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p进域多线性分数阶积分算子及其与CBMO函数交换子的有界性

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摘 要

本文证明了在p进域意义下，多线性分数次积分算子在Lebesgue空间上的有界性及其与CBMO函数生成的迭代交换子在中心Morrey空间上的有界性。

关键词

p进域，多线性分数次积分算子，中心Morrey空间

Boundedness of Multilinear Fractional Integral Operators and Their Commutators with CBMO Functions on p-Adic Fields

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Abstract

This paper establishes the boundedness of the multilinear fractional integral operator

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on p-adic Lebesgue spaces, as well as the boundedness of its iterated commutators with central BMO functions on p-adic central Morrey spaces.

Keywords

p-Adic Field, Multilinear Fractional Integral Operator, Central Morrey Space

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1. 引言

近年来，p进域广泛地应用在谐波分析和各种物理模型等方面, 并研究出了大量的学术成果. 例如, 对小范围的时空几何的探索、对自旋玻璃的研究，甚至关系到更加复杂的系统等这些都涉及p进域的理论知识. 下面对p进域作简要回顾。

设p为素数, 记$\mathbb{Q}_p$为p进制数形成的数域. p进制范数定义为

$$|a|_p = |p^{\theta}\frac{r}{s}|_p = p^{-\theta}, \mathbb{Q} \ni a \neq 0.$$

其中$\theta \in \mathbb{Z}, r \in \mathbb{Z}, s \in \mathbb{Z}$, r和s均不被p整除, 同时$\theta = \theta(a)$. 若$a = 0$, 则$|0|_p = 0$. 任意p进制数$a \in \mathbb{Q}$均可以唯一正则地表示为

$$a = p^{\theta}\sum_{j=0}^{\infty}\gamma_j p^j.$$

其中$\gamma_j, \theta \in \mathbb{Z}, \gamma_j \in \frac{\mathbb{Z}}{p\mathbb{Z}_p}, \gamma_0 \neq 0$. 由于$|\theta_j p^j|_p = p^{-j}$, 则上式在$p$进制范数下收敛. 由上述定义有

$$|ab|_p = |a|_p|b|_p, |a+b|_p \leq \max\{|a|_p, |b|_p\}.$$

设

$$B_\theta(x) = \{a \in \mathbb{Q}_p : |a-x|_p \leq p^{\theta}\}, S_\theta(x) = \{a \in \mathbb{Q}_p : |a-x|_p = p^{\theta}\},$$

则$B_\theta(x)$表示以$x \in \mathbb{Q}_p$为中心, p^θ为半径的球; $S_\theta(x)$表示以$x \in \mathbb{Q}_p$为中心, p^θ为半径的球面, 记$B_\theta(0) = B_\theta, S_\theta(0) = S_\theta$. 同时, 易得$S_\theta(x) = B_\theta(x) \setminus B_{\theta-1}(x), B_\theta(x) = \bigcup_{k\leq\gamma} S_k(x)$.

由于$\mathbb{Q}_p^n$在加法运算下是具有局部紧性的可交换群, 则存在一个Haar测度dz, 使得$\int_{B_0} dx = 1$. 特别地, 通过分析计算得到对于$\forall x \in \mathbb{Q}_p, |B_\theta(x)|_H = p^\theta, |S_\theta(x)|_H = p^\theta(1-p^{-1})$. 另外, $\mathbb{Q}_p$的n维向量

空间$\mathbb{Q}_p^n$的p进范数定义为

$$|a|_p = \max_{1\le j\le n} |a_j|_p.$$

p进Lebesgue空间$L^r(\mathbb{Q}_p^n)(1\le r<\infty)$由具有以下有限范数的所有可测函数组成

$$\|f\|_{L^r(\mathbb{Q}_p^n)} = \left(\int_{\mathbb{Q}_p^n} |f(x)|^r dy\right)^{\frac{1}{r}}.$$

1999年, Kenig和Stein [1]介绍了多线性分数阶积分算子$I_{\alpha,m}$

$$I_{\alpha,m}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1),\cdots,f_m(y_m)}{|(x-y_1,\cdots,x-y_m)|^{mn-\alpha}} d\vec{y}, 0<\alpha<mn,$$

其中$d\vec{y} = dy_1\cdots dy_m$.

而p进域多线性分数阶积分算子定义如下

定义1.1. 设$m\in\mathbb{N}$, $0<\alpha<mn, p$进域多线性分数阶积分算子$I^p_{\alpha,m}$定义为

$$I^p_{\alpha,m}(\vec{f})(x) = \int_{(\mathbb{Q}_p^n)^m} \frac{f_1(y_1)\cdots f_m(y_m)}{|(x-y_1,\cdots,x-y_m)|_p^{mn-\alpha}} d\vec{y}.$$

另外, 在上述条件下, 对于$i=1,\cdots,m$, 若$s_i>1, 0<u_i<\frac{1}{n}, \vec{b}=(b_1,\cdots,b_m)$, 则多线性分数阶积分算子与函数$b_i\in CBMO^{s_i,u_i}(\mathbb{Q}_p^n)$ 生成的迭代交换子定义如下

$$I^{p,\prod\vec{b}}_{\alpha,m}(\vec{f})(x) = \int_{(\mathbb{Q}_p^n)^m} \frac{\prod_{i=1}^m f_i(y_i)[b_i(x)-b_i(y_i)]}{|(x-y_1,\cdots,x-y_m)|_p^{mn-\alpha}} d\vec{y}.$$

其中$d\vec{y} = dy_1\cdots dy_m$, $|(x-y_1,\cdots,x-y_m)|_p = |x-y_1|_p+\cdots+|x-y_m|_p$.

为了叙述本文的结果, 还需回顾如下相关定义.

定义1.2. [2] 设$1<s<\infty, 0<u<\frac{1}{n}$，$\mathcal{M}^p(\mathbb{Q}_p^n)$表示勒贝格可测函数集. $CBMO^{s,u}(\mathbb{Q}_p^n)$空间定义为

$$CBMO^{s,u}(\mathbb{Q}_p^n) = \left\{f\in\mathcal{M}^p(\mathbb{Q}_p^n) : \|f\|_{CBMO^{s,u}(\mathbb{Q}_p^n)} < \infty\right\},$$

且

$$\|f\|_{CBMO^{s,u}(\mathbb{Q}_p^n)} = \sup_{\theta\in\mathbb{Z}} \frac{\|(f-f_{B_\theta})\chi_{B_\theta}\|_{L^s(\mathbb{Q}_p^n)}}{|B_\theta|_H^u \|\chi_{B_\theta}\|_{L^s(\mathbb{Q}_p^n)}}.$$

定义1.3. [3] 设$1<q<\infty, \lambda\in\mathbb{R}$, 则$p$进中心Morrey空间定义为

$$\dot{\mathfrak{B}}^{q,\lambda}(\mathbb{Q}_p^n) = \left\{f\in\mathcal{M}^p(\mathbb{Q}_p^n) : \|f\|_{\dot{\mathfrak{B}}^{q,\lambda}(\mathbb{Q}_p^n)} < \infty\right\},$$

且

$$\|f\|_{\dot{\mathfrak{B}}^{q,\lambda}(\mathbb{Q}_p^n)} = \sup_{\theta\in\mathbb{Z}} \frac{\|f\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)}}{|B_\theta|_H^{\lambda}\|\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)}}.$$

越来越多的学者关注多线性算子的研究. 例如, 2023年, Lu 和Zhou [4]研究了多线性算子与CBMO函数生成的交换子在中心Morrey空间的有界性. 同年, Tao和Shi [5]研究了在p进域下带粗糙核的分数阶积分算子与多线性广义Campanato函数的交换子在广义Morrey 空间的有界性. 2024年, Sarfraz和Aslam [3]给出了分数阶积分算子及其交换子在p进中心Morrey空间的有界性.

受上述文献启发, 本文将研究p进意义下多线性分数阶积分算子及其与CBMO函数的生成的迭代交换子在中心Morrey空间的有界性.

定理1.1. 设$m\in\mathbb{N}$, $0<\alpha<mn$, $1<q,t_i<\infty$, $i=1,2,\cdots,m$, $\frac{1}{q}=\sum_{i=1}^m\frac{1}{q_i}$, $\alpha=\sum_{i=1}^m\sum_{j=1}^n\frac{1}{t_{ij}}-\frac{1}{q_{ij}}$, 则$I_{\alpha,m}^p$是从$L^{t_1}(\mathbb{Q}_p^n)\times\cdots\times L^{t_m}(\mathbb{Q}_p^n)$到$L^q(\mathbb{Q}_p^n)$ 有界.

定理1.2. 设$m\in\mathbb{N}$, $0<\alpha<mn$,对于$i=1,\cdots,m$, 有$1<q,t_i<\infty$且$t_i<\frac{n}{\alpha}$, $t_i'<s_i$, $\alpha=\sum_{j=1}^n(\sum_{i=1}^m\frac{1}{s_{ij}}+\sum_{i=1}^m\frac{1}{t_{ij}}-\frac{1}{q_j})$, $\sum_{j=1}^n\sum_{i=1}^m\frac{1}{t_{ij}}>\alpha$, $\lambda=\sum_{i=1}^m(u_i+v_i)+\frac{\alpha}{n}$, 其中$u_i+v_i<-\frac{\alpha}{n}$, 则$I_{\alpha,m}^{p,\Pi\vec{b}}$是从$\dot{\mathfrak{B}}^{t_1,v_1}(\mathbb{Q}_p^n)\times\cdots\times\dot{\mathfrak{B}}^{t_m,v_m}(\mathbb{Q}_p^n)$ 到$\dot{\mathfrak{B}}^{q,\lambda}(\mathbb{Q}_p^n)$ 有界.

在本文中, 对于任意集合$E\subset\mathbb{Q}_p^n$, χ_E表示集合E的特征函数, E^c表示集合E在$\mathbb{Q}_p^n$ 中的补集. $|E|_H$记作$\mathbb{Q}_p^n$上可测集E的Haar测度. 用C表示与主要参数无关的常数, 且其取值在不同的位置可以不尽相同. 同时, 用$f\lesssim g$表示$f\le Cg$; 用$f\approx g$表示$C_1g\le f\le C_2g$. $p=(p_1,\cdots,p_n)$. 对于$j=1,\cdots,n,1<p<\infty$就意味着$1<p_j<\infty$且$p_j=(p_{j1},p_{j2},\cdots,p_{jn})$.

2. 预备知识

为了证明本文结果, 还需介绍如下引理.

引理2.1. [3] 设$\alpha\in(0,n)$, $1<r<q<\infty$且$\alpha=\sum_{j=1}^n\frac{1}{r_j}-\sum_{j=1}^n\frac{1}{q_j}$, 则$I_\alpha^p$ 从$L^r(\mathbb{Q}_p^n)$到$L^q(\mathbb{Q}_p^n)$有界.

引理2.2. [2] 设$b\in CBMO^{s,u}(\mathbb{Q}_p^n)$, $u\ge 0$且i, $j\in\mathbb{Z}$, 则

$$|b_{B_i}-b_{B_j}|\le p^n|i-j|\|b\|_{CBMO^{s,u}(\mathbb{Q}_p^n)}\max\{|B_i|_H^u,|B_j|_H^u\}.$$

引理2.3. [3] 设$\mathbb{Q}_p^n$为n维p进向量空间, 且$1\le r<\infty$, 则

$$\|\chi_{B_\theta(x)}\|_{L^r(\mathbb{Q}_p^n)}=|B_\theta(x)|_H^{\frac{1}{r}}=p^{\frac{n\theta}{r}}.$$

3. 定理的证明

定理1.1的证明 对于$i=1,2,\cdots,m$, $f_i\in L^{t_i}(\mathbb{Q}_p^n)$, 有

$$|I_{\alpha,m}^p(f_1,\ldots,f_m)(x)|=\left|\int_{(\mathbb{Q}_p^n)^m}\frac{f_1(y_1)\cdots f_m(y_m)}{(|x-y_1|_p+\cdots+|x-y_m|_p)^{mn-\alpha}}d\vec{y}\right|$$

$$\begin{aligned}
&\leq \int_{(\mathbb{Q}_p^n)^m} \frac{|f_1(y_1)|\cdots|f_m(y_m|}{|x-y_1|_p^{n-\alpha_1}\cdots|x-y_m|_p^{n-\alpha_m}} dy_1\cdots dy_m \\
&\leq \int_{\mathbb{Q}_p^n} \frac{|f_1(y_1)|}{|x-y_1|_p^{n-\alpha_1}} dy_1 \cdots \int_{\mathbb{Q}_p^n} \frac{|f_m(y_m)|}{|x-y_m|_p^{n-\alpha_m}} dy_m \\
&= \prod_{i=1}^m I_{\alpha_i}^p(|f_i|)(x),
\end{aligned}$$

由于$\frac{1}{q}=\sum_{i=1}^m \frac{1}{q_i}$，由引理2.1知$I_{\alpha_i}^p$从$L^{t_i}(\mathbb{Q}_p^n)$到$L^{q_i}(\mathbb{Q}_p^n)$有界，这里$\alpha_i=\sum_{j=1}^n \frac{1}{t_{ij}}-\frac{1}{q_{ij}}$. 则由Hölder不等式得

$$\begin{aligned}
\|I_{\alpha,m}^p(f_1,\ldots,f_m)\|_{L^q(Q_p^n)} &\leq \Big\| \prod_{i=1}^m I_{\alpha_i}^p(|f_i|) \Big\|_{L^q(\mathbb{Q}_p^n)} \\
&\leq \prod_{i=1}^m \| I_{\alpha_i}^p(|f_i|) \|_{L^{q_i}(\mathbb{Q}_p^n)} \\
&\leq \prod_{i=1}^m \| f_i \|_{L^{t_i}(\mathbb{Q}_p^n)} .
\end{aligned}$$

证毕.

定理1.2的证明 由于不失一般性, 仅考虑$m=2$的情形. 当$m\in\mathbb{Z}^+(m\geq 1)$时可类似证明.

当$m=2$时, 令$\vec{f}=(f_1,f_2)\in\dot{\mathfrak{B}}^{t_1,v_1}(\mathbb{Q}_p^n)\times\dot{\mathfrak{B}}^{t_2,v_2}(\mathbb{Q}_p^n)$, 对于$\theta\in\mathbb{Z}$, $B_\theta=B_\theta(0)=\{a\in\mathbb{Q}_p^n: |a-0|_p\leq p^\theta\}$, 有

$$\begin{aligned}
|I_{\alpha,2}^{p,\Pi\vec{b}}\vec{f}(x)| \lesssim\, & |[b_1(x)-(b_1)_{B_\theta}][b_2(x)-(b_2)_{B_\theta}]I_{\alpha,2}^p(f_1,f_2)(x)| \\
&+|[b_1(x)-(b_1)_{B_\theta}]I_{\alpha,2}^p(f_1,[b_2(\cdot)-(b_2)_{B_\theta}]f_2)(x)| \\
&+|[b_2(x)-(b_2)_{B_\theta}]I_{\alpha,2}^p([b_1(\cdot)-(b_1)_{B_\theta}]f_1,f_2)(x)| \\
&+|I_{\alpha,2}^p([b_1(\cdot)-(b_1)_{B_\theta}]f_1,[b_2(\cdot)-(b_2)_{B_\theta}]f_2)(x)| \\
:=\, & Y_1+Y_2+Y_3+Y_4.
\end{aligned}$$

由Minkowski不等式知

$$\begin{aligned}
&\|I_{\alpha,2}^{p,\Pi\vec{b}}\vec{f}\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)} \\
&\lesssim \|Y_1\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)}+\|Y_2\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)}+\|Y_3\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)}+\|Y_4\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)} \\
&:= J_1+J_2+J_3+J_4.
\end{aligned}$$

事实上，由于对称性知J_2,J_3的估计相似，因此只估计J_1,J_2,J_4.

设$f_i(y_i)=f_i(y_i)\chi_{B_\theta}+f_i(y_i)\chi_{B_\theta^c},(i=1,2)$. 由Minkowski不等式知

$$
\begin{aligned}
J_1 \lesssim & \|[b_1-(b_1)_{B_\theta}][b_2-(b_2)_{B_\theta}]I^p_{\alpha,2}(f_1\chi_{B_\theta},f_2\chi_{B_\theta})\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)}\\
&+\|[b_1-(b_1)_{B_\theta}][b_2-(b_2)_{B_\theta}]I^p_{\alpha,2}(f_1\chi_{B_\theta},f_2\chi_{B_\theta^c})\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)}\\
&+\|[b_1-(b_1)_{B_\theta}][b_2-(b_2)_{B_\theta}]I^p_{\alpha,2}(f_1\chi_{B_\theta^c},f_2\chi_{B_\theta})\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)}\\
&+\|[b_1-(b_1)_{B_\theta}][b_2-(b_2)_{B_\theta}]I^p_{\alpha,2}(f_1\chi_{B_\theta^c},f_2\chi_{B_\theta^c})\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)}\\
:=& J_{11}+J_{12}+J_{13}+J_{14}.
\end{aligned}
$$

对于J_{11}, 设$\frac{1}{q}=\sum_{i=1}^2\frac{1}{s_i}+\frac{1}{\delta}$, 则$\sum_{j=1}^n\frac{1}{\delta_j}=\sum_{j=1}^n\frac{1}{t_{1j}}+\sum_{j=1}^n\frac{1}{t_{2j}}-\alpha$. 由题设易知$\sum_{j=1}^n\frac{1}{t_{1j}}+\sum_{j=1}^n\frac{1}{t_{2j}}>\alpha$, 由Hölder不等式和$I^p_{\alpha,2}$从$L^{t_1}(\mathbb{Q}_p^n)\times L^{t_2}(\mathbb{Q}_p^n)$到$L^\delta(\mathbb{Q}_p^n)$有界知

$$
\begin{aligned}
J_{11} &\lesssim \|I^p_{\alpha,2}(f_1\chi_{B_\theta},f_2\chi_{B_\theta})\chi_{B_\theta}\|_{L^\delta(\mathbb{Q}_p^n)}\prod_{i=1}^2\|[b_i-\{b_i\}_{B_\theta}]\chi_{B_\theta}\|_{L^{s_i}(\mathbb{Q}_p^n)}\\
&\lesssim \prod_{i=1}^2\|f_i\chi_{B_\theta}\|_{L^{t_i}(\mathbb{Q}_p^n)}\|[b_i-\{b_i\}_{B_\theta}]\chi_{B_\theta}\|_{L^{s_i}(\mathbb{Q}_p^n)}\\
&\lesssim |B_\theta|_H^{v_1+v_2+u_1+u_2}\|\chi_{B_\theta}\|_{L^{t_1}(\mathbb{Q}_p^n)}\|\chi_{B_\theta}\|_{L^{t_2}(\mathbb{Q}_p^n)}\|\chi_{B_\theta}\|_{L^{s_1}(\mathbb{Q}_p^n)}\|\chi_{B_\theta}\|_{L^{s_2}(\mathbb{Q}_p^n)}\\
&\quad\times\prod_{i=1}^2\|f_i\|_{\dot{\mathfrak{B}}^{t_i,v_i}(\mathbb{Q}_p^n)}\|b_i\|_{CBMO^{s_i,u_i}(\mathbb{Q}_p^n)}\\
&\lesssim |B_\theta|_H^{\lambda}\|\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)}\prod_{i=1}^2\|f_i\|_{\dot{\mathfrak{B}}^{t_i,v_i}(\mathbb{Q}_p^n)}\|b_i\|_{CBMO^{s_i,u_i}(\mathbb{Q}_p^n)},
\end{aligned}
$$

对于J_{12}, 若$x\in B_\theta$, $y_2\in B_\theta^c$, 有$|x-y_2|_p>p^\theta$, 由向量的模长性质得$|(x-y_1,x-y_2)|_p^{2n-\alpha}\geq|x-y_1|_p^{2n-\alpha}$, 由Hölder不等式得

$$
\begin{aligned}
&|I^p_{\alpha,2}(f_1\chi_{B_\theta},f_2\chi_{B_\theta^c})(x)|\\
&\lesssim \int_{B_\theta}|f_1(y_1)|dy_1\int_{B_\theta^c}\frac{|f_2(y_2)|}{|x-y_2|_p^{2n-\alpha}}dy_2\\
&\lesssim \|f_1\chi_{B_\theta}\|_{L^{t_1}(\mathbb{Q}_p^n)}\|\chi_{B_\theta}\|_{L^{t_1'}(\mathbb{Q}_p^n)}\sum_{k=\theta}^\infty p^{-k(2n-\alpha)}\|f_2\chi_{B_k}\|_{L^{t_2}(\mathbb{Q}_p^n)}\|\chi_{B_k}\|_{L^{t_2'}(\mathbb{Q}_p^n)}\\
&\lesssim |B_\theta|_H^{v_1}\|\chi_{B_\theta}\|_{L^{t_1}(\mathbb{Q}_p^n)}\|\chi_{B_\theta}\|_{L^{t_1'}(\mathbb{Q}_p^n)}\|f_1\|_{\dot{\mathfrak{B}}^{t_1,v_1}(\mathbb{Q}_p^n)}\|f_2\|_{\dot{\mathfrak{B}}^{t_2,v_2}(\mathbb{Q}_p^n)}\sum_{k=\theta}^\infty|B_k|_H^{-2+\frac{\alpha}{n}}\\
&\quad\times|B_k|_H^{v_2}\|\chi_{B_k}\|_{L^{t_2}(\mathbb{Q}_p^n)}\|\chi_{B_k}\|_{L^{t_2'}(\mathbb{Q}_p^n)}\\
&\lesssim |B_\theta|_H^{v_1+v_2+\frac{\alpha}{n}}\prod_{i=1}^2\|f_i\|_{\dot{\mathfrak{B}}^{t_i,v_i}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

若$\frac{1}{q}=\frac{1}{s_1}+\frac{1}{s_2}+\frac{1}{\delta}$, 则由Hölder不等式及上式得

$$
\begin{aligned}
J_{12} &\lesssim |B_\theta|_H^{v_1+v_2+\frac{\alpha}{n}} \prod_{i=1}^{2} \|f_i\|_{\dot{\mathfrak{B}}^{t_i,v_i}(\mathbb{Q}_p^n)} \|[b_1-(b_1)_{B_\theta}][b_2-(b_2)_{B_\theta}]\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)} \\
&\lesssim |B_\theta|_H^{v_1+v_2+\frac{\alpha}{n}} \prod_{i=1}^{2} \|f_i\|_{\dot{\mathfrak{B}}^{t_i,v_i}(\mathbb{Q}_p^n)} \|[b_1-(b_1)_{B_\theta}]\chi_{B_\theta}\|_{L^{s_1}(\mathbb{Q}_p^n)} \\
&\quad \times \|[b_2-(b_2)_{B_\theta}]\chi_{B_\theta}\|_{L^{s_2}(\mathbb{Q}_p^n)} \|\chi_{B_\theta}\|_{L^{\delta}(\mathbb{Q}_p^n)} \\
&\lesssim |B_\theta|_H^{\lambda} \|\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)} \prod_{i=1}^{2} \|f_i\|_{\dot{\mathfrak{B}}^{t_i,v_i}(\mathbb{Q}_p^n)} \|b_i\|_{CBMO^{s_i,u_i}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

同理可证

$$
J_{13} \lesssim |B_\theta|_H^{\lambda} \|\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)} \prod_{i=1}^{2} \|f_i\|_{\dot{\mathfrak{B}}^{t_i,v_i}(\mathbb{Q}_p^n)} \|b_i\|_{CBMO^{s_i,u_i}(\mathbb{Q}_p^n)}.
$$

对于J_{14}, 若$x \in B_\theta$, $y_1, y_2 \in B_\theta^c$, 有$|x-y_i|_p > p^\theta$, $i=1,2$, 又由于$|(x-y_1, x-y_2)|_p^{2n} \geq |x-y_1|_p^n |x-y_2|_p^n$. 根据$J_{12}$ 的估计同理可得

$$
J_{14} \lesssim |B_\theta|_H^{\lambda} \|\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)} \prod_{i=1}^{2} \|f_i\|_{\dot{\mathfrak{B}}^{t_i,v_i}(\mathbb{Q}_p^n)} \|b_i\|_{CBMO^{s_i,u_i}(\mathbb{Q}_p^n)}.
$$

结合J_{11}, J_{12}, J_{13}, J_{14}有

$$
J_{1} \lesssim |B_\theta|_H^{\lambda} \|\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)} \prod_{i=1}^{2} \|f_i\|_{\dot{\mathfrak{B}}^{t_i,v_i}(\mathbb{Q}_p^n)} \|b_i\|_{CBMO^{s_i,u_i}(\mathbb{Q}_p^n)}.
$$

下面估计J_2, 由Minkowski不等式得

$$
\begin{aligned}
J_2 &\lesssim \|[b_1-(b_1)_{B_\theta}] I_{\alpha,2}^p (f_1\chi_{B_\theta}, [b_2(\cdot)-(b_2)_{B_\theta}] f_2\chi_{B_\theta})\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)} \\
&\quad + \|[b_1-(b_1)_{B_\theta}] I_{\alpha,2}^p (f_1\chi_{B_\theta}, [b_2(\cdot)-(b_2)_{B_\theta}] f_2\chi_{B_\theta^c})\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)} \\
&\quad + \|[b_1-(b_1)_{B_\theta}] I_{\alpha,2}^p (f_1\chi_{B_\theta^c}, [b_2(\cdot)-(b_2)_{B_\theta}] f_2\chi_{B_\theta})\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)} \\
&\quad + \|[b_1-(b_1)_{B_\theta}] I_{\alpha,2}^p (f_1\chi_{B_\theta^c}, [b_2(\cdot)-(b_2)_{B_\theta}] f_2\chi_{B_\theta^c})\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)} \\
&:= J_{21} + J_{22} + J_{23} + J_{24}.
\end{aligned}
$$

对于J_{21}, 设$\frac{1}{q} = \frac{1}{s_1} + \frac{1}{\eta}$, $\frac{1}{\tau} = \frac{1}{s_2} + \frac{1}{t_2}$, 则$\sum_{j=1}^n \frac{1}{\eta_j} = \sum_{j=1}^n \frac{1}{t_{1j}} + \sum_{j=1}^n \frac{1}{\tau_j} - \alpha$ 且$\sum_{j=1}^n \frac{1}{t_{1j}} + \sum_{j=1}^n \frac{1}{t_{2j}} > \alpha$, 由Hölder不等式和$I_{\alpha,2}^p$ 从$L^{t_1}(\mathbb{Q}_p^n) \times L^{\tau}(\mathbb{Q}_p^n)$到$L^{\eta}(\mathbb{Q}_p^n)$有界知

$$
\begin{aligned}
J_{21} &\lesssim \|[b_1-(b_1)_{B_\theta}]\chi_{B_\theta}\|_{L^{s_1}(\mathbb{Q}_p^n)} \|I_{\alpha,2}^p (f_1\chi_{B_\theta}, [b_2-(b_2)_{B_\theta}] f_2\chi_{B_\theta})\chi_{B_\theta}\|_{L^{\eta}(\mathbb{Q}_p^n)} \\
&\lesssim \|[b_1-(b_1)_{B_\theta}]\chi_{B_\theta}\|_{L^{s_1}(\mathbb{Q}_p^n)} \|f_1\chi_{B_\theta}\|_{L^{t_1}(\mathbb{Q}_p^n)} \|[b_2-(b_2)_{B_\theta}] f_2\chi_{B_\theta}\|_{L^{\tau}(\mathbb{Q}_p^n)} \\
&\lesssim \|[b_1-(b_1)_{B_\theta}]\chi_{B_\theta}\|_{L^{s_1}(\mathbb{Q}_p^n)} \|[b_2-(b_2)_{B_\theta}]\chi_{B_\theta}\|_{L^{s_2}(\mathbb{Q}_p^n)} \prod_{i=1}^{2} \|f_i\chi_{B_\theta}\|_{L^{t_i}(\mathbb{Q}_p^n)} \\
&\lesssim |B_\theta|_H^{\lambda} \|\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)} \prod_{i=1}^{2} \|f_i\|_{\dot{\mathfrak{B}}^{t_i,v_i}(\mathbb{Q}_p^n)} \|b_i\|_{CBMO^{s_i,u_i}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

对于J_{22}, 若$x \in B_\theta$, $y_2 \in B_\theta^c$, 有$|x-y_2|_p > p^\theta$, 由向量的模长性质得$|(x-y_1, x-y_2)|_p^{2n-\alpha} \geq |x-y_2|_p^{2n-\alpha}$, 由$\frac{1}{t_2'} = \frac{1}{s_2} + \frac{1}{\tau'}$, $u_2+v_2 < -\frac{\alpha}{n}$, 引理2.2及Hölder不等式得

$$\begin{aligned}
&|I_{\alpha,2}^p(f_1\chi_{B_\theta}, [b_2(\cdot)-\{b_2\}_{B_\theta}]f_2\chi_{B_\theta^c})(x)| \\
&\lesssim \int_{B_\theta} |f_1(y_1)|dy_1 \int_{B_\theta^c} \frac{|f_2(y_2)||b_2(y_2)-\{b_2\}_{B_\theta}|}{|x-y_2|_p^{2n-\alpha}} dy_2 \\
&\lesssim \|f_1\chi_{B_\theta}\|_{L^{t_1}(\mathbb{Q}_p^n)} \|\chi_{B_\theta}\|_{L^{t_1'}(\mathbb{Q}_p^n)} \sum_{k=\theta}^{\infty} p^{-k(2n-\alpha)} \\
&\quad \times \int_{B_k} |f_2(y_2)||b_2(y_2)-\{b_2\}_{B_k}+\{b_2\}_{B_k}-\{b_2\}_{B_\theta}|dy_2 \\
&\lesssim \|f_1\chi_{B_\theta}\|_{L^{t_1}(\mathbb{Q}_p^n)} \|\chi_{B_\theta}\|_{L^{t_1'}(\mathbb{Q}_p^n)} \sum_{k=\theta}^{\infty} p^{-k(2n-\alpha)} \|f_2\chi_{B_k}\|_{L^{t_2}(\mathbb{Q}_p^n)} \\
&\quad \times \left(\|[b_2-\{b_2\}_{B_k}]\chi_{B_k}\|_{L^{t_2'}(\mathbb{Q}_p^n)} + |\{b_2\}_{B_k}-\{b_2\}_{B_\theta}| \|\chi_{B_k}\|_{L^{t_2'}(\mathbb{Q}_p^n)} \right) \\
&\lesssim \|f_1\|_{\dot{\mathfrak{B}}^{t_1,v_1}(\mathbb{Q}_p^n)} |B_\theta|_H^{v_1} \|\chi_{B_\theta}\|_{L^{t_1}(\mathbb{Q}_p^n)} \|\chi_{B_\theta}\|_{L^{t_1'}(\mathbb{Q}_p^n)} \sum_{k=\theta}^{\infty} |B_k|_H^{-2+\frac{\alpha}{n}} \|f_2\|_{\dot{\mathfrak{B}}^{t_2,v_2}(\mathbb{Q}_p^n)} |B_k|_H^{v_2} \\
&\quad \times \|\chi_{B_k}\|_{L^{t_2}(\mathbb{Q}_p^n)} |B_k|_H^{u_2} \|\chi_{B_k}\|_{L^{t_2'}(\mathbb{Q}_p^n)} \|b_2\|_{CBMO^{s_2,u_2}(\mathbb{Q}_p^n)} \\
&\lesssim |B_\theta|_H^{v_1+v_2+u_2+\frac{\alpha}{n}} \|b_2\|_{CBMO^{s_2,u_2}(\mathbb{Q}_p^n)} \prod_{i=1}^{2} \|f_i\|_{\dot{\mathfrak{B}}^{t_i,v_i}(\mathbb{Q}_p^n)}.
\end{aligned}$$

由$\frac{1}{q} = \frac{1}{s_1} + \frac{1}{\eta}$ 及Hölder不等式得

$$\begin{aligned}
J_{22} &\lesssim |B_\theta|_H^{v_1+v_2+u_2+\frac{\alpha}{n}} \|b_2\|_{CBMO^{s_2,u_2}(\mathbb{Q}_p^n)} \\
&\quad \times \prod_{i=1}^{2} \|f_i\|_{\dot{\mathfrak{B}}^{t_i,v_i}(\mathbb{Q}_p^n)} \|[b_1-(b_1)_{B_\theta}]\chi_{B_\theta}\|_{L^{s_1}(\mathbb{Q}_p^n)} \|\chi_{B_\theta}\|_{L^{\eta}(\mathbb{Q}_p^n)} \\
&\lesssim |B_\theta|_H^{\lambda} \|\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)} \prod_{i=1}^{2} \|f_i\|_{\dot{\mathfrak{B}}^{t_i,v_i}(\mathbb{Q}_p^n)} \|b_i\|_{CBMO^{s_i,u_i}(\mathbb{Q}_p^n)}.
\end{aligned}$$

同理可证

$$J_{23} \lesssim |B_\theta|_H^{\lambda} \|\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)} \prod_{i=1}^{2} \|f_i\|_{\dot{\mathfrak{B}}^{t_i,v_i}(\mathbb{Q}_p^n)} \|b_i\|_{CBMO^{s_i,u_i}(\mathbb{Q}_p^n)}.$$

对于J_{24}, 由题设知$v_1 < -\frac{\alpha}{2n}$, $u_2+v_2 < -\frac{\alpha}{n}$, 由Hölder不等式得

$$|I_{\alpha,2}^p(f_1\chi_{B_\theta^c}, [b_2(\cdot)-\{b_2\}_{B_\theta}]f_2\chi_{B_\theta^c})(x)| \lesssim |B_\theta|_H^{v_1+v_2+u_2+\frac{\alpha}{n}} \|b_2\|_{CBMO^{s_2,u_2}(\mathbb{Q}_p^n)} \prod_{i=1}^{2} \|f_i\|_{\dot{\mathfrak{B}}^{t_i,v_i}(\mathbb{Q}_p^n)}.$$

与J_{22}同理，有

$$J_{24} \lesssim |B_\theta|_H^{\lambda} \|\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)} \prod_{i=1}^{2} \|f_i\|_{\dot{\mathfrak{B}}^{t_i,v_i}(\mathbb{Q}_p^n)} \|b_i\|_{CBMO^{s_i,u_i}(\mathbb{Q}_p^n)}.$$

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结合J_{21}，J_{22}，J_{23}，J_{24}有

$$J_2 \lesssim |B_\theta|_H^\lambda \|\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)} \prod_{i=1}^{2} \|f_i\|_{\dot{\mathfrak{B}}^{t_i,v_i}(\mathbb{Q}_p^n)} \|b_i\|_{CBMO^{s_i,u_i}(\mathbb{Q}_p^n)}.$$

最后估计J_4, 由Minkowski不等式得

$$\begin{aligned}
J_4 \lesssim & \|I_{\alpha,2}^p([b_1(\cdot)-(b_1)_{B_\theta}]f_1\chi_{B_\theta},[b_2(\cdot)-(b_2)_{B_\theta}]f_2\chi_{B_\theta})\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)} \\
&+\|I_{\alpha,2}^p([b_1(\cdot)-(b_1)_{B_\theta}]f_1\chi_{B_\theta},[b_2(\cdot)-(b_2)_{B_\theta}]f_2\chi_{B_\theta^c})\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)} \\
&+\|I_{\alpha,2}^p([b_1(\cdot)-(b_1)_{B_\theta}]f_1\chi_{B_\theta^c},[b_2(\cdot)-(b_2)_{B_\theta}]f_2\chi_{B_\theta})\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)} \\
&+\|I_{\alpha,2}^p([b_1(\cdot)-(b_1)_{B_\theta}]f_1\chi_{B_\theta^c},[b_2(\cdot)-(b_2)_{B_\theta}]f_2\chi_{B_\theta^c})\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)} \\
:=& J_{41}+J_{42}+J_{43}+J_{44}.
\end{aligned}$$

对于J_{41}, 设$\frac{1}{\gamma_i}=\frac{1}{s_i}+\frac{1}{t_i},(i=1,2)$, 则$\sum_{j=1}^n \frac{1}{q_j}=\sum_{j=1}^n \frac{1}{\gamma_{1j}}+\sum_{j=1}^n \frac{1}{\gamma_{2j}}-\alpha$. 由Hölder 不等式和$I_{\alpha,2}^p$从$L^{\gamma_1}(\mathbb{Q}_p^n)\times L^{\gamma_2}(\mathbb{Q}_p^n)$ 到$L^q(\mathbb{Q}_p^n)$有界知

$$\begin{aligned}
J_{41} &\lesssim \prod_{i=1}^{2}\|[b_i-(b_i)_{B_\theta}]f_i\chi_{B_\theta}\|_{L^{\gamma_i}(\mathbb{Q}_p^n)} \\
&\lesssim \prod_{i=1}^{2}\|[b_i-(b_i)_{B_\theta}]\chi_{B_\theta}\|_{L^{s_i}(\mathbb{Q}_p^n)}\|f_i\chi_{B_\theta}\|_{L^{t_i}(\mathbb{Q}_p^n)} \\
&\lesssim |B_\theta|_H^\lambda \|\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)} \prod_{i=1}^{2} \|f_i\|_{\dot{\mathfrak{B}}^{t_i,v_i}(\mathbb{Q}_p^n)} \|b_i\|_{CBMO^{s_i,u_i}(\mathbb{Q}_p^n)}.
\end{aligned}$$

对于J_{42}, 若$x\in B_\theta$, $y_2\in B_\theta^c$, 有$|x-y_2|_p>p^\theta$, 由向量的模长性质得$|(x-y_1,x-y_2)|_p^{2n-\alpha}\geq |x-y_2|_p^{2n-\alpha}$, 从而, 对于$i=1,2$, 由$\frac{1}{t_i'}=\frac{1}{s_i}+\frac{1}{\gamma_i'}$, $u_i+v_i<-\frac{\alpha}{n}$及Hölder不等式得

$$\begin{aligned}
&|I_{\alpha,2}^p([b_1(\cdot)-\{b_1\}_{B_\theta}]f_1\chi_{B_\theta},[b_2(\cdot)-\{b_2\}_{B_\theta}]f_2\chi_{B_\theta^c})(x)| \\
&\lesssim \int_{B_\theta}|b_1(y_1)-\{b_1\}_{B_\theta}||f_1(y_1)|dy_1\int_{B_\theta^c}\frac{|f_2(y_2)||b_2(y_2)-\{b_2\}_{B_\theta}|}{|x-y_2|_p^{2n-\alpha}}dy_2 \\
&\lesssim \|f_1\chi_{B_\theta}\|_{L^{t_1}(\mathbb{Q}_p^n)}\|[b_1-\{b_1\}_{B_\theta}]\chi_{B_\theta}\|_{L^{t_1'}(\mathbb{Q}_p^n)}\sum_{k=\theta}^{\infty}p^{-k(2n-\alpha)} \\
&\quad\times\int_{B_k}|f_2(y_2)||b_2(y_2)-\{b_2\}_{B_\theta}|dy_2 \\
&\lesssim |B_\theta|_H^{v_1}\|\chi_{B_\theta}\|_{L^{t_1}(\mathbb{Q}_p^n)}\|f_1\|_{\dot{\mathfrak{B}}^{t_1,v_1}(\mathbb{Q}_p^n)}\|[b_1-\{b_1\}_{B_\theta}]\chi_{B_\theta}\|_{L^{s_1}(\mathbb{Q}_p^n)}\|\chi_{B_\theta}\|_{L^{\gamma_1'}(\mathbb{Q}_p^n)} \\
&\quad\times\sum_{k=\theta}^{\infty}p^{-k(2n-\alpha)}\|f_2\chi_{B_k}\|_{L^{t_2}(\mathbb{Q}_p^n)}\|[b_2-\{b_2\}_{B_\theta}]\chi_{B_k}\|_{L^{t_2'}(\mathbb{Q}_p^n)}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim |B_\theta|_H^{v_1+u_1+1}\|f_1\|_{\dot{\mathfrak{B}}^{t_1,v_1}(\mathbb{Q}_p^n)}\|b_1\|_{CBMO^{s_1,u_1}(\mathbb{Q}_p^n)}\sum_{k=\theta}^{\infty}|B_k|^{-2+\frac{\alpha}{n}}\|f_2\|_{\dot{\mathfrak{B}}^{t_2,v_2}(Q_p^n)}|B_k|_H^{v_2}\\
&\quad\times\|\chi_{B_k}\|_{L^{t_2}(\mathbb{Q}_p^n)}\|b_2\|_{CBMO^{s_2,u_2}(\mathbb{Q}_p^n)}|B_k|_H^{u_2}\|\chi_{B_k}\|_{L^{s_2}(\mathbb{Q}_p^n)}\|\chi_{B_k}\|_{L^{\gamma_2'}(\mathbb{Q}_p^n)}\\
&\lesssim |B_\theta|_H^{v_1+u_1+1}\|f_1\|_{\dot{\mathfrak{B}}^{t_1,v_1}(\mathbb{Q}_p^n)}\|b_1\|_{CBMO^{s_1,u_1}(\mathbb{Q}_p^n)}\|f_2\|_{\dot{\mathfrak{B}}^{t_2,v_2}(\mathbb{Q}_p^n)}\|b_2\|_{CBMO^{s_2,u_2}(\mathbb{Q}_p^n)}\\
&\quad\times\sum_{k=\theta}^{\infty}|B_k|^{-1+u_2+v_2+\frac{\alpha}{n}}\\
&\lesssim |B_\theta|_H^{\lambda}\prod_{i=1}^{2}\|f_i\|_{\dot{\mathfrak{B}}^{t_i,v_i}(\mathbb{Q}_p^n)}\|b_i\|_{CBMO^{s_i,u_i}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

因此

$$J_{42}\lesssim |B_\theta|_H^{\lambda}\|\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)}\prod_{i=1}^{2}\|f_i\|_{\dot{\mathfrak{B}}^{t_i,v_i}(\mathbb{Q}_p^n)}\|b_i\|_{CBMO^{s_i,u_i}(\mathbb{Q}_p^n)}.$$

同理可证

$$J_{43}\lesssim |B_\theta|_H^{\lambda}\|\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)}\prod_{i=1}^{2}\|f_i\|_{\dot{\mathfrak{B}}^{t_i,v_i}(\mathbb{Q}_p^n)}\|b_i\|_{CBMO^{s_i,u_i}(\mathbb{Q}_p^n)}.$$

对于J_{44}, 类似于J_{42}的估计得

$$
\begin{aligned}
&|I_{\alpha,2}^{p}([b_1(\cdot)-\{b_1\}_{B_\theta}]f_1\chi_{B_\theta^c},[b_2(\cdot)-\{b_2\}_{B_\theta}]f_2\chi_{B_\theta^c})(x)|\\
&\lesssim |B_\theta|_H^{\lambda}\|\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)}\prod_{i=1}^{2}\|f_i\|_{\dot{\mathfrak{B}}^{t_i,v_i}(\mathbb{Q}_p^n)}\|b_i\|_{CBMO^{s_i,u_i}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

所以

$$J_{44}\lesssim |B_\theta|_H^{\lambda}\|\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)}\prod_{i=1}^{2}\|f_i\|_{\dot{\mathfrak{B}}^{t_i,v_i}(\mathbb{Q}_p^n)}\|b_i\|_{CBMO^{s_i,u_i}(\mathbb{Q}_p^n)}.$$

结合J_{41}, J_{42}, J_{43}, J_{44}有

$$J_{4}\lesssim |B_\theta|_H^{\lambda}\|\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)}\prod_{i=1}^{2}\|f_i\|_{\dot{\mathfrak{B}}^{t_i,v_i}(\mathbb{Q}_p^n)}\|b_i\|_{CBMO^{s_i,u_i}(\mathbb{Q}_p^n)}.$$

综上所述

$$\|I_{\alpha,2}^{p,\prod\vec{b}}(\vec{f})\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)}\lesssim |B_\theta|_H^{\lambda}\|\chi_{B_\theta}\|_{L^q(\mathbb{Q}_p^n)}\prod_{i=1}^{2}\|f_i\|_{\dot{\mathfrak{B}}^{t_i,v_i}(\mathbb{Q}_p^n)}\|b_i\|_{CBMO^{s_i,u_i}(\mathbb{Q}_p^n)}.$$

即

$$\|I_{\alpha,2}^{p,\prod\vec{b}}(\vec{f})\|_{\dot{\mathfrak{B}}^{q,\lambda}(\mathbb{Q}_p^n)}\lesssim\prod_{i=1}^{2}\|f_i\|_{\dot{\mathfrak{B}}^{t_i,v_i}(\mathbb{Q}_p^n)}\|b_i\|_{CBMO^{s_i,u_i}(\mathbb{Q}_p^n)}.$$

证毕.

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