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基于三次函数不等式的时变时滞系统稳定性分析

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摘　要

时滞系统的稳定性分析一直是学术研究的焦点。本文采用Lyapunov-Krasovskii泛函(LKF)方法，对时变时滞系统的稳定性进行了深入研究，并提出了一个互凸三次矩阵不等式。首先，利用辅助函数积分不等式和互凸三次矩阵不等式，对LKF导数中的积分项进行了有效估计。随后，基于三次函数负定方法，以线性矩阵不等式(LMI)的形式给出了时变时滞系统渐近稳定的稳定性准则，以确保系统的稳定性。最后，通过数值例子，验证了所提出方法的可行性和优越性。

关键词

时滞系统，稳定性，三次不等式，积分不等式

Stability Analysis of Time-Varying Time-Delay Systems Based on Cubic Function Inequality

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Abstract

The stability analysis of time-delay systems has always been a focus of academic research. This article adopts the Lyapunov Krasovskii functional (LKF) method to conduct in-depth research on the stability of time-varying time-delay systems, and proposes a mutually convex cubic matrix inequality. Firstly,

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the integral terms in the LKF derivative are effectively estimated using the auxiliary function integral inequality and the convex cubic matrix inequality. Subsequently, based on the negative definite method of cubic functions, a stability criterion for asymptotic stability of time-varying time-delay systems is presented in the form of linear matrix inequality (LMI) to ensure the stability of the system. Finally, the feasibility and superiority of the proposed method are verified through numerical examples.

Keywords

Time Delayed Systems, Stability, Cubic Inequality, Integral Inequality

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1. 引言

时滞系统的稳定性分析是控制领域的一个重要研究方向。时滞对系统的稳定性造成了一些不可避免的挑战，例如时滞的存在会限制系统的稳定性，这使得时滞系统的稳定性分析更加复杂。因此，在时滞的最大允许范围内，确保系统仍能保持稳定运行至关重要。近年来，时变时滞系统的稳定性分析引起了许多研究人员的广泛关注，并取得了许多研究成果[1]-[20]。如，文献[11]基于 Jensen 不等式提出了 Wirtinger 不等式，用于获得时滞系统的稳定性；文献[13]提出贝塞尔－勒让德不等式，研究了时变时滞系统的时滞相关稳定性。时变时滞系统的稳定性分析中，主流方法是 Lyapunov-Krasovskii 泛函(LKF)方法[21]，相关研究主要从构造适当的 LKF 与严格估计 LKF 的时间导数两个方面进行考虑。

LKF 的时间导数中的积分项通常采用积分不等式来估计，如 Jensen 不等式、Wirtinger 不等式[11] [12]、贝塞尔－勒让德不等式[13]、二重积分不等式[14] [15]以及其它改进的积分不等式[16] [17] [18] [19] [20]等。积分不等式用于估计积分项虽有效但也会出现一个新问题，即估计得到的项具有非凸性。为了应对这个问题，学者们提出了诸多互凸方法，例如，Park 等人提出了逆凸引理(Reciprocally Convex Lemma，简称 RCL) [22]；Zhang 等人提出了参数相关的互凸矩阵不等式[23] [24]；Zeng 等人提出了互凸二次矩阵不等式[25]；Yang 等人提出了参数相关的互凸矩阵不等式，其中包含了更多自由矩阵、时滞二次项以及时滞导数的信息[20]。鉴于时变时滞三次项的信息的探索相对较少，自然提出一问题：能否提出一种新的互凸三次矩阵不等式，以获得更为精确的结果，从而避免产生过于保守的结果？

为了获得保守性较低的稳定性准则，研究者们提出了多种构造 LKF 的方法，如多重积分 LKF [26]、增广 LKF [12]以及时滞乘积 LKF [27]等。此外，有时候构造 LKF 需要借助新颖的积分不等式和互凸不等式来完成。LKF 的构造方式、积分项的估计、凸式的估计以及不等式的处理技巧等因素都可能对稳定性准则的保守性产生影响。在推导保守性较低的稳定性准则时，往往会从多个角度改进稳定性分析方法，这导致很难确定哪些因素对结论产生了影响。基于此进一步探讨：能否基于简单的 LKF、三阶贝塞尔－勒让德积分不等式等，以研究新提出的互凸三次矩阵不等式能否获得相对保守较小的结果？

在本文中，R^n 表示 n 维列向量，$R^{m\times n}$ 表示 m 行和 n 列的所有实矩阵的集合，上标 T 和−1 分别表示矩阵的转置和逆，R 表示实数集，I_n 是 n 维单位矩阵，$*$ 表示对称矩阵中的对称项并且 $Sym\{X\} = X + X^{\mathrm{T}}$。

2. 系统描述

考虑以下具有时变时滞的线性系统：

$$\begin{cases}\dot{x}(t)=Ax(t)+Bx(t-d(t)), & t>0\\ x(t)=\varphi(t), & t\in[-h_2,0]\end{cases} \tag{1}$$

其中，$x(t)\in R^n$ 是状态向量，$\varphi(t)\in R^n$ 是初始值，$A,B\in R^{n\times n}$ 是常数矩阵，h_1,h_2 是常数，时变可微函数 $d(t)$ 满足以下条件：

$$0\le h_1\le d(t)\le h_2. \tag{2}$$

利用以下引理来推导时变时滞系统的稳定性准则。

引理 1 [13](三阶贝塞尔－勒让德积分不等式)：给定任意一个对称正定矩阵 $W\in R^{n\times n}$，标量 a 和 b 满足 $b>a$，对于任一可微向量函数 $x:[a,b]\to R^n$，有

$$(b-a)\int_a^b \dot{x}^{\mathrm{T}}(u)W\dot{x}(u)\mathrm{d}u\ge\left[r1^{\mathrm{T}},r2^{\mathrm{T}},r3^{\mathrm{T}}\right]\begin{bmatrix}W & & \\ & 3W & \\ & & 5W\end{bmatrix}\begin{bmatrix}r1\\ r2\\ r3\end{bmatrix}, \tag{3}$$

其中，

$$\begin{aligned}&r1=x(b)-x(a),\\ &r2=x(b)+x(a)-\frac{2}{b-a}\int_a^b x(u)\mathrm{d}u,\\ &r3=x(b)-x(a)+\frac{6}{b-a}\int_a^b x(u)\mathrm{d}u-\frac{12}{(b-a)^2}\int_a^b\int_\theta^b x(u)\mathrm{d}u\mathrm{d}\theta.\end{aligned} \tag{4}$$

引理 2 [25]：给定任意一个对称正定矩阵 $W\in R^{n\times n}$，标量 $\alpha,\beta\in(0,1)$ 且 $\alpha+\beta=1$，如果存在实对称矩阵 $X_1,X_2,X_3,X_4\in R^{n\times n}$，实矩阵 $Y_1,Y_2,Y_3,Y_4\in R^{n\times n}$，若

$$\begin{bmatrix}W & 0\\ 0 & W\end{bmatrix}-\alpha\begin{bmatrix}X_1 & Y_1\\ * & 0\end{bmatrix}-\beta\begin{bmatrix}0 & Y_2\\ * & X_2\end{bmatrix}-\alpha^2\begin{bmatrix}X_3 & Y_3\\ * & 0\end{bmatrix}-\beta^2\begin{bmatrix}0 & Y_4\\ * & X_4\end{bmatrix}\ge 0, \tag{5}$$

则以下不等式成立：

$$\begin{bmatrix}\frac{1}{\alpha}W & 0\\ 0 & \frac{1}{\beta}W\end{bmatrix}\ge\begin{bmatrix}W+T_1 & T_2\\ * & W+T_3\end{bmatrix}, \tag{6}$$

其中，$T_1=\beta X_1+\alpha\beta X_3$，$T_2=\alpha Y_1+\beta Y_2+\alpha^2Y_3+\beta^2Y_4$，$T_3=\alpha X_2+\alpha\beta X_4$。

引理 3 [28]：令三次函数 $f(s)=a_0+a_1s+a_2s^2+a_3s^3$，其中，$a_i\in R$，$i=0,1,2,3$，$s\in[h_1,h_2]$，对任意 $s\in[h_1,h_2]$，若满足以下不等式，则 $f(s)<0$。

$$\begin{aligned}&\text{(i) } f(h_1)<0,\ f(h_2)<0,\\ &\text{(ii) } f(h_1)-a_2h_1^2<0,\ f(h_2)-a_2h_2^2<0,\\ &\text{(iii) } f(h_1)-a_3h_1^3<0,\ f(h_2)-a_3h_2^3<0,\\ &\text{(iv) } a_3(h_2-h_1)^2+f(h_2)-(h_2-h_1)^3(2a_3h_2+a_3h_1+a_2)<0.\end{aligned} \tag{7}$$

引理 4 [29]：令二次函数 $f(s)=a_0+a_1s+a_2s^2$，其中，$a_i\in R$，$i=0,1,2$，$s\in[h_1,h_2]$，对任意 $s\in[h_1,h_2]$，若满足以下不等式，则 $f(s)<0$。

$$\begin{aligned}&\text{(i) } f(h_1)<0,\\ &\text{(ii) } f(h_2)<0,\\ &\text{(iii) } f(h_1)-a_2(h_2-h_1)^2<0.\end{aligned} \tag{8}$$

3. 互凸三次矩阵不等式

本节的目的在于推导一个新的互凸矩阵不等式，具体表达如下：

引理 5：给定任意一个对称正定矩阵$W \in R^{n\times n}$，标量$\alpha,\beta\in(0,1)$且$\alpha+\beta=1$，如果存在实对称矩阵$X_1,X_2,X_3,X_4,X_5,X_6 \in R^{n\times n}$，实矩阵$Y_1,Y_2,Y_3,Y_4,Y_5,Y_6 \in R^{n\times n}$，若

$$\begin{aligned}&\begin{bmatrix}W & 0\\ 0 & W\end{bmatrix}-\alpha\begin{bmatrix}X_1 & Y_1\\ * & 0\end{bmatrix}-\beta\begin{bmatrix}0 & Y_2\\ * & X_2\end{bmatrix}-\alpha^2\begin{bmatrix}X_3 & Y_3\\ * & 0\end{bmatrix}\\ &-\beta^2\begin{bmatrix}0 & Y_4\\ * & X_4\end{bmatrix}-\alpha^3\begin{bmatrix}X_5 & Y_5\\ * & 0\end{bmatrix}-\beta^3\begin{bmatrix}0 & Y_6\\ * & X_6\end{bmatrix}\geq 0,\end{aligned}\tag{9}$$

则以下不等式成立：

$$\begin{bmatrix}\frac{1}{\alpha}W & 0\\ 0 & \frac{1}{\beta}W\end{bmatrix}\geq\begin{bmatrix}W+T_4 & T_5\\ * & W+T_6\end{bmatrix},\tag{10}$$

其中，$T_4=\beta X_1+\alpha\beta X_3+\alpha^2\beta X_5$，$T_5=\alpha Y_1+\beta Y_2+\alpha^2 Y_3+\beta^2 Y_4+\alpha^3 Y_5+\beta^3 Y_6$，$T_6=\alpha X_2+\alpha\beta X_4+\alpha\beta^2 X_6$。

证明：将式(9)左右两端分别乘P^{T},P，则得到以下不等式：

$$P^{\mathrm{T}}\begin{bmatrix}W-\alpha X_1-\alpha^2X_3-\alpha^3X_5 & -\alpha Y_1-\beta Y_2-\alpha^2Y_3-\beta^2Y_4-\alpha^3Y_5-\beta^3Y_6\\ * & W-\beta X_2-\beta^2X_4-\beta^3X_6\end{bmatrix}P\geq 0,\tag{11}$$

其中，$P=\begin{bmatrix}\sqrt{\frac{1-\alpha}{\alpha}}\\ \sqrt{\frac{\alpha}{1-\alpha}}\end{bmatrix}$。通过对上述式(11)进行化简和移项，即可完成证明。

注 1：不等式(9)的左侧是一个三次函数。为了便于描述，将不等式(9)改写为一般形式，

$$\begin{aligned}&\alpha^3\begin{bmatrix}-X_5 & -Y_5+Y_6\\ * & X_6\end{bmatrix}+\alpha^2\begin{bmatrix}-X_3 & -Y_3-Y_4-3Y_6\\ * & -X_4-3X_6\end{bmatrix}\\ &+\alpha\begin{bmatrix}-X_1 & Y_1+Y_2+2Y_4+3Y_6\\ * & X_2+2X_4+3X_6\end{bmatrix}+\begin{bmatrix}W & -Y_2-Y_4-Y_6\\ * & W-X_2-X_4-X_6\end{bmatrix}\geq 0.\end{aligned}\tag{12}$$

注 2：本文所提出的互凸三次矩阵不等式相比许多现有方法更具普适性。例如，当条件$X_5=0,X_6=0,Y_5=0,Y_6=0$成立时，所得不等式(10)将演化为引理 2；在此基础上，当条件$X_3=0,X_4=0,Y_3=0,Y_4=0$成立时，不等式(10)将演化为[21]中的引理 1。

注 3：在引理 5 中，我们提出了一种互凸三次矩阵不等式，通过引入时变时滞的三次项信息，以获得更为保守的稳定性准则。值得注意的是，这一方法可以适用于各种时滞系统。

4. 稳定性准则

基于引理[1] [2] [3] [4]，本文对于时变时滞系统(1)提出了一个具有较低保守性的稳定性准则。为简洁起见，定义以下符号，

$$\begin{aligned}\zeta(t)=\Big[&x^{\mathrm{T}}(t)\quad x^{\mathrm{T}}(t-h_1)\quad x^{\mathrm{T}}(t-h(t))\quad x^{\mathrm{T}}(t-h_2)\quad \frac{1}{h_1}\int_{t-h_1}^{t}x^{\mathrm{T}}(s)\mathrm{d}s\\ &\frac{1}{h(t)-h_1}\int_{t-h(t)}^{t-h_1}x^{\mathrm{T}}(s)\mathrm{d}s\quad \frac{1}{h_2-h(t)}\int_{t-h_2}^{t-h(t)}x^{\mathrm{T}}(s)\mathrm{d}s\quad \frac{1}{h_1^2}\int_{t-h_1}^{t}\int_{\theta}^{t}x^{\mathrm{T}}(s)\mathrm{d}s\mathrm{d}\theta\end{aligned}$$

$$\frac{1}{\left(h(t)-h_1\right)^2}\int_{t-h(t)}^{t-h_1}\int_{\theta}^{t}x^{\mathrm{T}}(s)\mathrm{d}s\mathrm{d}\theta \quad \frac{1}{\left(h_2-h(t)\right)^2}\int_{t-h_2}^{t-h(t)}\int_{\theta}^{t-h(t)}x^{\mathrm{T}}(s)\mathrm{d}s\mathrm{d}\theta$$
$$\left.\int_{t-h(t)}^{t-h_1}x^{\mathrm{T}}(s)\mathrm{d}s \quad \int_{t-h_2}^{t-h(t)}x^{\mathrm{T}}(s)\mathrm{d}s\right]^{\mathrm{T}},$$
$$\xi_1(t)=\left[x^{\mathrm{T}}(t) \quad \int_{t-h_1}^{t}x^{\mathrm{T}}(s)\mathrm{d}s \quad \int_{t-h_2}^{t-h_1}x^{\mathrm{T}}(s)\mathrm{d}s \quad \int_{t-h_1}^{t}\int_{\theta}^{t}x^{\mathrm{T}}(s)\mathrm{d}s\mathrm{d}\theta \quad \int_{t-h_2}^{t-h_1}\int_{\theta}^{t-h_1}x^{\mathrm{T}}(s)\mathrm{d}s\mathrm{d}\theta\right]^{\mathrm{T}},$$
$$\xi_2(t,s)=\left[x^{\mathrm{T}}(s) \quad x^{\mathrm{T}}(t) \quad \int_{s}^{t-h_1}x^{\mathrm{T}}(\theta)\mathrm{d}\theta \quad \int_{t-h_2}^{s}x^{\mathrm{T}}(\theta)\mathrm{d}\theta\right]^{\mathrm{T}},$$
$$e_i=\left[0_{n\times(i-1)n} \quad I_n \quad 0_{n\times(12-i)n}\right],\quad i=1,2,\cdots,12.$$

定理 1 给定标量 h_1,h_2，对于满足式(2)的系统(1)是渐近稳定的，如果存在对称正定矩阵 $P\in R^{5n\times5n}$，$W_1\in R^{n\times n}$，$W\in R^{4n\times4n}$，$S_1,S_2\in R^{n\times n}$，实对称矩阵 $X_1,X_2,X_3,X_4,X_5,X_6\in R^{3n\times3n}$，任意矩阵 $Z_1,Z_2\in R^{12n\times n}$，$Y_1,Y_2,Y_3,Y_4,Y_5,Y_6\in R^{3n\times3n}$，使当 $h(t)\in[h_1,h_2]$ 时，(13)和(14)成立，

$$\Sigma_1=h^3(t)\Pi_3+h^2(t)\Pi_2+h(t)\Pi_1+\Pi_0, \tag{13}$$

$$\Omega_1=h^3(t)\Psi_3+h^2(t)\Psi_2+h(t)\Psi_1+\Psi_0, \tag{14}$$

其中，$P=\begin{bmatrix}P_1 & P_2 & P_3 & P_4 & P_5\\ & P_6 & P_7 & P_8 & P_9\\ & & P_{10} & P_{11} & P_{12}\\ & & & P_{13} & P_{14}\\ & & & & P_{15}\end{bmatrix}$，其余符号见附录。

证明：构造如下 LKF，

$$V=V_1(t)+V_2(t)+V_3(t),$$

其中，

$$V_1(t)=\xi_1^{\mathrm{T}}(t)P\xi_1(t),$$
$$V_2(t)=\int_{t-h_1}^{t}x^{\mathrm{T}}(s)W_1x(s)\mathrm{d}s+\int_{t-h_2}^{t-h_1}\xi_2^{\mathrm{T}}(t,s)W\xi_2(t,s)\mathrm{d}s,$$
$$V_3(t)=h_1\int_{-h_1}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)S_1\dot{x}(s)\mathrm{d}s\mathrm{d}\theta+(h_2-h_1)\int_{-h_2}^{-h_1}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)S_2\dot{x}(s)\mathrm{d}s\mathrm{d}\theta,$$

对 $V(t)$ 求导有以下结果

$$\dot{V}_1(t)=\zeta^{\mathrm{T}}(t)\left[h^2(t)b_2+h(t)b_1+(\Xi_1+b_0)\right]\zeta(t),$$
$$\dot{V}_2(t)=\zeta^{\mathrm{T}}(t)\left[h^2(t)\varpi_{33}^{\mathrm{T}}WL_1+h(t)\varpi_{32}^{\mathrm{T}}WL_1+\left[e_1^{\mathrm{T}}W_1e_1-e_2^{\mathrm{T}}W_1e_2+\varpi_1^{\mathrm{T}}W\varpi_1-\varpi_2^{\mathrm{T}}W\varpi_2+sym\left\{\varpi_{31}^{\mathrm{T}}WL_1\right\}\right]\right]\zeta(t),$$
$$\dot{V}_3(t)=\zeta^{\mathrm{T}}(t)\Xi_2\zeta(t)-h_1\int_{t-h_1}^{t}\dot{x}^{\mathrm{T}}(\theta)S_1\dot{x}(\theta)\mathrm{d}\theta-(h_2-h_1)\int_{t-h_2}^{t-h_1}\dot{x}^{\mathrm{T}}(\theta)S_2\dot{x}(\theta)\mathrm{d}\theta,$$

利用引理 1 对积分项进行估计，有

$$-h_1\int_{t-h_1}^{t}\dot{x}^{\mathrm{T}}(\theta)S_1\dot{x}(\theta)\mathrm{d}\theta\le-\zeta^{\mathrm{T}}(t)\eta_1^{\mathrm{T}}\tilde{S}_1\eta_1\zeta(t),$$

$$-(h_2-h_1)\int_{t-h_2}^{t-h_1}\dot{x}^{\mathrm{T}}(\theta)S_2\dot{x}(\theta)\mathrm{d}\theta$$
$$=-(h_2-h_1)\left[\int_{t-h(t)}^{t-h_1}\dot{x}^{\mathrm{T}}(\theta)S_2\dot{x}(\theta)\mathrm{d}\theta+\int_{t-h_2}^{t-h(t)}\dot{x}^{\mathrm{T}}(\theta)S_2\dot{x}(\theta)\mathrm{d}\theta\right]$$
$$\le-\zeta^{\mathrm{T}}(t)\left[\frac{h_2-h_1}{h(t)-h_1}\eta_2^{\mathrm{T}}\tilde{S}_2\eta_2+\frac{h_2-h_1}{h_2-h(t)}\eta_3^{\mathrm{T}}\tilde{S}_2\eta_3\right]\zeta(t)=-\zeta^{\mathrm{T}}(t)\left[\frac{1}{\alpha}\eta_2^{\mathrm{T}}\tilde{S}_2\eta_2+\frac{1}{\beta}\eta_3^{\mathrm{T}}\tilde{S}_2\eta_3\right]\zeta(t),$$

其中，$\alpha=\dfrac{h(t)-h_1}{h_2-h_1}$，$\beta=\dfrac{h_2-h(t)}{h_2-h_1}$。根据引理 5 的结论，有如下凸式估计

$$-\zeta^{\mathrm{T}}(t)\begin{bmatrix}\eta_2^{\mathrm{T}} & \eta_3^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\frac{1}{\alpha}\tilde{S}_2 & 0\\ 0 & \frac{1}{\beta}\tilde{S}_2\end{bmatrix}\begin{bmatrix}\eta_2\\ \eta_3\end{bmatrix}\zeta(t)\le\zeta^{\mathrm{T}}(t)\left[c_3h^3(t)+c_2h^2(t)+c_1h(t)+c_0\right]\zeta(t),$$

综上所述，

$$\begin{aligned}\dot{V}(t)\le\zeta^{\mathrm{T}}(t)\Big[&h^3(t)c_3+h^2(t)\left(b_2+c_2+\varpi_{33}^{\mathrm{T}}WL_1\right)+h(t)\left(b_1+c_1+\varpi_{32}^{\mathrm{T}}WL_1\right)\\&+\left(\Xi_1+\Xi_2+b_0+c_0+e_1^{\mathrm{T}}W_1e_1-e_2^{\mathrm{T}}W_1e_2+\varpi_1^{\mathrm{T}}W\varpi_1-\varpi_2^{\mathrm{T}}W\varpi_2+sym\left\{\varpi_{31}^{\mathrm{T}}WL_1\right\}-\eta_1^{\mathrm{T}}\tilde{S}_1\eta_1\right)\Big]\zeta(t),\end{aligned}\tag{15}$$

对于任意矩阵 $Z_1,Z_2\in R^{12n\times n}$，以下等式成立，

$$2\zeta^{\mathrm{T}}(t)\left[Z_1\left(\left(h(t)-h_1\right)e_6-e_{11}\right)+Z_2\left(\left(h_2-h(t)\right)e_7-e_{12}\right)\right]\zeta(t)=0,\tag{16}$$

将上述等式增加到式(15)中，有

$$\begin{aligned}\dot{V}(t)\le\zeta^{\mathrm{T}}(t)\Big[&h^3(t)c_3+h^2(t)\left(b_2+c_2+\varpi_{33}^{\mathrm{T}}WL_1\right)+h(t)\left(b_1+c_1+\varpi_{32}^{\mathrm{T}}WL_1+sym\left\{Z_1e_6-Z_2e_7\right\}\right)\\&+\Big(\Xi_1+\Xi_2+b_0+c_0+e_1^{\mathrm{T}}W_1e_1-e_2^{\mathrm{T}}W_1e_2+\varpi_1^{\mathrm{T}}W\varpi_1-\varpi_2^{\mathrm{T}}W\varpi_2\\&+sym\left\{\varpi_{31}^{\mathrm{T}}WL_1-Z_1\left(h_1e_6+e_{11}\right)+Z_2\left(h_2e_7-e_{12}\right)\right\}-\eta_1^{\mathrm{T}}\tilde{S}_1\eta_1\Big)\Big]\zeta(t)\\=\ &\zeta^{\mathrm{T}}(t)\Sigma_1\zeta(t),\end{aligned}\tag{17}$$

对于 $h(t)\in[h_1,h_2]$，如果式(17)成立，那么 $\dot{V}(t)<0$，证明完成。

注 4：条件(13)和(14)是为了确定 $h(t)\in[h_1,h_2]$ 的三次函数的负性。然而，这不能直接使用现有的线性矩阵不等式(LMI)工具来解决。现使用引理 3，使这些条件以 LMI 的形式呈现。

依据引理 3，可得以下稳定性准则。

定理 2 给定标量 h_1,h_2，对于满足式(2)的系统(1)是渐近稳定的，如果存在对称正定矩阵 $P\in R^{5n\times 5n}$，$W_1\in R^{n\times n}$，$W\in R^{4n\times 4n}$，$S_1,S_2\in R^{n\times n}$，对称矩阵 $X_1,X_2,X_3,X_4,X_5,X_6\in R^{3n\times 3n}$，任意矩阵 $Z_1,Z_2\in R^{12n\times n}$，$Y_1,Y_2,Y_3,Y_4,Y_5,Y_6\in R^{3n\times 3n}$，使当 $h(t)\in[h_1,h_2]$时，下述不等式成立，

$$\begin{aligned}&h_i^3\Pi_3+h_i^2\Pi_2+h_i\Pi_1+\Pi_0<0\quad i=1,2,\\&h_i^3\Pi_3+h_i\Pi_1+\Pi_0<0\quad i=1,2,\\&h_i^2\Pi_2+h_i\Pi_1+\Pi_0<0\quad i=1,2,\\&h_2^3\Pi_3+h_2^2\Pi_2+h_2\Pi_1+\Pi_0-\Pi_3\left(h_2-h_1\right)^2-\left(h_2-h_1\right)^3\left(2\Pi_3h_2+\Pi_3h_1+\Pi_2\right)<0,\\&h_i^3\Psi_3+h_i^2\Psi_2+h_i\Psi_1+\Psi_0<0\quad i=1,2,\\&h_i^3\Psi_3+h_i\Psi_1+\Psi_0<0\quad i=1,2,\\&h_i^2\Psi_2+h_i\Psi_1+\Psi_0<0\quad i=1,2,\\&h_2^3\Psi_3+h_2^2\Psi_2+h_2\Psi_1+\Psi_0-\Psi_3\left(h_2-h_1\right)^2-\left(h_2-h_1\right)^3\left(2\Psi_3h_2+\Psi_3h_1+\Psi_2\right)<0.\end{aligned}$$

为了评估本文引理 5 相对于现有方法的优越性，并验证本文方法的有效性和可行性，我们使用引理 2 对凸式进行估计，并结合引理 4 得到以下稳定性准则(定理 3)。

定理 3 给定标量 h_1,h_2，对于满足式(2)的系统(1)是渐近稳定的，如果存在对称正定矩阵 $P\in R^{5n\times 5n}$，$W_1\in R^{n\times n}$，$W\in R^{4n\times 4n}$，$S_1,S_2\in R^{n\times n}$，对称矩阵 $X_1,X_2,X_3,X_4\in R^{3n\times 3n}$，任意矩阵 $Z_1,Z_2\in R^{12n\times n}$，

$Y_1, Y_2, Y_3, Y_4 \in R^{3n\times 3n}$，当 $h(t)\in[h_1,h_2]$ 时，下述不等式成立，

$$\begin{aligned}
&h_i^2\Pi_2' + h_i\Pi_1' + \Pi_0' < 0 \quad i = 1,2,\\
&h_1^2\Pi_2' + h_1\Pi_1' + \Pi_0' - (h_2 - h_1)^2\Pi_2' < 0,\\
&\Psi_0' < 0,\\
&\Psi_2' + \Psi_1' + \Psi_0' < 0,\\
&\Psi_0' - \Psi_2' < 0.
\end{aligned}$$

5. 数值实例

在本节中，应用一个经常用到的数值例子来验证所提出的互凸三次矩阵不等式的优越性。考虑系统(1)，该系统的参数如下：

$$A = \begin{bmatrix} 0 & 1.0 \\ -10 & -1.0 \end{bmatrix},\quad B = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}.$$

依据定理 2，计算出给定 h_1 对应的 h_2 的最大上界，并将结果列于表 1 中。通过观察表格可以发现，定理 2 所提供的 h_2 值相对较大。同时，本文依据定理 3 给出的相应的稳定性准则，同样计算出给定 h_1 对应的 h_2 的最大上界。通过数值实验的对比结果，不仅突显了互凸二次矩阵不等式的优越性，同时也彰显了互凸三次矩阵不等式的优势所在。

Table 1. Upper bound h_2 given by h_1
表 1. h_1 给定下的上界 h_2

h_1	0.0	0.3	0.7	1.0
[12]	1.59	2.01	2.41	2.62
[30]	1.64	2.13	2.70	2.96
定理 2	1.59	2.99	3.09	3.38
定理 3	1.98	2.42	2.90	3.09

注 5：数值模拟过程并不涉及实验参数的选择，因此，所得结果具有一般性。值得注意的是，我们不仅能够求解在满足定理设定的时滞最大允许上界，而且还能给出最大允许上界情况下的矩阵(即定理中设定的存在矩阵)。然而，所求解出的矩阵并不影响最大允许上界。因此，在本节中我们未提供这些矩阵的具体数值。

注 6：本节所采用的仿真实验平台为 MATLAB，其中内置的线性矩阵不等式(LMI)工具箱。

6. 结论

本文针对时变时滞系统的稳定性问题，提出了一个新的互凸三次矩阵不等式，并据此导出了保守性较低的稳定性准则。此外，给出了在互凸二次矩阵不等式下的稳定性准则，以验证互凸三次矩阵不等式的优越性。实验结果表明，所提出的方法具有可行性和优越性。

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附　录

$$
\begin{aligned}
b_2 = sym\{ & e_1^{\mathrm{T}} A^{\mathrm{T}} P_5 e_9 + e_1^{\mathrm{T}} A^{\mathrm{T}} P_5 e_{10} + e_1^{\mathrm{T}} A^{\mathrm{T}} P_5 e_6 + e_3^{\mathrm{T}} B^{\mathrm{T}} P_5 e_9 + e_3^{\mathrm{T}} B^{\mathrm{T}} P_5 e_{10} + e_3^{\mathrm{T}} B^{\mathrm{T}} P_5 e_6 + e_1^{\mathrm{T}} P_9 e_9 \\
& + e_1^{\mathrm{T}} P_9 e_{10} + e_1^{\mathrm{T}} P_9 e_6 - e_2^{\mathrm{T}} P_9 e_9 - e_2^{\mathrm{T}} P_9 e_6 + e_2^{\mathrm{T}} P_{12} e_9 + e_2^{\mathrm{T}} P_{12} e_{10} + e_2^{\mathrm{T}} P_{12} e_6 - e_4^{\mathrm{T}} P_{12} e_9 - e_4^{\mathrm{T}} P_{12} e_{10} \\
& - e_4^{\mathrm{T}} P_{12} e_6 + h_1 e_1^{\mathrm{T}} P_{14} e_9 + h_1 e_1^{\mathrm{T}} P_{14} e_{10} + h_1 e_1^{\mathrm{T}} P_{14} e_6 - h_1 e_5^{\mathrm{T}} P_{14} e_9 - h_1 e_5^{\mathrm{T}} P_{14} e_{10} - h_1 e_5^{\mathrm{T}} P_{14} e_6 \\
& + (h_2 - h_1) e_9^{\mathrm{T}} P_{15} e_2 + (h_2 - h_1) e_{10}^{\mathrm{T}} P_{12} e_2 + (h_2 - h_1) e_6^{\mathrm{T}} P_{15} e_2 - e_9^{\mathrm{T}} P_{15} e_{11} - e_9^{\mathrm{T}} P_{15} e_{12} - e_{10}^{\mathrm{T}} P_{15} e_{11} \\
& - e_{10}^{\mathrm{T}} P_{15} e_{12} - e_6^{\mathrm{T}} P_{15} e_{11} - e_6^{\mathrm{T}} P_{15} e_{12} \},
\end{aligned}
$$

$$
\begin{aligned}
b_1 = sym\{ & -2h_1 e_1^{\mathrm{T}} A^{\mathrm{T}} P_5 e_9 - 2h_2 e_1^{\mathrm{T}} A^{\mathrm{T}} P_5 e_{10} - 2h_2 e_1^{\mathrm{T}} A^{\mathrm{T}} P_5 e_6 - 2h_1 e_3^{\mathrm{T}} B^{\mathrm{T}} P_5 e_9 - 2h_2 e_3^{\mathrm{T}} B^{\mathrm{T}} P_5 e_{10} \\
& - 2h_2 e_3^{\mathrm{T}} B^{\mathrm{T}} P_5 e_6 - 2h_1 e_1^{\mathrm{T}} P_9 e_9 - 2h_2 e_1^{\mathrm{T}} P_9 e_{10} - 2h_2 e_1^{\mathrm{T}} P_9 e_6 + 2h_1 e_2^{\mathrm{T}} P_9 e_9 + 2h_2 e_2^{\mathrm{T}} P_9 e_{10} + 2h_2 e_2^{\mathrm{T}} P_9 e_6 \\
& - 2h_1 e_2^{\mathrm{T}} P_{12} e_9 - 2h_2 e_2^{\mathrm{T}} P_{12} e_{10} - 2h_2 e_2^{\mathrm{T}} P_{12} e_6 + 2h_1 e_4^{\mathrm{T}} P_{12} e_9 + 2h_2 e_4^{\mathrm{T}} P_{12} e_{10} + 2h_2 e_4^{\mathrm{T}} P_{12} e_6 \\
& - 2h_1 h_2 e_1^{\mathrm{T}} P_{14} e_{10} - 2h_1 h_2 e_1^{\mathrm{T}} P_{14} e_6 + 2h_1^2 e_5^{\mathrm{T}} P_{14} e_9 + 2h_1 h_2 e_5^{\mathrm{T}} P_{14} e_{10} + 2h_1 h_2 e_5^{\mathrm{T}} P_{14} e_6 \\
& - 2h_2 (h_2 - h_1) e_{10}^{\mathrm{T}} P_{12} e_2 - 2h_2 (h_2 - h_1) e_6^{\mathrm{T}} P_{15} e_2 + 2h_1 e_9^{\mathrm{T}} P_{15} e_{11} + 2h_1 e_9^{\mathrm{T}} P_{15} e_{12} + 2h_2 e_{10}^{\mathrm{T}} P_{15} e_{11} \\
& + 2h_2 e_6^{\mathrm{T}} P_{15} e_{11} + 2h_2 e_6^{\mathrm{T}} P_{15} e_{12} - 2h_1^2 e_1^{\mathrm{T}} P_{14} e_9 - 2h_1 (h_2 - h_1) e_9^{\mathrm{T}} P_{15} e_2 + 2h_2 e_{10}^{\mathrm{T}} P_{15} e_{12} \},
\end{aligned}
$$

$$
\begin{aligned}
b_0 = sym\{ & h_1^2 e_1^{\mathrm{T}} A^{\mathrm{T}} P_5 e_9 + h_2^2 e_1^{\mathrm{T}} A^{\mathrm{T}} P_5 e_{10} + h_2^2 e_1^{\mathrm{T}} A^{\mathrm{T}} P_5 e_6 + h_1^2 e_3^{\mathrm{T}} B^{\mathrm{T}} P_5 e_9 + h_2^2 e_3^{\mathrm{T}} B^{\mathrm{T}} P_5 e_{10} \\
& + h_2^2 e_3^{\mathrm{T}} B^{\mathrm{T}} P_5 e_6 + h_1^2 e_1^{\mathrm{T}} P_9 e_9 + h_2^2 e_1^{\mathrm{T}} P_9 e_{10} + h_2^2 e_1^{\mathrm{T}} P_9 e_6 - h_1^2 e_2^{\mathrm{T}} P_9 e_9 - h_2^2 e_2^{\mathrm{T}} P_9 e_{10} - h_2^2 e_2^{\mathrm{T}} P_9 e_6 \\
& + h_1^2 e_2^{\mathrm{T}} P_{12} e_9 + h_2^2 e_2^{\mathrm{T}} P_{12} e_{10} + h_2^2 e_2^{\mathrm{T}} P_{12} e_9 - h_1^2 e_4^{\mathrm{T}} P_{12} e_9 - h_2^2 e_4^{\mathrm{T}} P_{12} e_{10} - h_2^2 e_4^{\mathrm{T}} P_{12} e_6 + h_1^3 e_1^{\mathrm{T}} P_{14} e_9 \\
& + h_1 h_2^2 e_1^{\mathrm{T}} P_{14} e_{10} + h_1 h_2^2 e_1^{\mathrm{T}} P_{14} e_6 - h_1^3 e_5^{\mathrm{T}} P_{14} e_9 - h_1 h_2^2 e_5^{\mathrm{T}} P_{14} e_{10} - h_1 h_2^2 e_5^{\mathrm{T}} P_{14} e_6 + h_1^2 (h_2 - h_1) e_9^{\mathrm{T}} P_{15} e_2 \\
& + h_1^2 (h_2 - h_1) e_{10}^{\mathrm{T}} P_{12} e_2 + h_2^2 (h_2 - h_1) e_6^{\mathrm{T}} P_{15} e_2 - h_1^2 e_9^{\mathrm{T}} P_{15} e_{11} - h_1^2 e_9^{\mathrm{T}} P_{15} e_{12} - h_2^2 e_{10}^{\mathrm{T}} P_{15} e_{11} \\
& - h_2^2 e_{10}^{\mathrm{T}} P_{15} e_{12} - h_2^2 e_6^{\mathrm{T}} P_{15} e_{11} - h_2^2 e_6^{\mathrm{T}} P_{15} e_{12} \},
\end{aligned}
$$

$$
\begin{aligned}
\Xi_1 = sym\{ & e_1^{\mathrm{T}} P_1 A e_1 + e_1^{\mathrm{T}} P_1 B e_3 + h_1 e_1^{\mathrm{T}} A^{\mathrm{T}} P_2 e_5 + h_1 e_3^{\mathrm{T}} B^{\mathrm{T}} P_2 e_5 + e_1^{\mathrm{T}} P_2 e_1 - e_1^{\mathrm{T}} P_2 e_2 + e_1^{\mathrm{T}} A^{\mathrm{T}} P_3 e_{11} \\
& + e_1^{\mathrm{T}} A^{\mathrm{T}} P_3 e_{12} + e_3^{\mathrm{T}} B^{\mathrm{T}} P_3 e_{11} + e_3^{\mathrm{T}} B^{\mathrm{T}} P_3 e_{12} + e_1^{\mathrm{T}} P_3 e_2 - e_1^{\mathrm{T}} P_3 e_4 + h_1^2 e_1^{\mathrm{T}} A^{\mathrm{T}} P_4 e_8 + h_1^2 e_3^{\mathrm{T}} B^{\mathrm{T}} P_4 e_8 \\
& + h_1 e_1^{\mathrm{T}} P_4 e_1 - h_1 e_1^{\mathrm{T}} P_4 e_5 + (h_2 - h_1) e_1^{\mathrm{T}} P_5 e_2 - e_1^{\mathrm{T}} P_5 e_{11} - e_1^{\mathrm{T}} P_5 e_{12} + h_1 e_5^{\mathrm{T}} P_6 e_1 - h_1 e_5^{\mathrm{T}} P_6 e_2 \\
& + e_1^{\mathrm{T}} P_7 e_{11} + e_1^{\mathrm{T}} P_7 e_{12} - e_2^{\mathrm{T}} P_7 e_{11} - e_2^{\mathrm{T}} P_7 e_{12} + h_1 e_5^{\mathrm{T}} P_7 e_2 - h_1 e_5^{\mathrm{T}} P_7 e_4 + h_1^2 e_1^{\mathrm{T}} P_8 e_8 - h_1^2 e_2^{\mathrm{T}} P_8 e_8 \\
& + h_1^2 e_5^{\mathrm{T}} P_8 e_1 - h_1^2 e_5^{\mathrm{T}} P_8 e_5 + h_1 (h_2 - h_1) e_5^{\mathrm{T}} P_9 e_2 - h_1 e_5^{\mathrm{T}} P_9 e_{11} - h_1 e_5^{\mathrm{T}} P_9 e_{12} + e_{11}^{\mathrm{T}} P_{10} e_2 - e_{11}^{\mathrm{T}} P_{10} e_4 \\
& + e_{12}^{\mathrm{T}} P_{10} e_2 - e_{12}^{\mathrm{T}} P_{10} e_4 + h_1^2 e_2^{\mathrm{T}} P_{11} e_8 - h_1^2 e_4^{\mathrm{T}} P_{11} e_8 + h_1 e_{11}^{\mathrm{T}} P_{11} e_1 + h_1 e_{12}^{\mathrm{T}} P_{11} e_1 - h_1 e_{11}^{\mathrm{T}} P_{11} e_5 \\
& - h_1 e_{12}^{\mathrm{T}} P_{11} e_5 + (h_2 - h_1) e_{11}^{\mathrm{T}} P_{12} e_2 + (h_2 - h_1) e_{12}^{\mathrm{T}} P_{12} e_2 - e_{11}^{\mathrm{T}} P_{12} e_{11} - e_{11}^{\mathrm{T}} P_{12} e_{12} - e_{12}^{\mathrm{T}} P_{12} e_{11} \\
& - e_{12}^{\mathrm{T}} P_{12} e_{12} + h_1^3 e_8^{\mathrm{T}} P_{13} e_1 - h_1^3 e_8^{\mathrm{T}} P_{13} e_5 + h_1^2 (h_2 - h_1) e_8^{\mathrm{T}} P_{14} e_2 - h_1^2 e_8^{\mathrm{T}} P_{14} e_{11} - h_1^2 e_8^{\mathrm{T}} P_{14} e_{12} \},
\end{aligned}
$$

$$\varpi_1 = \begin{bmatrix} e_2^{\mathrm{T}} & e_1^{\mathrm{T}} & 0 & e_{11}^{\mathrm{T}} + e_{12}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$

$$\varpi_1 = \begin{bmatrix} e_4^{\mathrm{T}} & e_1^{\mathrm{T}} & e_{11}^{\mathrm{T}} + e_{12}^{\mathrm{T}} & 0 \end{bmatrix}^{\mathrm{T}},$$

$$\varpi_{31} = \begin{bmatrix} e_{11}^{\mathrm{T}} + e_{12}^{\mathrm{T}} & (h_2 - h_1) e_1^{\mathrm{T}} & h_1^2 e_9^{\mathrm{T}} + h_2^2 e_{10}^{\mathrm{T}} + h_2 e_{11}^{\mathrm{T}} & (h_2 - h_1)\left(e_{11}^{\mathrm{T}} + e_{12}^{\mathrm{T}}\right) - h_1^2 e_9^{\mathrm{T}} - h_2^2 e_{10}^{\mathrm{T}} - h_2 e_{11}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$

$$\varpi_{32} = \begin{bmatrix} 0 & 0 & -2h_1 e_9^{\mathrm{T}} - 2h_2 e_{10}^{\mathrm{T}} - e_{11}^{\mathrm{T}} & 2h_1 e_9^{\mathrm{T}} + 2h_2 e_{10}^{\mathrm{T}} + e_{11}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$

$$\varpi_{33} = \begin{bmatrix} 0 & 0 & e_9^{\mathrm{T}} + e_{10}^{\mathrm{T}} & -e_9^{\mathrm{T}} - e_{10}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$

$$L_1 = \begin{bmatrix} 0 & e_0^{\mathrm{T}} & e_2^{\mathrm{T}} & -e_4^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$

$$\Xi_2 = h_1^2 e_1^{\mathrm{T}} A^{\mathrm{T}} S_1 A e_1 + h_1^2 e_3^{\mathrm{T}} B^{\mathrm{T}} S_1 B e_3 + h_1^2 sym\left\{e_1^{\mathrm{T}} A^{\mathrm{T}} S_1 B e_3\right\} + \left(h_2 - h_1\right)^2 e_1^{\mathrm{T}} A^{\mathrm{T}} S_2 A e_1$$
$$+\left(h_2 - h_1\right)^2 e_3^{\mathrm{T}} B^{\mathrm{T}} S_2 B e_3 + \left(h_2 - h_1\right)^2 sym\left\{e_1^{\mathrm{T}} A^{\mathrm{T}} S_2 B e_3\right\},$$

$$\eta_1 = \begin{bmatrix} e_1 - e_2 \\ e_1 + e_2 - 2e_5 \\ e_1 - e_2 + 6e_5 - 12e_8 \end{bmatrix} \quad \eta_2 = \begin{bmatrix} e_2 - e_3 \\ e_2 + e_3 - 2e_6 \\ e_2 - e_3 + 6e_6 - 12e_9 \end{bmatrix} \quad \eta_3 = \begin{bmatrix} e_3 - e_4 \\ e_3 + e_4 - 2e_7 \\ e_3 - e_4 + 6e_7 - 12e_{10} \end{bmatrix},$$

$$\tilde{S}_1 = \begin{bmatrix} S_1 & 0 & 0 \\ 0 & 3S_1 & 0 \\ 0 & 0 & 5S_1 \end{bmatrix} \quad \tilde{S}_2 = \begin{bmatrix} S_2 & 0 & 0 \\ 0 & 3S_2 & 0 \\ 0 & 0 & 5S_2 \end{bmatrix},$$

$$c_3 = \frac{1}{\left(h_2 - h_1\right)^3}\left[\eta_2^{\mathrm{T}} X_5 \eta_2 - \eta_3^{\mathrm{T}} X_6 \eta_3 + sym\left\{\eta_2^{\mathrm{T}}\left(Y_6 - Y_5\right)\eta_3\right\}\right],$$

$$c_2 = \frac{1}{\left(h_2 - h_1\right)^2}\left[\eta_2^{\mathrm{T}} X_3 \eta_2 + \frac{-h_2 - 2h_1}{h_2 - h_1}\eta_2^{\mathrm{T}} X_5 \eta_2 + \eta_3^{\mathrm{T}} X_4 \eta_3 + \frac{h_1 + 2h_2}{h_2 - h_1}\eta_3^{\mathrm{T}} X_6 \eta_3\right.$$
$$\left.+ sym\left\{\eta_2^{\mathrm{T}}\left(-Y_3 - Y_4\right)\eta_3 + \frac{3h_1}{h_2 - h_1}\eta_2^{\mathrm{T}} Y_5 \eta_3 - \frac{3h_2}{h_2 - h_1}\eta_2^{\mathrm{T}} Y_6 \eta_3\right\}\right],$$

$$c_1 = \frac{1}{h_2 - h_1}\left[\eta_2^{\mathrm{T}} X_1 \eta_2 + \frac{-\left(h_1 + h_2\right)}{h_2 - h_1}\eta_2^{\mathrm{T}} X_3 \eta_2 + \frac{2h_1 h_2 + h_1^2}{\left(h_2 - h_1\right)^2}\eta_2^{\mathrm{T}} X_5 \eta_2 - \eta_3^{\mathrm{T}} X_2 \eta_3\right.$$
$$- \frac{h_2 + h_1}{h_2 - h_1}\eta_3^{\mathrm{T}} X_4 \eta_3 - \frac{h_2^2 + 2h_1 h_2}{\left(h_2 - h_1\right)^2}\eta_3^{\mathrm{T}} X_6 \eta_3 + sym\left\{\eta_2^{\mathrm{T}}\left(Y_2 - Y_1\right)\eta_3 + \frac{2h_1}{h_2 - h_1}\eta_2^{\mathrm{T}} Y_3 \eta_3\right.$$
$$\left.\left. + \frac{2h_2}{h_2 - h_1}\eta_2^{\mathrm{T}} Y_4 \eta_3 - \frac{3h_1^2}{\left(h_2 - h_1\right)^2}\eta_2^{\mathrm{T}} Y_5 \eta_3 + \frac{3h_2^2}{\left(h_2 - h_1\right)^2}\eta_2^{\mathrm{T}} Y_6 \eta_3\right\}\right],$$

$$c_0 = -\eta_2^{\mathrm{T}} \tilde{S}_2 \eta_2 - \frac{h_2}{h_2 - h_1}\eta_2^{\mathrm{T}} X_1 \eta_2 + \frac{h_1 h_2}{\left(h_2 - h_1\right)^2}\eta_2^{\mathrm{T}} X_3 \eta_2 - \frac{h_1^2 h_2}{\left(h_2 - h_1\right)^3}\eta_2^{\mathrm{T}} X_5 \eta_2 - \eta_3^{\mathrm{T}} \tilde{S}_2 \eta_3$$
$$+ \frac{h_1}{h_2 - h_1}\eta_3^{\mathrm{T}} X_2 \eta_3 + \frac{h_1 h_2}{\left(h_2 - h_1\right)^2}\eta_3^{\mathrm{T}} X_4 \eta_3 + \frac{h_1 h_2^2}{\left(h_2 - h_1\right)^3}\eta_3^{\mathrm{T}} X_6 \eta_3 + sym\left\{\frac{h_1}{h_2 - h_1}\eta_2^{\mathrm{T}} Y_1 \eta_3\right.$$
$$- \frac{h_2}{h_2 - h_1}\eta_2^{\mathrm{T}} Y_2 \eta_3 - \frac{h_1^2}{\left(h_2 - h_1\right)^2}\eta_2^{\mathrm{T}} Y_3 \eta_3 - \frac{h_2^2}{\left(h_2 - h_1\right)^2}\eta_2^{\mathrm{T}} Y_4 \eta_3 + \frac{h_1^3}{\left(h_2 - h_1\right)^3}\eta_2^{\mathrm{T}} Y_5 \eta_3$$
$$\left.- \frac{h_2^3}{\left(h_2 - h_1\right)^3}\eta_2^{\mathrm{T}} Y_6 \eta_3\right\},$$

$$\Psi_3 = \frac{1}{\left(h_2 - h_1\right)^3}\begin{bmatrix} X_5 & Y_5 - Y_6 \\ * & -X_6 \end{bmatrix},$$

$$\Psi_2 = \frac{1}{\left(h_2 - h_1\right)^2}\begin{bmatrix} X_3 - \frac{3h_1}{h_2 - h_1}X_5 & Y_3 + Y_4 - \frac{3h_1}{h_2 - h_1}Y_5 + \frac{3h_2}{h_2 - h_1}Y_6 \\ * & X_4 + \frac{3h_2}{h_2 - h_1}X_6 \end{bmatrix},$$

$$\Psi_1=\frac{1}{h_2-h_1}\begin{bmatrix} X_1-\frac{2h_1}{h_2-h_1}X_3+\frac{3h_1^2X_5}{(h_2-h_1)^2} & Y_1-Y_2+\frac{-2h_1Y_3-2h_2Y_4}{h_2-h_1}+\frac{3h_1^2Y_5-3h_2^2Y_6}{(h_2-h_1)^2} \\ * & -X_2-\frac{2h_2}{h_2-h_1}X_4-\frac{3h_2^2}{(h_2-h_1)^2}X_6 \end{bmatrix},$$

$$\Psi_0=\begin{bmatrix}\Psi_{011} & \Psi_{012}\\ \Psi_{012}^{\mathrm{T}} & \Psi_{022}\end{bmatrix},$$

$$\Psi_{011}=-\tilde{S}_2-\frac{h_1}{h_2-h_1}X_1+\frac{h_1^2}{(h_2-h_1)^2}X_3-\frac{h_1^3}{(h_2-h_1)^3}X_5,$$

$$\Psi_{012}=-\frac{h_1}{h_2-h_1}Y_1+\frac{h_2}{h_2-h_1}Y_2+\frac{h_1^2}{(h_2-h_1)^2}Y_3+\frac{h_2^2}{(h_2-h_1)^2}Y_4-\frac{h_1^3}{(h_2-h_1)^3}Y_5+\frac{h_2^3}{(h_2-h_1)^3}Y_6,$$

$$\Psi_{022}=-\tilde{S}_2+\frac{h_2}{h_2-h_1}X_2+\frac{h_2^2}{(h_2-h_1)^2}X_4+\frac{h_2^3}{(h_2-h_1)^3}X_6,$$

$$\Sigma_1=h^3(t)\Pi_3+h^2(t)\Pi_2+h(t)\Pi_1+\Pi_0,$$

$$\Pi_3=c_3\quad \Pi_2=b_2+c_2+\varpi_{33}^{\mathrm{T}}WL_1\quad \Pi_1=b_1+c_1+\varpi_{32}^{\mathrm{T}}WL_1+sym\{Z_1e_6-Z_2e_7\},$$

$$\Pi_0=\Xi_1+\Xi_2+b_0+c_0+e_1^{\mathrm{T}}W_1e_1-e_2^{\mathrm{T}}W_1e_2+\varpi_1^{\mathrm{T}}W\varpi_1-\varpi_2^{\mathrm{T}}W\varpi_2-\eta_1^{\mathrm{T}}\tilde{S}_1\eta_1$$
$$+sym\{\varpi_{31}^{\mathrm{T}}RL_1-Z_1(h_1e_6+e_{11})+Z_2(h_2e_7-e_{12})\},$$

$$\Pi_2'=b_2+d_2+\varpi_{33}^{\mathrm{T}}WL_1,$$

$$\Pi_1'=b_1+d_1+\varpi_{32}^{\mathrm{T}}WL_1,$$

$$\Pi_0'=\Xi_1+\Xi_2+b_0+d_0+e_1^{\mathrm{T}}W_1e_1-e_2^{\mathrm{T}}W_1e_2+\varpi_1^{\mathrm{T}}W\varpi_1-\varpi_2^{\mathrm{T}}W\varpi_2+sym\{\varpi_{31}^{\mathrm{T}}WL_1\}-\eta_1^{\mathrm{T}}\tilde{S}_1\eta_1,$$

$$d_2=\frac{1}{(h_2-h_1)^2}\left[\eta_2^{\mathrm{T}}X_3\eta_2+\eta_3^{\mathrm{T}}X_4\eta_3+sym\{\eta_2^{\mathrm{T}}(-Y_3-Y_4)\eta_3\}\right],$$

$$d_1=\frac{1}{h_2-h_1}\Bigg[\eta_2^{\mathrm{T}}X_1\eta_2-\frac{h_1+h_2}{h_2-h_1}\eta_2^{\mathrm{T}}X_3\eta_2-\eta_3^{\mathrm{T}}X_2\eta_3-\frac{h_1+h_2}{h_2-h_1}\eta_3^{\mathrm{T}}X_4\eta_3$$
$$+sym\left\{\eta_2^{\mathrm{T}}\left(-Y_1+Y_2+\frac{2h_1Y_3+2h_2Y_4}{h_2-h_1}\right)\eta_3\right\}\Bigg],$$

$$d_0=\eta_2^{\mathrm{T}}\left(-\tilde{S}_2-\frac{h_2X_3}{h_2-h_1}\right)\eta_2+\eta_3^{\mathrm{T}}\left(-\tilde{S}_2+\frac{h_1X_2}{h_2-h_1}+\frac{h_1h_2X_4}{(h_2-h_1)^2}\right)\eta_3$$
$$+sym\left\{\eta_2^{\mathrm{T}}\left(\frac{h_1Y_1-h_2Y_2}{h_2-h_1}-\frac{h_1^2Y_3+h_2^2Y_4}{(h_2-h_1)^2}\right))\eta_3\right\},$$

$$\Psi_2'=\begin{bmatrix}X_3 & Y_3+Y_4\\ * & X_4\end{bmatrix}\quad \Psi_1'=\begin{bmatrix}X_1 & Y_1-Y_2-2Y_4\\ * & -X_2-2X_4\end{bmatrix}\quad \Psi_0'=\begin{bmatrix}-\tilde{S}_2 & Y_2+Y_4\\ * & -\tilde{S}_2+X_2+X_4\end{bmatrix}.$$