Pure Mathematics 理论数学, 2024, 14(5), 219-225
Published Online May 2024 in Hans. https://www.hanspub.org/journal/pm
https://doi.org/10.12677/pm.2024.145179

Hans汉斯

椭圆链式KP系统的可积性分析

王辛乙

上海理工大学理学院，上海

收稿日期：2024年3月25日；录用日期：2024年4月29日；发布日期：2024年5月28日

摘 要

我们以前已经深入研究了求解椭圆链式KP方程的直接线性化方法，而求解该方程的柯西矩阵方法只是近年来的一个热门主题。本文将讨论基于柯西矩阵方法的椭圆链式KP方程的可积性，为下一步的求解做准备。本文首先从辅助向量 $\boldsymbol{u}^{(2i)}=(\boldsymbol{I}+\boldsymbol{XC})^{-1}\boldsymbol{P}^{i}\boldsymbol{c}$ 出发得到椭圆链式KP系统的lax组，然后从Lax组中推导出椭圆链式KP系统来完成闭环。

关键词

椭圆链式KP系统，Lax组，柯西矩阵法

Integrability Analysis of Elliptical Chain KP System

Xinyi Wang

School of Science, University of Shanghai for Science and Technology, Shanghai

Received: Mar. 25th, 2024; accepted: Apr. 29th, 2024; published: May 28th, 2024

Abstract

We have previously delved into direct linearization methods for solving the elliptic lattice KP system, and the Cauchy matrix method for solving this equation has only been a hot topic in recent years. This article will discuss the integrability of the elliptic lattice KP system based on the Cauchy matrix method, in preparation for the next step of solution. Firstly, starting from the auxiliary vector $\boldsymbol{u}^{(2i)}=(\boldsymbol{I}+\boldsymbol{XC})^{-1}\boldsymbol{P}^{i}\boldsymbol{c}$, we obtain the lax system of the elliptic lattice KP system, and then derive the elliptic lattice KP system from the lax system to complete the closed-loop process.

文章引用: 王辛乙. 椭圆链式 KP 系统的可积性分析[J]. 理论数学, 2024, 14(5): 219-225.
DOI: 10.12677/pm.2024.145179

Keywords

Elliptic Lattice KP System, Lax Triplet, Cauchy Matrix Method

1. 引言

椭圆可积系统被认为是“顶部的可积系统”[1]，通过周期性退化，它们可以简化为与有理函数相关的可积方程。现已知的例子是链式 Landau-Lifshitz 方程[2]，Adler 的链式 krichhever-novikov 系统[3]，也称为 Adler-bobenko-suris 四边形方程[4]中的 Q4 方程，Adler-yamilov 系统[5]和椭圆 Korteweg-de Vries 系统[6]。

本节要研究的椭圆 KP 方程是 3 + 1 维的，它是比 2 + 1 维椭圆 Kdv 方程更高阶的存在，当然由椭圆 Kadomtsev-Petviashvili (KP)方程可以约化为椭圆 Kdv 方程，因此我们可以把椭圆 KP 方程看作是椭圆 Kdv 方程的推广。

椭圆链式 KP 系统的 Lax 表示与辅助向量在离散演化密切相关。在之前的研究中，我们已经通过直接线性化方法求出过椭圆 KP 的离散 Lax 对[7]，本文将介绍另一种方法——柯西矩阵法，通过 Lax 对的推导，能够更直观的感受椭圆 KP 方程的可积性。

2. 预备知识

本文考虑与椭圆曲线相关的链势 Kadomtsev-Petviashvili (KP)方程的推广[8]：

$$\begin{aligned}&(a-\tilde{u})\left(b-c+\tilde{\dot{u}}-\tilde{\hat{u}}\right)+(b-\hat{u})\left(c-a+\hat{\tilde{u}}-\hat{\dot{u}}\right)+(c-\dot{u})\left(a-b+\dot{\hat{u}}-\dot{\tilde{u}}\right)\\&=g\left(\tilde{\hat{s}}'(\tilde{s}-\hat{s})+\hat{\dot{s}}'(\hat{s}-\dot{s})+\dot{\tilde{s}}'(\dot{s}-\tilde{s})\right),\end{aligned}\tag{2.1a}$$

$$\begin{aligned}&\frac{(a-\dot{u})\dot{\tilde{s}}-(b+\dot{u})\hat{\dot{s}}+\hat{\dot{w}}-\tilde{\dot{w}}}{\dot{s}}+\frac{(b+\tilde{u})\hat{\tilde{s}}-(c+\tilde{u})\tilde{\dot{s}}+\tilde{\dot{w}}-\hat{\tilde{w}}}{\tilde{s}}\\&+\frac{(c+\hat{u})\dot{\hat{s}}-(a+\hat{u})\hat{\tilde{s}}+\hat{\tilde{w}}-\hat{\dot{w}}}{\hat{s}}=0,\end{aligned}\tag{2.1b}$$

$$\begin{aligned}&\frac{\left(a-\tilde{\dot{u}}\right)\dot{s}'-\left(c-\tilde{\dot{u}}\right)\tilde{s}'+\dot{w}'-\tilde{w}'}{\tilde{\dot{s}}'}+\frac{\left(c-\hat{\dot{u}}\right)\hat{s}'-\left(b-\hat{\dot{u}}\right)\dot{s}'+\hat{w}'-\dot{w}'}{\hat{\dot{s}}'}\\&+\frac{\left(b-\tilde{\hat{u}}\right)\tilde{s}'-\left(a-\hat{\tilde{u}}\right)\hat{s}'+\tilde{w}'-\hat{w}'}{\hat{\tilde{s}}'}=0,\end{aligned}\tag{2.1c}$$

$$\left(a+u-\frac{\tilde{w}}{\tilde{s}}\right)\left(a-\tilde{u}+\frac{w}{s}\right)=a^2+\left(\underline{\tilde{y}}'-\tilde{y}\right)-\left(y'-y\right)+\frac{\tilde{w}}{\tilde{s}}\left(\tilde{u}-\underline{\tilde{u}}\right)-\left(\frac{1}{\tilde{s}\underline{s}'}+3e_1+g\tilde{s}'s\right),\tag{2.1d}$$

$$\left(b+u-\frac{\hat{w}}{\hat{s}}\right)\left(b-\hat{u}+\frac{w}{s}\right)=b^2+\left(\underline{\hat{y}}'-\hat{y}\right)-\left(y'-y\right)+\frac{\hat{w}}{\hat{s}}\left(\hat{u}-\underline{\hat{u}}\right)-\left(\frac{1}{\hat{s}\underline{s}'}+3e_1+g\hat{s}'s\right),\tag{2.1e}$$

$$\left(c+u-\frac{\dot{w}}{\dot{s}}\right)\left(c-\dot{u}+\frac{w}{s}\right)=c^2+\left(\underline{\dot{y}}'-\dot{y}\right)-\left(y'-y\right)+\frac{\dot{w}}{\dot{s}}\left(\dot{u}-\underline{\dot{u}}\right)-\left(\frac{1}{\dot{s}\underline{s}'}+3e_1+g\dot{s}'s\right),\tag{2.1f}$$

$$s'\overline{w}=w'\overline{s}.\tag{2.1g}$$

其中 a,b,c 是常数，符号~，∧，˙，¯分别表示相对于链方向 n,m,l,M 的位移[9]。(2.1)中的 $e_1,g\in\mathbb{C}$ 是椭圆曲线的模

$$y^2=R(x)=\frac{1}{x}+3e_1+gx.$$

我们将引入 $\boldsymbol{c}$ 和 $\boldsymbol{r}$ 上的色散关系：

$$\tilde{\boldsymbol{c}}=(a\boldsymbol{I}+\boldsymbol{p}),\ \hat{\boldsymbol{c}}=(b\boldsymbol{I}+\boldsymbol{p})\boldsymbol{c},\ \dot{\boldsymbol{c}}=(b\boldsymbol{I}+\boldsymbol{p})\boldsymbol{c},\ \overline{\boldsymbol{c}}=-\boldsymbol{Pc}\,, \tag{2.2a}$$

$$\tilde{\boldsymbol{r}}=\boldsymbol{r}(a\boldsymbol{I}-\boldsymbol{q})^{-1},\ \hat{\boldsymbol{r}}=\boldsymbol{r}(b\boldsymbol{I}-\boldsymbol{q})^{-1},\ \dot{\boldsymbol{r}}=\boldsymbol{r}(c\boldsymbol{I}-\boldsymbol{q})^{-1},\ \overline{\boldsymbol{r}}=-\boldsymbol{rQ}^{-1}\,, \tag{2.2b}$$

其中 a,b,c 称为链参数，分别与链方向 n,m,l 相关，M 方向没有链参数。其中矩阵 $\boldsymbol{p},\boldsymbol{P}\in\mathbb{C}_{N\times N}$，$\boldsymbol{q},\boldsymbol{Q}\in\mathbb{C}_{N'\times N'}$，$\boldsymbol{X}\in\mathbb{C}_{N\times N'}$，列向量 $\boldsymbol{c}\in\mathbb{C}_{N\times 1}$，行向量 $\boldsymbol{r}\in\mathbb{C}_{1\times N'}$。

考虑一组标量，用 $S^{(i,j)}$ 表示，其中 $i,j\in\mathbb{Z}$，写作：

$$\boldsymbol{S}^{(2i,2j)}=\boldsymbol{rQ}^j\boldsymbol{C}(\boldsymbol{I}+\boldsymbol{XC})^{-1}\boldsymbol{P}^i\boldsymbol{c}\,, \tag{2.3a}$$

$$\boldsymbol{S}^{(2i+1,2j)}=\boldsymbol{rQ}^j\boldsymbol{C}(\boldsymbol{I}+\boldsymbol{XC})^{-1}\boldsymbol{P}^i\boldsymbol{pc}\,, \tag{2.3b}$$

其中，$\boldsymbol{C}\in\mathbb{C}_{N'\times N}$ 是一个辅助常数矩阵，使得 $\boldsymbol{I}+\boldsymbol{XC}$ 可逆。并且引入一些辅助向量[10]：

$$\boldsymbol{u}^{(2i)}=(\boldsymbol{I}+\boldsymbol{XC})^{-1}\boldsymbol{P}^i\boldsymbol{c},\ \boldsymbol{u}^{(2i+1)}=(\boldsymbol{I}+\boldsymbol{XC})^{-1}\boldsymbol{P}^i\boldsymbol{pc}\,, \tag{2.4}$$

因此 $S^{(i,j)}$ 可以被写成

$$\boldsymbol{S}^{(2i,2j)}=\boldsymbol{rQ}^j\boldsymbol{Cu}^{(2i)},\quad \boldsymbol{S}^{(2i+1,2j)}=\boldsymbol{rQ}^j\boldsymbol{Cu}^{(2i+1)}\,, \tag{2.5}$$

$\boldsymbol{u}^{(i)}$ 位移关系的显式计算可以利用式(2.3)实现。根据式(2.2)和式(2.4)的结构，结合 $\boldsymbol{u}^{(i)}$ 的定义，可以推断出 $\boldsymbol{u}^{(i)}$ 在 n,m,l 方向上的位移关系具有相似性，可以通过交换位移和链参数从一个方向变换到另一个方向。

3. 椭圆 KP 方程的 Lax 组

通过对(2.3)两边对于矩阵的变换，我们可以得到可以得到 $\boldsymbol{u}^{(i)}$ 的位移关系：

$$\tilde{\boldsymbol{u}}^{(2i)}=a\boldsymbol{u}^{(2i)}+\boldsymbol{u}^{(2i+1)}-\boldsymbol{u}^{(0)}\tilde{S}^{(2i,0)}+g\boldsymbol{u}^{(-2)}\tilde{S}^{(2i,-2)}\,, \tag{3.1a}$$

$$\tilde{\boldsymbol{u}}^{(2i+1)}=a\boldsymbol{u}^{(2i+1)}+\boldsymbol{u}^{(2i+2)}+3e_1\boldsymbol{u}^{(2i)}+g\boldsymbol{u}^{(2i-2)}-\boldsymbol{u}^{(0)}\tilde{S}^{(2i+1,0)}+g\boldsymbol{u}^{(-2)}\tilde{S}^{(2i+1,-2)}\,, \tag{3.1b}$$

$$\overline{\boldsymbol{u}}^{(2i)}=-\boldsymbol{u}^{(2i+2)}+\boldsymbol{u}^{(1)}\overline{S}^{(2i,0)}-\boldsymbol{u}^{(0)}\overline{S}^{(2i,1)}\,, \tag{3.1c}$$

$$\overline{\boldsymbol{u}}^{(2i+1)}=-\boldsymbol{u}^{(2i+3)}+\boldsymbol{u}^{(1)}\overline{S}^{(2i+1,0)}-\boldsymbol{u}^{(0)}\overline{S}^{(2i+1,1)}\,。 \tag{3.1d}$$

接着，让我们引入标量函数：

$$u=S^{(0,0)},\ s=S^{(-2,0)},\ s'=S^{(0,-2)},\ h=S^{(-2,-2)},$$
$$v=1-S^{(-1,0)},\ v'=1-S^{(0,1)},\ w=1+S^{(-2,1)},\ w'=1+S^{(1,-2)}.$$

结合式(3.1a)~式(3.1b)中 $i=0$ 的具体情况，可以形成两个耦合线性方程：

$$\tilde{\boldsymbol{u}}^{(0)}=(a-\tilde{u})\boldsymbol{u}^{(0)}+\boldsymbol{u}^{(1)}-g\tilde{s}'w\underline{\boldsymbol{u}}^{(0)}+g\tilde{s}'s\underline{\boldsymbol{u}}^{(1)}\,, \tag{3.2a}$$

$$\tilde{\boldsymbol{u}}^{(1)}=(3e_1-\overline{y}-\tilde{y}')\boldsymbol{u}^{(0)}+(a+\overline{u})\boldsymbol{u}^{(1)}-g\tilde{w}'w\underline{\boldsymbol{u}}^{(0)}+g\tilde{w}'s\underline{\boldsymbol{u}}^{(1)}-\overline{\boldsymbol{u}}^{(0)}\,。 \tag{3.2b}$$

在一个 2 × 1 的块矩阵 $\Phi=\begin{pmatrix}\boldsymbol{u}^{(0)}\\ \boldsymbol{u}^{(1)}\end{pmatrix}$，每个式中，每个块都包含一个 N 阶的列向量，则可将上述两种关

系结合为 Φ 式，即：

$$\tilde{\Phi} = A_1\Phi + B_1\underline{\Phi} + J\bar{\Phi}\,, \tag{3.3a}$$

其中：

$$A_1 = \begin{pmatrix} a-\tilde{u} & 1 \\ 3e_1-\bar{y}-\tilde{y}' & a+\bar{u} \end{pmatrix},\quad B_1 = g\begin{pmatrix} -\tilde{s}'w & \tilde{s}'s \\ -\tilde{w}'w & \tilde{w}'s \end{pmatrix},\quad J = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \tag{3.3b}$$

类似地，我们有“−”和“˙”位移的类似方程

$$\hat{\Phi} = A_2\Phi + B_2\underline{\Phi} + J\bar{\Phi}\,, \tag{3.3c}$$

$$\dot{\Phi} = A_3\Phi + B_3\underline{\Phi} + J\bar{\Phi}\,, \tag{3.3d}$$

其中：

$$A_2 = \begin{pmatrix} b-\hat{u} & 1 \\ 3e_1-\bar{y}-\hat{y}' & b+\bar{u} \end{pmatrix},\quad B_2 = g\begin{pmatrix} -\hat{s}'w & \hat{s}'s \\ -\hat{w}'w & \hat{w}'s \end{pmatrix}, \tag{3.3e}$$

$$A_3 = \begin{pmatrix} c-\dot{u} & 1 \\ 3e_1-\bar{y}-\dot{y}' & c+\bar{u} \end{pmatrix},\quad B_3 = g\begin{pmatrix} -\dot{s}'w & \dot{s}'s \\ -\dot{w}'w & \dot{w}'s \end{pmatrix}。 \tag{3.3f}$$

上述关系维度为 $2N \times 2N$ 的矩阵。当然它们也可以被理解为独立矩阵系统的集合，每个系统操作一个双分量向量，表示为 N 个解耦的 2×2 矩阵系统，其中：

$$\tilde{\Phi}_i = A_1\Phi_i + B_1\underline{\Phi}_i + J\bar{\Phi}_i\,, \tag{3.4a}$$

$$\hat{\Phi}_i = A_2\Phi_i + B_2\underline{\Phi}_i + J\bar{\Phi}_i\,, \tag{3.4b}$$

$$\dot{\Phi}_i = A_3\Phi_i + B_3\underline{\Phi}_i + J\bar{\Phi}_i\,, \tag{3.4c}$$

其中：

$$\Phi_i = \begin{pmatrix} \left(\boldsymbol{u}^{(0)}\right)_i \\ \left(\boldsymbol{u}^{(1)}\right)_i \end{pmatrix},\ i = 1, 2, \cdots, N. \tag{3.5}$$

$\left(\boldsymbol{u}^{(j)}\right)_i$ 是 N 个分量向量 $\boldsymbol{u}^{(j)}$ 的第 i 个分量，线性关系(3.55)的相容条件为 $\hat{\tilde{\Phi}}_i = \tilde{\hat{\Phi}}_i$， $\dot{\tilde{\Phi}}_i = \tilde{\dot{\Phi}}_i$，得到：

$$\begin{aligned}&\left(\hat{A}_1J + J\bar{A}_2 - \tilde{A}_2J - J\bar{A}_1\right)\bar{\Phi} + \left(\hat{A}_1A_2 + \hat{B}_1J + J\bar{B}_2 - \tilde{A}_2A_1 - \tilde{B}_2J - J\bar{B}_1\right)\Phi \\ &+\left(\hat{A}_1B_2 + \hat{B}_1\underline{A}_2 - \tilde{A}_2B_1 - \tilde{B}_2\underline{A}_1\right)\underline{\Phi} + \left(\hat{B}_1\underline{B}_2 - \tilde{B}_2\underline{B}_1\right)\underline{\underline{\Phi}} = 0\end{aligned}, \tag{3.6a}$$

$$\begin{aligned}&\left(\dot{A}_1J + J\bar{A}_3 - \tilde{A}_3J - J\bar{A}_1\right)\bar{\Phi} + \left(\dot{A}_1A_3 + \dot{B}_1J + J\bar{B}_3 - \tilde{A}_3A_1 - \tilde{B}_3J - J\bar{B}_1\right)\Phi \\ &+\left(\dot{A}_1B_3 + \dot{B}_1\underline{A}_3 - \tilde{A}_3B_1 - \tilde{B}_3\underline{A}_1\right)\underline{\Phi} + \left(\dot{B}_1\underline{B}_3 - \tilde{B}_3\underline{B}_1\right)\underline{\underline{\Phi}} = 0\end{aligned}, \tag{3.6b}$$

$$\begin{aligned}&\left(\hat{A}_3J + J\bar{A}_2 - \dot{A}_2J - J\bar{A}_3\right)\bar{\Phi} + \left(\hat{A}_3A_2 + \hat{B}_3J + J\bar{B}_2 - \dot{A}_2A_3 - \dot{B}_2J - J\bar{B}_3\right)\Phi \\ &+\left(\hat{A}_3B_2 + \hat{B}_3\underline{A}_2 - \dot{A}_2B_3 - \dot{B}_2\underline{A}_3\right)\underline{\Phi} + \left(\hat{B}_3\underline{B}_2 - \dot{B}_2\underline{B}_3\right)\underline{\underline{\Phi}} = 0\end{aligned}。 \tag{3.6c}$$

由此得到了椭圆链式 KP 系统的 Lax 对，或者更准确地说是用 Lax 三重态来积分的证据[11]。

4. 由 Lax 组到 KP 方程

由相容条件(3.6a)可得：

$$g\hat{s}\tilde{\hat{s}}' + g\tilde{s}\tilde{\hat{s}}' - (b-\hat{u})\left(a-\tilde{\hat{u}}\right) + (a-\tilde{u})\left(b-\tilde{\hat{u}}\right) + \hat{y}' - \tilde{y}' = 0, \tag{4.1a}$$

$$\begin{aligned} & g\bar{\hat{s}}'\bar{w} - g\bar{\tilde{s}}'\bar{w} - g\hat{s}\tilde{\hat{w}}' + g\tilde{s}\tilde{\hat{w}}' + \left(a+\bar{\hat{u}}\right)(3e_1-\bar{y}-\hat{y}') - \left(b+\bar{\tilde{u}}\right)(3e_1-\bar{y}-\tilde{y}') \\ & +(b-\hat{u})\left(3e_1-\bar{\hat{y}}-\tilde{\hat{y}}'\right) - (a-\tilde{u})\left(3e_1-\bar{\tilde{y}}-\tilde{\hat{y}}'\right) = 0 \end{aligned}, \tag{4.1b}$$

$$-gs\hat{s}' + gs\tilde{s}' + (b+u)(a+\hat{u}) - (a+u)(b+\tilde{u}) - \hat{y} + \tilde{y} = 0, \tag{4.1c}$$

$$\begin{aligned} & \frac{-\hat{s}'\left(a-\hat{\tilde{u}}\right) + \tilde{s}'\left(b-\hat{\tilde{u}}\right) - \hat{w}' + \tilde{w}'}{\hat{\tilde{s}}'} \\ & + \frac{-(b-\underline{\hat{u}})\hat{w} + (a-\underline{\tilde{u}})\tilde{w} + \hat{s}\left(3e_1-y-\underline{\hat{y}}'\right) - \tilde{s}\left(3e_1-y-\underline{\tilde{y}}'\right)}{w} = 0 \end{aligned}, \tag{4.1d}$$

$$\frac{\tilde{s}(a+u) - \hat{s}(b+u) + \hat{w} - \tilde{w}}{s} + \frac{\tilde{s}'\left(b-\hat{\tilde{u}}\right) - \hat{s}'\left(a-\hat{\tilde{u}}\right) - \hat{w}' + \tilde{w}'}{\hat{\tilde{s}}'} = 0, \tag{4.1e}$$

$$\begin{aligned} & -\left(a+\hat{\bar{u}}\right)w\hat{w}' + \left(b+\tilde{\bar{u}}\right)w\tilde{w}' - (b-\underline{\hat{u}})\hat{w}\hat{\tilde{w}}' + (a-\underline{\tilde{u}})\tilde{w}\hat{\tilde{w}}' + \hat{s}\hat{\tilde{w}}'\left(3e_1-y-\underline{\hat{y}}'\right) \\ & -\tilde{s}\hat{\tilde{w}}'\left(3e_1-y-\underline{\tilde{y}}'\right) - \hat{s}'w\left(3e_1-\hat{\bar{y}}-\hat{\tilde{y}}'\right) + \tilde{s}'w\left(3e_1-\tilde{\bar{y}}-\hat{\tilde{y}}'\right) = 0 \end{aligned}, \tag{4.1f}$$

$$\begin{aligned} & s\left(a\hat{w}' + \hat{\bar{u}}\hat{w}' - b\tilde{w}' - \tilde{\bar{u}}\tilde{w}' + \hat{s}'\left(3e_1-\hat{\bar{y}}-\hat{\tilde{y}}'\right) + \tilde{s}'\left(-3e_1+\tilde{\bar{y}}+\hat{\tilde{y}}'\right)\right) \\ & +\left(-\tilde{s}(a+u) + \hat{s}(b+u) - \hat{w} + \tilde{w}\right)\hat{\tilde{w}}' = 0 \end{aligned}, \tag{4.1g}$$

$$s'\bar{w} = w'\bar{s}\text{。} \tag{4.1h}$$

由相容条件(3.6b)可得：

$$-g\dot{s}\tilde{\dot{s}}' + g\tilde{s}\tilde{\dot{s}}' - (c-\dot{u})\left(a-\tilde{\dot{u}}\right) - (a-\tilde{u})\left(b-\tilde{\dot{u}}\right) + \dot{y}' - \tilde{y}' = 0, \tag{4.2a}$$

$$\begin{aligned} & g\bar{\dot{s}}'\bar{w} - g\bar{\tilde{s}}'\bar{w} - g\dot{s}\tilde{\dot{w}}' + g\tilde{s}\tilde{\dot{w}}' + \left(a+\bar{\dot{u}}\right)(3e_1-\bar{y}-\dot{y}') - \left(c+\bar{\tilde{u}}\right)(3e_1-\bar{y}-\tilde{y}') \\ & +(c-\dot{u})\left(3e_1-\bar{\dot{y}}-\tilde{\dot{y}}'\right) - (a-\tilde{u})\left(3e_1-\bar{\tilde{y}}-\tilde{\dot{y}}'\right) = 0 \end{aligned}, \tag{4.2b}$$

$$-gs\dot{s}' + gs\tilde{s}' + (c+u)(a+\dot{u}) - (a+u)(c+\tilde{u}) - \dot{y} + \tilde{y} = 0, \tag{4.2c}$$

$$\begin{aligned} & \frac{-\tilde{s}'\left(c-\dot{\tilde{u}}\right) + \dot{s}'\left(a-\dot{\tilde{u}}\right) - \tilde{w}' + \dot{w}'}{\hat{\tilde{s}}'} \\ & + \frac{-(a-\underline{\tilde{u}})\tilde{w} + (c-\underline{\dot{u}})\dot{w} + \tilde{s}\left(3e_1-y-\underline{\tilde{y}}'\right) - \dot{s}\left(3e_1-y-\underline{\dot{y}}'\right)}{w} = 0 \end{aligned}, \tag{4.2d}$$

$$\frac{-\tilde{s}(a+u) + \dot{s}(c+u) - \dot{w} + \tilde{w}}{s} + \frac{\dot{s}'\left(a-\dot{\tilde{u}}\right) + \tilde{s}'\left(-c+\dot{\tilde{u}}\right) + \dot{w}' - \tilde{w}'}{\dot{\tilde{s}}'} = 0, \tag{4.2e}$$

$$\begin{aligned} & -\left(a+\dot{\bar{u}}\right)w\dot{w}' + \left(c+\tilde{\bar{u}}\right)w\tilde{w}' - (c-\underline{\dot{u}})\dot{w}\dot{\tilde{w}}' + (a-\underline{\tilde{u}})\tilde{w}\dot{\tilde{w}}' + \dot{s}\dot{\tilde{w}}'\left(3e_1-y-\underline{\dot{y}}'\right) \\ & -\tilde{s}\dot{\tilde{w}}'\left(3e_1-y-\underline{\tilde{y}}'\right) - \dot{s}'w\left(3e_1-\dot{\bar{y}}-\dot{\tilde{y}}'\right) + \tilde{s}'w\left(3e_1-\tilde{\bar{y}}-\dot{\tilde{y}}'\right) = 0 \end{aligned}, \tag{4.2f}$$

$$\begin{aligned} & s\left(a\dot{w}' + \dot{\bar{u}}\dot{w}' - c\tilde{w}' - \tilde{\bar{u}}\tilde{w}' + \dot{s}'\left(3e_1-\dot{\bar{y}}-\dot{\tilde{y}}'\right) + \tilde{s}'\left(-3e_1+\tilde{\bar{y}}+\dot{\tilde{y}}'\right)\right) \\ & +\left(-\tilde{s}(a+u) + \dot{s}(c+u) - \dot{w} + \tilde{w}\right)\dot{\tilde{w}}' = 0 \end{aligned}, \tag{4.2g}$$

$$s'\bar{w} = w'\bar{s}\text{。} \tag{4.2h}$$

由相容条件(3.6c)可得：

$$-g\hat{s}\dot{\hat{s}}' + g\dot{s}\dot{\hat{s}}' + (b-\hat{u})\left(c-\dot{\hat{u}}\right) - (c-\dot{u})\left(b-\dot{\hat{u}}\right) - \hat{y}' + \dot{y}' = 0\ , \tag{4.3a}$$

$$\begin{aligned} &g\overline{\hat{s}}'\overline{w} - g\overline{\dot{s}}'\overline{w} - g\hat{s}\dot{\hat{w}}' + g\dot{s}\dot{\hat{w}}' + \left(c+\overline{\hat{u}}\right)(3e_1 - \overline{y} - \hat{y}') - \left(b+\overline{\dot{u}}\right)(3e_1 - \overline{y} - \dot{y}') \\ &+(b-\hat{u})\left(3e_1 - \overline{\hat{y}} - \dot{\hat{y}}'\right) - (c-\dot{u})\left(3e_1 - \overline{\dot{y}} - \dot{\hat{y}}'\right) = 0 \end{aligned}\ , \tag{4.3b}$$

$$\begin{aligned} &-gs\hat{s}' + gs\dot{s}' + (b+u)(c+\hat{u}) - (c+u)(b+\dot{u}) - \hat{y} + \dot{y} = 0, \\ &\frac{-\dot{s}'\left(b-\dot{\hat{u}}\right) + \hat{s}'\left(c-\dot{\hat{u}}\right) - \dot{w}' + \hat{w}'}{\dot{\hat{s}}'}, \end{aligned} \tag{4.3c}$$

$$+\frac{-(c-\underline{\dot{u}})\dot{w} + (b-\underline{\hat{u}})\hat{w} + \dot{s}\left(3e_1 - y - \underline{\dot{y}}'\right) - \hat{s}\left(3e_1 - y - \underline{\hat{y}}'\right)}{w} = 0\ , \tag{4.3d}$$

$$\frac{-\dot{s}(c+u) + \hat{s}(b+u) - \hat{w} + \dot{w}}{s} + \frac{\hat{s}'\left(c-\dot{\hat{u}}\right) + \dot{s}'\left(-b+\dot{\hat{u}}\right) + \hat{w}' - \dot{w}'}{\dot{\hat{s}}'} = 0\ , \tag{4.3e}$$

$$\begin{aligned} &-\left(c+\hat{\overline{u}}\right)w\hat{w}' + \left(b+\dot{\overline{u}}\right)w\dot{w}' - (b-\underline{\hat{u}})\hat{w}\dot{\hat{w}}' + (c-\underline{\dot{u}})\dot{w}\dot{\hat{w}}' + \hat{s}\dot{\hat{w}}'\left(3e_1 - y - \underline{\hat{y}}'\right) \\ &-\tilde{s}\hat{\tilde{w}}'\left(3e_1 - y - \underline{\tilde{y}}'\right) - \dot{s}'w\left(3e_1 - \hat{\overline{y}} - \dot{\hat{y}}'\right) + \dot{s}'w\left(3e_1 - \dot{\overline{y}} - \dot{\hat{y}}'\right) = 0 \end{aligned}\ , \tag{4.3f}$$

$$\begin{aligned} &s\left(c\hat{w}' + \hat{\overline{u}}\hat{w}' - b\dot{w}' - \dot{\overline{u}}\dot{w}' + \hat{s}'\left(3e_1 - \hat{\overline{y}} - \dot{\hat{y}}'\right) + \dot{s}'\left(-3e_1 + \dot{\overline{y}} + \dot{\hat{y}}'\right)\right) \\ &+\left(-\dot{s}(c+u) + \hat{s}(b+u) - \hat{w} + \dot{w}\right)\dot{\hat{w}}' = 0 \end{aligned}\ , \tag{4.3g}$$

$$s'\overline{w} = w'\overline{s}\ 。 \tag{4.3h}$$

通过方程式(4.1a)、(4.2a)和(4.3a)或(4.1c)$^{\cdot}$，(4.2c)$^{\wedge}$以及(4.3c)$^{\sim}$我们可以得到(2.1a)。方程(1.1b)可以通过用方程(4.1e)$^{\cdot}$，(4.2e)$^{\wedge}$和(4.3e)$^{\sim}$得到。将方程(4.1d)、(4.2d)和(4.3d)相加，或(4.1e)、(4.2e)和(4.3e)相加，得到(2.1c)。最后，计算(4.1c) + (4.1d) − (4.1e)、(4.2c) + (4.2d) − (4.2e)和(4.3c) + (4.3d) − (4.3e)将得到(1.1d)、(1.1e)和(1.1f)。

5. 结论

本文首先给出了椭圆 KP 方程的色散关系和辅助向量 $\boldsymbol{u}^{(i)}$，从辅助向量 $\boldsymbol{u}^{(i)}$ 出发通过矩阵变换得到了 $\boldsymbol{u}^{(i)}$ 的位移关系式，当 $\boldsymbol{u}^{(i)}$ 里 $i=0$ 可求出两个耦合线性方程，进一步得到了椭圆 KP 方程的 Lax 组，接着我们从 Lax 组反推出了椭圆 KP 方程。

参考文献

[1] Hietarinta, J., Joshi, N. and Nijhoff, F.W. (2016) Discrete Systems an Integrability. Cambridge University Press, Cambridge. https://doi.org/10.1017/CBO9781107337411

[2] Nijhoff, F.W. and Papageorgiou, V. (1989) Lattice Equations Associated with the Landau-Lifschitz Equations. *Physics Letters A*, **141**, 269-274. https://doi.org/10.1016/0375-9601(89)90483-0

[3] Adler, V.E. (1998) Bäcklund Transformation for the Krichever-Novikov Equation. *International Mathematics Research Notices*, **1**, 1-4.

[4] Adler, V.E., Bobenko, A.I. and Suris, Y.B. (2003) Classification of Integrable Equations on Quad-Graphs, the Consistency Approach. *Communications in Mathematical Physics*, **233**, 513-543. https://doi.org/10.1007/s00220-002-0762-8

[5] Adler, V.E. and Yamilov, R.I. (1994) Explicit Auto-Transformations of Integrable Chains. *Journal of Physics A*: *Mathematical and General*, **27**, 477-492. https://doi.org/10.1088/0305-4470/27/2/030

[6] Nijhoff, F.W. and Puttock, S.E. (2003) On a Two-Parameter Extension of the Lattice KdV System Associated with an Elliptic Curve. *Journal of Nonlinear Mathematical Physics*, **10**, 107-123. https://doi.org/10.2991/jnmp.2003.10.s1.8

[7] Nijhoff, F.W., Capel, H.W., Wiersma, G.L. and Quispel, G.R.W. (1984) Bäcklund Transformations and Three Dimen-

sional Lattice Equations. *Physics Letters A*, **105**, 267-272. https://doi.org/10.1016/0375-9601(84)90994-0

[8] Jennings, P. and Nijhoff, F.W. (2014) On an Elliptic Extension of the Kadomtsev-Petviashvili Equation. *Journal of Physics A*: *Mathematical and Theoretical*, **47**, 055205. https://doi.org/10.1088/1751-8113/47/5/055205

[9] Xu, D.D., Zhang, D.J. and Zhao, S.L. (2014) The Sylvester Equation and Integrable Equations, I. The Kortewegde Vries System and Sine-Gordon Equation. *Journal of Nonlinear Mathematical Physics*, **21**, 382-406. https://doi.org/10.1080/14029251.2014.936759

[10] Sun, Y.Y., Zhang, D.J. and Nijhoff, F.W. (2017) The Sylvester Equation and the Elliptic Korteweg-De Vries System. *Journal of Mathematical Physics,* **58**, 033504. https://doi.org/10.1063/1.4977477

[11] Nijhoff, F.W., Atkinson, J. and Hietarinta, J. (2009) Soliton Solutions for ABS Lattice Equations, I. Cauchy Matrix Approach. *Journal of Physics A*: *Mathematical and Theoretical*, **42**, 404005. https://doi.org/10.1088/1751-8113/42/40/404005