2}\frac{1}{(2-N)^2}\frac{(N-2)^2}{(2-q_1)(4-q_1-N)} \\
&= \frac{l\gamma cC_2R^{2-q_2}}{4(2-q_1)(4-q_1-N)} \\
&= 5\gamma c.
\end{aligned}$$

则 $\min_{t\in[\frac{1}{4},\frac{3}{4}]}Tu(t) \ge \frac{1}{4}\|Tu\|_0 \ge \frac{1}{4}Tu(\frac{1}{4}) \ge \gamma\|Tu\|$. 对 $t \in [\frac{1}{4},\frac{3}{4}]$. 因此,引理 1 的条件 (3) 满足.

由引理 1 可知, T 至少有一个不动点 $u^* \in \bar{P}_c$, 并且 $u^* \ge 0$. 从而问题 (1) 至少有一个正解.

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