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一类*k*-Hessian系统*k*-允许解的存在性

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摘 要

研究一类带非线性算子和梯度项的多参数*k*-Hessian系统Dirichlet问题，运用Krasnosel'skii-Precup不动点定理获得了该系统非平凡*k*-允许径向解的存在性。

关键词

k-Hessian系统，*k*-允许解，不动点定理

The Existence of *k*-Admissible Solution for a Class of *k*-Hessian System

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Abstract

In this paper, we investigate the Dirichlet problem of a *k*-Hessian system with a nonlinear operator and gradients. Several findings concerning the existence of *k*-admissible radial solutions are established via Krasnosel'skii-Precup fixed point theorem.

Keywords

Coupled *k*-Hessian Equations, *k*-Admissible Solutions, Fixed Point Theorem

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1. 引言

本文拟研究一类带非线性算子和梯度项的多参数 k-Hessian 系统

$$\begin{cases} \mathfrak{B}\left(K_1^{\frac{1}{k}}\right)K_1^{\frac{1}{k}} = \mu_1 h_1\left(|x|\right) f_1\left(|x|,-u_1,-u_2\right), & x\in B, \\ \mathfrak{B}\left(K_2^{\frac{1}{k}}\right)K_2^{\frac{1}{k}} = \mu_2 h_2\left(|x|\right) f_2\left(|x|,-u_1,-u_2\right), & x\in B, \\ u_1 = u_2 = 0, & x\in\partial B \end{cases} \tag{1.1}$$

k-允许径向解的存在性，其中 $k\in\{1,2,\cdots,N\}, N\ge 2$， $K_i = S_k\left(\lambda\left(D^2u_i+\alpha|\nabla u_i|I\right)\right)$， $\mu_i>0$ 是参数，$i=1,2$， $B=\left\{x\in R^N : |x|<1\right\}, \alpha\ge 0$， ∇u 表示 u 的梯度，I 是单位矩阵，$\mathfrak{B}$ 是非线性算子且具有如下性质：

$$\mathcal{X} = \left\{\mathfrak{B}\in C^2\left([0,\infty),[0,\infty)\right): \text{存在常数 } \sigma>0, \text{对任意的 } 0<c<1 \text{ 都有 } \mathfrak{B}(cs)\le c^{\sigma}\mathfrak{B}(s)\right\}。$$

对 $i=1,2$， h_i 和 f_i 满足：

(**A**1) $h_i\in C\left([0,1],R_+\right)$ 是一个非负函数，在 $[0,1]$ 的任何子区间上 $h_i\not\equiv 0$；

(**A**2) $f_i\left(r,v_1,v_2\right)\in C\left([0,1]\times R_+^2, R_+\right)$， $R_+ := [0,+\infty)$。

k-Hessian 算子 $S_k\left(D^2u\right)$ 是 Hessian 矩阵 $D^2u=\left(\dfrac{\partial^2 u}{\partial x_i\partial x_j}\right)_{N\times N}$ 的 k 迹或特征值的 k 阶初等对称多项式。一般地，设 $\Omega\subset R^N$ 是一个开集，函数 $u\in C^2(\Omega)$ 对于 $k=1,2,\cdots,N$， k-Hessian 算子被定义为

$$S_k\left(D^2u\right) := S_k\left(\lambda\left(D^2u\right)\right) = \sum_{1\le i_1<i_2<\cdots<i_k\le N}\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}，\quad 1\le k\le N，$$

其中 $\lambda_1,\lambda_2,\cdots,\lambda_N$ 是 D^2u 的特征值， $\lambda\left(D^2u\right)=\left(\lambda_1,\lambda_2,\cdots,\lambda_N\right)$ 称为 D^2u 的特征向量， $S_k\left(\lambda\left(D^2u\right)\right)$ 是二阶完全非线性算子。特别地，当 $k=1$ 时，k-Hessian 算子退化为经典的 Laplace 算子 Δu [1]；当 $k=N$ 时，k-Hessian 算子即为 Monge-Ampère 算子 $\det\left(D^2u\right)$ [2]。

令 $u\in C^2(\Omega)\cup C^0(\bar{\Omega})$ 且 $\Gamma_k := \left\{\lambda\in R^N : S_i(\lambda)>0, i=1,2,\cdots,k\right\}$ 是一个顶点在原点的凸锥。若 $\lambda\left(D^2u+\alpha|Du|I\right)\in\bar{\Gamma}_k$， 则 u 是 k-允许的，详见[3]。

k-Hessian 方程起源于几何学，流体力学和其他应用学科，是一类与超曲面的高斯曲率有关的方程。关于 k-Hessian 方程解的存在性的研究一直以来都是众多学者研究的热点，他们运用不动点理论[4]-[6]，分歧方法[7] [8]，变分方法[9]，单调迭代方法[10] [11]，上下解方法[12] [13]以及移动平面法[14]等方法，讨论了 k-Hessian 方程不同形式解的存在性，多解性以及在边界附近的渐近行为等性质，详见文献[15] [16]。该方程解的存在性使得很多实际问题具有一定的理论价值和实际意义，如种群动力学和人口流动模型等问题。

2021 年，Zhang 等[17]运用单调迭代方法研究了 k-Hessian 系统

$$\begin{cases} \mathfrak{B}\left(S_k^{\frac{1}{k}}\left(\lambda\left(D^2z_1\right)\right)\right)S_k^{\frac{1}{k}}\left(\lambda\left(D^2z_1\right)\right) = b\left(|x|\right)\varphi\left(z_1,z_2\right), & x\in R^N, \\ \mathfrak{B}\left(S_k^{\frac{1}{k}}\left(\lambda\left(D^2z_2\right)\right)\right)S_k^{\frac{1}{k}}\left(\lambda\left(D^2z_2\right)\right) = h\left(|x|\right)\psi\left(z_1\right), & x\in R^N \end{cases}$$

全局正 k-凸径向解的存在性，其中 $\mathfrak{B}$ 是非线性算子，权函数 $b,h\in C\left(\left[0,+\infty\right),\left[0,+\infty\right)\right)$，$\varphi\in C\left(\left[0,\infty\right)\times\left[0,\infty\right),\left(0,\infty\right)\right)$ 关于每个变量是递增的且 $\varphi\left(t_1,t_2\right)>0$，$t_1,t_2>0$，$\psi\in C\left(\left[0,\infty\right),\left(0,\infty\right)\right)$ 是递增的且 $\psi\left(0\right)>0$ 当 $h\left(\left|x\right|\right)\psi\left(z_1\right)=h\left(\left|x\right|\right)\psi\left(z_1,z_2\right)$ 时，Wang 等[11]运用单调迭代方法，研究了该系统有界径向解的存在性以及全局爆破解的存在性和不存在性，其中 $\varphi,\psi\in C\left(\left[0,\infty\right)\times\left[0,\infty\right),\left(0,\infty\right)\right)$ 是递增的。

2024 年，Yang 等[18]研究了一类带梯度项的 k-Hessian 系统

$$\begin{cases} S_k\left(\lambda\left(D^2u_i+\alpha\left|\nabla u_i\right|I\right)\right)=\varphi_i\left(\left|x\right|,-u_1,-u_2,\cdots,-u_n\right),\ x\in B,\\ u_i=0,\ x\in\partial B,\ i=1,2,\cdots,n,\end{cases}$$

其中 $B=\left\{x\in R^N:\left|x\right|<1\right\}$，$\alpha\ge 0$，$n\ge 2$，$N\ge 2$，$1\le k\le N$，$\nabla u$ 表示 u 的梯度，I 是单位矩阵且 $\varphi_i\in C^2\left(\left[0,1\right]\times\left[0,\infty\right)^n,\left[0,\infty\right)\right)$，$i=1,2,\cdots,n$。作者运用 R_+^n 单调矩阵方法以及不动点定理获得了负径向 k-凸解的存在性与多解性。Ji [19]获得了该系统对应的单个 k-Hessian 问题全局 k-允许解存在的充分必要条件。

受上述文献的启发，本文在不要求非线性项具有单调性的情况下，考虑一类带有非线性算子和梯度项的多参数 k-Hessian 系统。同时，运用类似的方法可以获得对应的单个 k-Hessian 问题解的存在性。此外，运用该方法可以进一步得到 k-允许径向解的多解性。

2. 预备知识

引理 2.1 [20]设 $\mathfrak{R}\left(x\right)=x\mathfrak{B}\left(x\right)$，若 $\mathfrak{B}\in\mathcal{X}$，则

(1) $\mathfrak{R}$ 有严格递增的非负逆映射 $\mathfrak{R}^{-1}\left(x\right)$；

(2) 当 $0<b<1$ 时，有 $\mathfrak{R}^{-1}\left(bx\right)\ge b^{\frac{1}{1+\sigma}}\mathfrak{R}^{-1}\left(x\right)$；

(3) 当 $b\ge 1$ 时，有 $\mathfrak{R}^{-1}\left(bx\right)\le b^{\frac{1}{1+\sigma}}\mathfrak{R}^{-1}\left(x\right)$。

令 $r=\left|x\right|=\sqrt{x_1^2+x_2^2+\cdots+x_N^2}<1$。假设 $u\left(r\right)\in C^2\left[0,1\right]$ 是径向对称函数且满足 $u'\left(0\right)=0$，则当 $u\left(\left|x\right|\right)=u\left(r\right)\in C^2\left(B\right)$ 时，有

$$\lambda\left(D^2u+\alpha\left|\nabla u\right|I\right)=\begin{cases}\left(u''\left(r\right)+\alpha u'\left(r\right),\dfrac{1+\alpha r}{r}u'\left(r\right),\cdots,\dfrac{1+\alpha r}{r}u'\left(r\right)\right), & r\in\left(0,1\right],\\ \left(u''\left(0\right),u''\left(0\right),\cdots,u''\left(0\right)\right), & r=0,\end{cases}$$

$$S_k\left(\lambda\left(D^2u+\alpha\left|\nabla u\right|I\right)\right)=\begin{cases}C_{N-1}^{k-1}\left(u''\left(r\right)+\alpha u'\left(r\right)\right)\left(\dfrac{1+\alpha r}{r}u'\left(r\right)\right)^{k-1}+C_{N-1}^{k}\left(\dfrac{1+\alpha r}{r}u'\left(r\right)\right)^{k}, & r\in\left(0,1\right],\\ C_N^k\left(u''\left(0\right)\right)^k, & r=0.\end{cases}\tag{2.1}$$

若 $\mathbf{u}=\left(u_1,u_2\right)$ 是问题(1.1)的径向解当且仅当 $\mathbf{v}=\left(v_1,v_2\right)$ 是常微分方程边值问题

$$\begin{cases}\left[r^{N-k}e^{N\alpha r}\left(-v_1'\left(r\right)\right)^k\right]'=\dfrac{k}{C_{N-1}^{k-1}}\dfrac{e^{N\alpha r}r^{N-1}}{\left(1+\alpha r\right)^{k-1}}\left[\mathfrak{R}^{-1}\left(\mu_1h_1\left(r\right)f_1\left(r,\mathbf{v}\right)\right)\right]^k, & r\in\left(0,1\right),\\ \left[r^{N-k}e^{N\alpha r}\left(-v_2'\left(r\right)\right)^k\right]'=\dfrac{k}{C_{N-1}^{k-1}}\dfrac{e^{N\alpha r}r^{N-1}}{\left(1+\alpha r\right)^{k-1}}\left[\mathfrak{R}^{-1}\left(\mu_2h_2\left(r\right)f_2\left(r,\mathbf{v}\right)\right)\right]^k, & r\in\left(0,1\right),\\ v_1'\left(0\right)=0,\ v_2'\left(0\right)=0,\ v_1\left(1\right)=0,\ v_2\left(1\right)=0\end{cases}\tag{2.2}$$

的解。

显然，问题(2.2)等价于如下的积分系统

$$
\begin{cases}
v_1(r)=\int_r^1\left(\frac{k}{C_{N-1}^{k-1}}\frac{s^{k-N}}{\mathrm{e}^{N\alpha s}}\int_0^s\frac{\mathrm{e}^{N\alpha\tau}\tau^{N-1}}{(1+\alpha\tau)^{k-1}}\left[\Re^{-1}\left(\mu_1 h_1(\tau)f_1(\tau,\mathbf{v})\right)\right]^k\mathrm{d}\tau\right)^{\frac{1}{k}}\mathrm{d}s, \quad r\in[0,1],\\
v_2(r)=\int_r^1\left(\frac{k}{C_{N-1}^{k-1}}\frac{s^{k-N}}{\mathrm{e}^{N\alpha s}}\int_0^s\frac{\mathrm{e}^{N\alpha\tau}\tau^{N-1}}{(1+\alpha\tau)^{k-1}}\left[\Re^{-1}\left(\mu_2 h_2(\tau)f_2(\tau,\mathbf{v})\right)\right]^k\mathrm{d}\tau\right)^{\frac{1}{k}}\mathrm{d}s, \quad r\in[0,1].
\end{cases}
\tag{2.3}
$$

设 $E=C[0,1]\times C[0,1]$，则 E 按范数 $\|\mathbf{v}\|=\|(v_1,v_2)\|=\|v_1\|+\|v_2\|$ 构成 Banach 空间，其中 $\|v_i\|=\max_{r\in[0,1]}|v_i(r)|(i=1,2)$。定义 $C[0,1]$ 上的锥 H_i

$$H_i=\left\{v_i\in C[0,1]:v_i(r)\geq 0,r\in[0,1],\min_{r\in[\delta,1-\delta]}v_i(r)\geq\delta\|v_i\|\right\},$$

其中 $\delta\in\left(0,\frac{1}{2}\right)$。

令 $H=H_1\times H_2$。对于 $\mathbf{v}=(v_1,v_2)\in H$，定义

$$\mathfrak{F}_i(\mathbf{v}(r))=\int_r^1\left(\frac{k}{C_{N-1}^{k-1}}\frac{s^{k-N}}{\mathrm{e}^{N\alpha s}}\int_0^s\frac{\mathrm{e}^{N\alpha\tau}\tau^{N-1}}{(1+\alpha\tau)^{k-1}}\left[\Re^{-1}\left(\mu_i h_i(\tau)\varphi_i(\tau,v_1(\tau),v_2(\tau))\right)\right]^k\mathrm{d}\tau\right)^{\frac{1}{k}}\mathrm{d}s. \tag{2.4}$$

令 $\mathfrak{F}=(\mathfrak{F}_1,\mathfrak{F}_2)$。容易验证 $\mathfrak{F}$ 是全连续算子且(2.4)等价于不动点方程

$$\mathbf{v}=\mathfrak{F}(\mathbf{v}),\quad \mathbf{v}\in H。$$

引理 2.2 [5]设 $(X,\|\cdot\|)$ 是 Banach 空间，H_1,H_2 是 X 上的两个锥且 $H:=H_1\times H_2$，$c,C\in R_+^2$ 满足 $0<c_i<C_i$，$H_{c,C}=\{\mathbf{v}=(v_1,v_2)\in H:c_i\leq\|v_i\|\leq C_i,i=1,2\}$。令 $N:H_{c,C}\to H$，$N=(N_1,N_2)$ 是一个紧算子。假设在 $H_{c,C}$ 上以下条件之一成立：

(i) 若 $\|v_i\|=c_i$，则 $N_i(\mathbf{v})-v_i\notin H_i$ 且若 $\|v_i\|=C_i$，则 $v_i-N_i(\mathbf{v})\notin H_i$；

(ii) 若 $\|v_i\|=c_i$，则 $v_i-N_i(\mathbf{v})\notin H_i$ 且若 $\|v_i\|=C_i$，则 $N_i(\mathbf{v})-v_i\notin H_i$，

则 N 在 H 上有不动点 $\mathbf{v}=(v_1,v_2)$ 且满足 $c_i\leq\|v_i\|\leq C_i$，$i=1,2$。

引理 2.3 [18]设 z 是 $[a,b]$ 上非减的连续函数，则

(1) 对于 $0<\beta\leq 1$，有

$$\left(\int_a^b z(t)\mathrm{d}t\right)^\beta\geq(b-a)^{\beta-1}\int_a^b z^\beta(t)\mathrm{d}t；$$

(2) 对于 $\beta\geq 1$，有

$$\left(\int_a^b z(t)\mathrm{d}t\right)^\beta\leq(b-a)^{\beta-1}\int_a^b z^\beta(t)\mathrm{d}t。$$

为方便起见，定义如下标记：

$$M_i=\max\{f_i(r,v_1,v_2):0\leq r\leq 1,0\leq v_i\leq C_i\},$$
$$m_i=\min\{f_i(r,v_1,v_2):\delta\leq r\leq 1-\delta,\delta c_i\leq v_i\leq C_i\},$$
$$N_1=\left(\frac{k}{\mathrm{e}^{N\alpha}(1+\alpha)^{k-1}C_{N-1}^{k-1}}\right)^{\frac{1}{k}},\quad N_2=\left(\frac{k}{C_{N-1}^{k-1}}\right)^{\frac{q}{k}},$$
$$Q_i=\left(N_2\int_0^1\left[\Re^{-1}(h_i(s))\right]^q\mathrm{d}s\right)^{\frac{1}{q}},\quad P_i=N_1\int_\delta^{1-\delta}\int_\delta^s\tau^{\frac{N-1}{k}}\left[\Re^{-1}(h_i(\tau))\right]\mathrm{d}\tau\mathrm{d}s,$$

其中 $q\geq 1$ 是一个常数且满足 $\frac{q}{k}\geq 1$。

3. *k*-允许解的存在性

本节运用引理 2.2 来建立 *k*-Hessian 系统(1.1) *k*-允许径向解的存在性结果并给出解存在时参数的范围。

定理 3.1 假设(**A**1)和(**A**2)成立。若存在常数 $c_i, C_i > 0$ 满足 $0 < c_i < C_i \left(i = 1,2\right)$ 使得对 $\mu_i \in \left[\frac{1}{M_i}, \frac{1}{m_i}\right)$，$\sigma > 0$，有

$$\mu_i m_i > \left(\frac{c_i}{P}\right)^{1+\sigma}, \quad \mu_i M_i < \left(\frac{C_i}{Q}\right)^{1+\sigma}$$

成立，则问题(1.1)至少有一个 *k*-允许径向解 $\mathbf{u} = \left(u_1, u_2\right) = \left(-v_1, -v_2\right)$ 且满足 $c_i \le \left\|u_i\right\| \le C_i$，$i = 1,2$。

证明对 $\mathbf{v} \in H_{c,C}$，有 $c_1 \le \left\|v_2\right\| \le C_1$ 和 $c_2 \le \left\|v_2\right\| \le C_2$ 由 H_i 的定义可得

$$\delta c_i \le v_i\left(r\right) \le C_i, \quad r \in \left[\delta, 1-\delta\right]。$$

若 $\left\|v_i\right\| = c_i$，则有 $v_i - \mathfrak{F}_i\left(\mathbf{v}\right) \notin H_i$。事实上，假设 $v_i - \mathfrak{F}_i\left(\mathbf{v}\right) \in H_i$，则根据引理 2.3 中(1)可得

$$\begin{aligned}
v_i\left(\delta\right) &\ge \mathfrak{F}_i\left(\mathbf{v}\left(\delta\right)\right) \\
&= \int_\delta^1 \left(\frac{k}{C_{N-1}^{k-1}} \frac{s^{k-N}}{\mathrm{e}^{N\alpha s}} \int_0^s \frac{\mathrm{e}^{N\alpha\tau}\tau^{N-1}}{\left(1+\alpha\tau\right)^{k-1}} \left[\Re^{-1}\left(\mu_i h_i\left(\tau\right) f_i\left(\tau, \mathbf{v}\right)\right)\right]^k \mathrm{d}\tau\right)^{\frac{1}{k}} \mathrm{d}s \\
&= \left(\frac{k}{C_{N-1}^{k-1}}\right)^{\frac{1}{k}} \int_\delta^1 \left(\frac{s^{k-N}}{\mathrm{e}^{N\alpha s}}\right)^{\frac{1}{k}} \left(\int_0^s \frac{\mathrm{e}^{N\alpha\tau}\tau^{N-1}}{\left(1+\alpha\tau\right)^{k-1}} \left[\Re^{-1}\left(\mu_i h_i\left(\tau\right) f_i\left(\tau, \mathbf{v}\right)\right)\right]^k \mathrm{d}\tau\right)^{\frac{1}{k}} \mathrm{d}s \\
&\ge \left(\frac{k}{C_{N-1}^{k-1}}\right)^{\frac{1}{k}} \int_\delta^1 \left(\frac{s^{k-N}}{\mathrm{e}^{N\alpha s}}\right)^{\frac{1}{k}} s^{\frac{1}{k}-1} \int_0^s \left(\frac{\mathrm{e}^{N\alpha\tau}\tau^{N-1}}{\left(1+\alpha\tau\right)^{k-1}}\right)^{\frac{1}{k}} \left[\Re^{-1}\left(\mu_i h_i\left(\tau\right) f_i\left(\tau, \mathbf{v}\right)\right)\right] \mathrm{d}\tau \mathrm{d}s \\
&\ge \left(\frac{k}{\left(1+\alpha\right)^{k-1} C_{N-1}^{k-1}}\right)^{\frac{1}{k}} \int_\delta^1 \left(\frac{s^{k-N}}{\mathrm{e}^{N\alpha s}}\right)^{\frac{1}{k}} \int_0^s \left(\mathrm{e}^{N\alpha\tau}\tau^{N-1}\right)^{\frac{1}{k}} \left[\Re^{-1}\left(\mu_i h_i\left(\tau\right) f_i\left(\tau, \mathbf{v}\right)\right)\right] \mathrm{d}\tau \mathrm{d}s \\
&\ge \left(\frac{k}{\left(1+\alpha\right)^{k-1} C_{N-1}^{k-1}}\right)^{\frac{1}{k}} \int_\delta^1 \left(\frac{1}{\mathrm{e}^{N\alpha}}\right)^{\frac{1}{k}} \int_0^s \tau^{\frac{N-1}{k}} \left[\Re^{-1}\left(\mu_i h_i\left(\tau\right) f_i\left(\tau, \mathbf{v}\right)\right)\right] \mathrm{d}\tau \mathrm{d}s \\
&= \left(\frac{k}{\mathrm{e}^{N\alpha}\left(1+\alpha\right)^{k-1} C_{N-1}^{k-1}}\right)^{\frac{1}{k}} \int_\delta^1 \int_0^s \tau^{\frac{N-1}{k}} \left[\Re^{-1}\left(\mu_i h_i\left(\tau\right) f_i\left(\tau, \mathbf{v}\right)\right)\right] \mathrm{d}\tau \mathrm{d}s \\
&= N_1 \int_\delta^1 \int_0^s \tau^{\frac{N-1}{k}} \left[\Re^{-1}\left(\mu_i h_i\left(\tau\right) f_i\left(\tau, \mathbf{v}\right)\right)\right] \mathrm{d}\tau \mathrm{d}s \\
&\ge N_1 \int_\delta^{1-\delta} \int_\delta^s \tau^{\frac{N-1}{k}} \left[\Re^{-1}\left(\mu_i h_i\left(\tau\right) m_i\right)\right] \mathrm{d}\tau \mathrm{d}s \\
&> N_1 \int_\delta^{1-\delta} \int_\delta^s \tau^{\frac{N-1}{k}} \left[\Re^{-1}\left(h_i\left(\tau\right)\left(\frac{c_i}{P_i}\right)^{1+\sigma}\right)\right] \mathrm{d}\tau \mathrm{d}s \\
&\ge N_1 \frac{c_i}{P_i} \int_\delta^{1-\delta} \int_\delta^s \tau^{\frac{N-1}{k}} \left[\Re^{-1}\left(h_i\left(\tau\right)\right)\right] \mathrm{d}\tau \mathrm{d}s \\
&= c_i.
\end{aligned}$$

因此，$c_i > c_i$ 矛盾！

当$\|v_i\|=C_i$时，则$\mathfrak{F}_i(\mathbf{v})-v_i\notin H_i$。反设$\mathfrak{F}_i(\mathbf{v})-v_i\in H_i$，由引理2.3中(2)可得

$$
\begin{aligned}
v_i^q(r) &\le \mathfrak{F}_i^q(\boldsymbol{v}(r)) \\
&=\left[\int_r^1\left(\frac{k}{C_{N-1}^{k-1}}\frac{s^{k-N}}{e^{N\alpha s}}\int_0^s\frac{e^{N\alpha\tau}\tau^{N-1}}{(1+\alpha\tau)^{k-1}}\left[\mathfrak{R}^{-1}\left(\mu_i h_i(\tau) f_i(\tau,\mathbf{v})\right)\right]^k \mathrm{d}\tau\right)^{\frac{1}{k}}\mathrm{d}s\right]^q \\
&\le (1-r)^{q-1}\int_r^1\left(\frac{k}{C_{N-1}^{k-1}}\frac{s^{k-N}}{e^{N\alpha s}}\int_0^s\frac{e^{N\alpha\tau}\tau^{N-1}}{(1+\alpha\tau)^{k-1}}\left[\mathfrak{R}^{-1}\left(\mu_i h_i(\tau) f_i(\tau,\mathbf{v})\right)\right]^k \mathrm{d}\tau\right)^{\frac{q}{k}}\mathrm{d}s \\
&\le \left(\frac{k}{C_{N-1}^{k-1}}\right)^{\frac{q}{k}}\int_r^1\left(\frac{s^{k-N}}{e^{N\alpha s}}\right)^{\frac{q}{k}}\left(\int_0^s\frac{e^{N\alpha\tau}\tau^{N-1}}{(1+\alpha\tau)^{k-1}}\left[\mathfrak{R}^{-1}\left(\mu_i h_i(\tau) f_i(\tau,\mathbf{v})\right)\right]^k \mathrm{d}\tau\right)^{\frac{q}{k}}\mathrm{d}s \\
&\le \left(\frac{k}{C_{N-1}^{k-1}}\right)^{\frac{q}{k}}\int_r^1\left(\frac{s^{k-N}}{e^{N\alpha s}}\right)^{\frac{q}{k}} s^{\frac{q}{k}-1}\int_0^s\left(\frac{e^{N\alpha\tau}\tau^{N-1}}{(1+\alpha\tau)^{k-1}}\right)^{\frac{q}{k}}\left[\mathfrak{R}^{-1}\left(\mu_i h_i(\tau) f_i(\tau,\mathbf{v})\right)\right]^q \mathrm{d}\tau\mathrm{d}s \\
&\le \left(\frac{k}{C_{N-1}^{k-1}}\right)^{\frac{q}{k}}\int_r^1\left(\frac{s^{k-N}}{e^{N\alpha s}}\right)^{\frac{q}{k}}\left(e^{N\alpha s}s^{N-1}\right)^{\frac{q}{k}}\int_0^s\left[\mathfrak{R}^{-1}\left(\mu_i h_i(\tau) f_i(\tau,\mathbf{v})\right)\right]^q \mathrm{d}\tau\mathrm{d}s \\
&= \left(\frac{k}{C_{N-1}^{k-1}}\right)^{\frac{q}{k}}\int_r^1 s^{\frac{q(k-1)}{k}}\int_0^s\left[\mathfrak{R}^{-1}\left(\mu_i h_i(\tau) f_i(\tau,\mathbf{v})\right)\right]^q \mathrm{d}\tau\mathrm{d}s \\
&\le \left(\frac{k}{C_{N-1}^{k-1}}\right)^{\frac{q}{k}}\int_r^1\int_0^s\left[\mathfrak{R}^{-1}\left(\mu_i h_i(\tau) f_i(\tau,\mathbf{v})\right)\right]^q \mathrm{d}\tau\mathrm{d}s \\
&= N_2\int_r^1\int_0^s\left[\mathfrak{R}^{-1}\left(\mu_i h_i(\tau) f_i(\tau,\mathbf{v})\right)\right]^q \mathrm{d}\tau\mathrm{d}s \\
&\le N_2\int_0^1\int_0^s\left[\mathfrak{R}^{-1}\left(\mu_i h_i(\tau) M_i\right)\right]^q \mathrm{d}\tau\mathrm{d}s \\
&< N_2\int_0^1\int_0^s\left[\mathfrak{R}^{-1}\left(h_i(\tau)\left(\frac{C_i}{Q_i}\right)^{1+\sigma}\right)\right]^q \mathrm{d}\tau\mathrm{d}s \\
&\le N_2\left(\frac{C_i}{Q_i}\right)^q\int_0^1\int_0^1\left[\mathfrak{R}^{-1}\left(h_i(\tau)\right)\right]^q \mathrm{d}\tau\mathrm{d}s \\
&= N_2\left(\frac{C_i}{Q_i}\right)^q\int_0^1\left[\mathfrak{R}^{-1}\left(h_i(s)\right)\right]^q \mathrm{d}s \\
&= C_i^q.
\end{aligned}
$$

因此，$C_i<C_i$矛盾！

由引理2.2 (ii)可得系统(2.3)至少存在一个解$\mathbf{v}=(v_1,v_2)$。因此，问题(1.1)至少存在一个k-允许径向解$\mathbf{u}=-\mathbf{v}$。

推论3.1 假设(**A**1)和(**A**2)成立。若存在两个正序列$c_i^j, C_i^j>0\ (i=1,2;\ j=1,2,\cdots,m)$对$\mu_i\in\left[\frac{1}{M_i^j},\frac{1}{m_i^j}\right)$，$\sigma>0$，$C_i^{j-1}\le c_i^j$，有

$$
\mu_i m_i^j>\left(\frac{c_i^j}{P}\right)^{1+\sigma},\quad \mu_i M_i^j<\left(\frac{C_i^j}{Q}\right)^{1+\sigma}
$$

成立，则问题(1.1)至少存在 m 个 k-允许径向解 $\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_m$ 满足 $\mathbf{u}_j=\left(-v_1^j,-v_2^j\right)$， $c_i^j\le\left\|v_i^j\right\|\le C_i^j$ 。

证明首先，定义如下的算子 $\mathfrak{F}_i^j\mathbf{v}_j:H_i\to H_i$，

$$\mathfrak{F}_i^j\left(\mathbf{v}_j(r)\right)=\int_r^1\left(\frac{k}{C_{N-1}^{k-1}}\frac{s^{k-N}}{\mathrm{e}^{N\alpha s}}\int_0^s\frac{\mathrm{e}^{N\alpha\tau}\tau^{N-1}}{(1+\alpha\tau)^{k-1}}\left[\mathfrak{R}^{-1}\mu_i h_i(\tau)f_i\left(\tau,\mathbf{v}_j\right)\right]^k\mathrm{d}\tau\right)^{\frac{1}{k}}\mathrm{d}s\text{。}\tag{3.1}$$

显然， $\mathfrak{F}_i^j\mathbf{v}_j:H_i\to H_i\,(i=1,2;j=1,2,\cdots,m)$ 是全连续算子，并且由 H_{c^j,C^j} 和 H 的定义可知，$\mathfrak{F}^j\mathbf{v}_j:H_{c^j,C^j}\to H$ 也是全连续算子。对 $j=1,2,\cdots,m$，$\mathbf{v}_j$ 是 $\mathfrak{F}^j$ 的一个不动点当且仅当 $\mathbf{v}_j$ 是(3.1)的一个解。对任意的 $\mathbf{v}_j\in H_{c^j,C^j}$，有 $c_i^j\le\left\|v_i^j\right\|\le C_i^j$ 且满足 $0<c_i^j<C_i^j$。对 $i=1,2$ 和 $j=1,2,\cdots,m$，$\left(c_i^{j-1},C_i^{j-1}\right)\cap\left(c_i^j,C_i^j\right)=\varnothing$ 成立。由推论 3.1 中类似的证明可知，对每个 $j=1,2,\cdots,m$，都可以得到 $\mathfrak{F}^j$ 有一个不动点 $\mathbf{v}_j$ 这说明系统(2.4)至少存在 m 个不同的解 $\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_m$， 即问题(1.1)至少存在 m 个 k-允许径向解 $\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_m$。

4. 数值例子

例子 4.1. 考虑如下 3-Hessian 系统的 Dirichlet 问题 k-允许径向解的存在性：

$$\begin{cases}\left(S_3^{\frac{1}{3}}\left(\lambda\left(Du_1+\left|\nabla u_1\right|I\right)\right)\right)^2S_3^{\frac{1}{3}}\left(\lambda\left(Du_1+\left|\nabla u_1\right|I\right)\right)=|x|^3\left(u_1u_2\right)^{\frac{1}{2}}, & x\in\partial B,\\ \left(S_3^{\frac{1}{3}}\left(\lambda\left(Du_2+\left|\nabla u_2\right|I\right)\right)\right)^2S_3^{\frac{1}{3}}\left(\lambda\left(Du_2+\left|\nabla u_2\right|I\right)\right)=|x|^6\left(-u_1^2u_2\right)^{\frac{1}{3}}, & x\in B,\\ u_1=u_2=0, & x\in\partial B,\end{cases}\tag{4.1}$$

其中 $B=\left\{x\in R^N:|x|<1\right\}$，则 3-Hessian 问题(4.1)至少有一个 k-允许径向解。

证明事实上，$k=3$，$N=4$，$q=4$，$\alpha=1$，$\sigma=2$，$\delta=\frac{1}{4}$，$h_1(r)=r^3$，$h_2(r)=r^6$，$f_1\left(r,v_1,v_2\right)=\left(v_1v_2\right)^{\frac{1}{2}}$，$f_2\left(r,v_1,v_2\right)=\left(v_1^2v_2\right)^{\frac{1}{3}}$。令 $\mu_1=100$， $\mu_2=1000$， $c_1=c_2=0.001$， $C_1=10$， $C_2=15$ 且 $\mathfrak{B}(x)=x^2$ 因此，$\mathfrak{R}(x)=x^3$。通过计算，我们可以得到

$$N_1=\left(\frac{k}{\mathrm{e}^{N\alpha}(1+\alpha)^{k-1}C_{N-1}^{k-1}}\right)^{\frac{1}{k}}=\sqrt[3]{\frac{1}{4\times\mathrm{e}^4}},\ N_2=\left(\frac{k}{C_{N-1}^{k-1}}\right)^{\frac{q}{k}}=1,$$

$$P_1=N_1\int_\delta^{1-\delta}\int_\delta^s\tau^{\frac{N-1}{k}}\left[\mathfrak{R}^{-1}\left(h_1(\tau)\right)\right]\mathrm{d}\tau\mathrm{d}s=\sqrt[3]{\frac{1}{4\times\mathrm{e}^4}}\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^s\tau\left[\mathfrak{R}^{-1}\left(\tau^3\right)\right]\mathrm{d}\tau\mathrm{d}s=\frac{3\times4^{\frac{2}{3}}\times\left(\frac{1}{\mathrm{e}^4}\right)^{\frac{1}{3}}}{512},$$

$$P_2=N_1\int_\delta^{1-\delta}\int_\delta^s\tau^{\frac{N-1}{k}}\left[\mathfrak{R}^{-1}\left(h_2(\tau)\right)\right]\mathrm{d}\tau\mathrm{d}s=\sqrt[3]{\frac{1}{4\times\mathrm{e}^4}}\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^s\tau\left[\mathfrak{R}^{-1}\left(\tau^6\right)\right]\mathrm{d}\tau\mathrm{d}s=\frac{29\times4^{\frac{2}{3}}\times\left(\frac{1}{\mathrm{e}^4}\right)^{\frac{1}{3}}}{10240},$$

$$Q_1=\left(N_2\int_0^1\left[\mathfrak{R}^{-1}\left(h_i(s)\right)\right]^q\mathrm{d}s\right)^{\frac{1}{q}}=\left(\int_0^1\left[\mathfrak{R}^{-1}\left(s^3\right)\right]^4\mathrm{d}s\right)^{\frac{1}{4}}=\frac{5^{\frac{3}{4}}}{5},$$

$$Q_2=\left(N_2\int_0^1\left[\mathfrak{R}^{-1}\left(h_i(s)\right)\right]^q\mathrm{d}s\right)^{\frac{1}{q}}=\left(\int_0^1\left[\mathfrak{R}^{-1}\left(s^6\right)\right]^4\mathrm{d}s\right)^{\frac{1}{4}}=\frac{9^{\frac{3}{4}}}{9},$$

则有

$$m_1=\min\left\{f_1(r,v_1,v_2):\delta\le r\le 1-\delta,\delta c_1\le v_i\le C_1\right\}=f_1(\delta,\delta c_1,\delta c_1)=\delta c_1,$$
$$M_1=\max\left\{f_1(r,v_1,v_2):0\le r\le 1,0\le v_i\le C_1\right\}=f_1(1,C_1,C_1)=C_1,$$
$$\mu_1 m_1=100\times\frac{1}{4}\times 0.001=0.025,\ \left(\frac{c_1}{P_1}\right)^3\approx 0.017,$$
$$\mu_1 M_1=100\times 10=1000,\ \left(\frac{C_1}{Q_1}\right)^3=3343.70$$

和

$$m_2=\min\left\{f_2(r,v_1,v_2):\delta\le r\le 1-\delta,\delta c_2\le v_i\le C_2\right\}=f_2(\delta,\delta c_2,\delta c_2)=\delta c_2,$$
$$M_2=\max\left\{f_2(r,v_1,v_2):0\le r\le 1,0\le v_i\le C_2\right\}=f_2(1,C_2,C_2)=C_2,$$
$$\mu_2 m_2=1000\times\frac{1}{4}\times 0.001=0.25,\ \left(\frac{c_2}{P_2}\right)^3\approx 0.15,$$
$$\mu_2 M_2=1000\times 15=15000,\ \left(\frac{C_2}{Q_2}\right)^3\approx 17537.01.$$

所以，对于 $\mu_1\in\left[\frac{1}{10},4000\right)$，$\mu_2\in\left[\frac{1}{15},4000\right)$，有如下不等式成立：

$$\mu_1 m_1>\left(\frac{c_1}{P_1}\right)^3，\ \mu_2 m_2>\left(\frac{c_2}{P_2}\right)^3，$$
$$\mu_1 M_1<\left(\frac{C_1}{Q_1}\right)^3，\ \mu_2 M_2<\left(\frac{C_2}{Q_2}\right)^3。$$

根据定理 3.1，我们可以得到问题(4.1)至少有一个 k-允许径向解。

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