Pure Mathematics 理论数学, 2026, 16(5), 30-42
Published Online May 2026 in Hans. https://www.hanspub.org/journal/pm
https://doi.org/10.12677/pm.2026.165128

经典 Adams 谱序列 E_2-项中的非平凡乘积

黄 郅

广西师范大学数学与统计学院，广西 桂林

收稿日期：2026 年 3 月 31 日；录用日期：2026 年 5 月 10 日；发布日期：2026 年 5 月 25 日

摘 要

经典 Adams 谱序列是研究球面稳定同伦群 $\pi_* S$ 的最基本工具，利用 May 谱序列的相关理论对 Adams 谱序列的 E_2-项进行研究，具体给出了 $\tilde{\delta}_{s+4}h_0h_n \in Ext_A^{s+7,tq+s}(Z_p, Z_p)$ 在 Adams 谱序列中的非平凡性，其中 $p \geq 11$ $n \geq 2$ $0 \leq s \leq p-5$ $t = p^n + (s+4)p^3 + (s+3)p^2 + (s+2)p + (s+2)$ $q = 2(p-1)$

关键词

Adams 谱序列，May 谱序列，非平凡性，$\tilde{\delta}_s$ 元素族

Non-Trivial Products in the E_2-Term of the Classical Adams Spectral Sequence

Zhi Huang

School of Mathematics and Statistics, Guangxi Normal University, Guilin Guangxi

Received: March 31, 2026; accepted: May 10, 2026; published: May 25, 2026

Abstract

The classical Adams spectral sequence is the most fundamental tool for studying the

文章引用: 黄郅. 经典 Adams 谱序列 E_2-项中的非平凡乘积 [J]. 理论数学, 2026, 16(5): 30-42.
DOI: 10.12677/pm.2026.165128

stable homotopy groups of spheres π_*S. By using the relevant theories of the May spectral sequence, we study the E_2-term of the Adams spectral sequence, and specifically give the non-triviality of $\tilde{\delta}_{s+4}h_0h_n \in Ext_A^{s+7,tq+s}(Z_p,Z_p)$ in the Adams spectral sequence, where $p \geq 11, n \geq 1, 0 \leq s \leq p-5, t = p^n+(s+4)p^3+(s+3)p^2+(s+2)p+(s+2), q = 2(p-1)$.

Keywords

Adams Spectral Sequence, May Spectral Sequence, Nontriviality, $\tilde{\delta}_s$ Family Elements

1. 引言

在稳定同伦论中，利用 Adams 谱序列 (ASS) 发掘球面和其他空间的稳定同伦群中的非平凡元素是一种经典且有效的方法之一. 令 A 是模 p 的 Sreenrod 代数，S 表示在素数 p 处局部化的球谱，固定 $q=2(p-1)$. J.F.Adams 在文献 [1] 中证明了存在谱序列 $\{E_r^{s,t},d_r\}$, 该谱序列的 $E_2^{s,t} \cong Ext_A^{s,t}(Z_p,Z_p) \Rightarrow \pi_{t-s}S$. 这里 Adams 谱序列的 $E_2^{s,t}$-项是谱的上同调, 具有 Adams 微分 $d_r^{s,t}: E_r^{s,t} \longrightarrow E_r^{s+r,t+r-1}(r \geq 2)$. 因此，对于一个具有收敛性的 Adams 谱序列的非平凡元素 $x_i \in E_2^{s,*}$ 对应球面稳定同伦群的一个非零元 $f_i \in \pi_*S$.

由文献 [2] 知，当 $p \geq 11, 0 < s < p-4, \tilde{t}(s) = q[(s+4)p^3+(s+3)p^2+(s+2)p+(s+1)]+s$ 时, 存在第四希腊字母元素 $0 \neq \tilde{\alpha}_s^{(4)} = \tilde{\delta}_{s+4} \in Ext_A^{s+4,\tilde{t}(s)}(Z_p,Z_p)$. 进一步，由文献 [3] 可知 $Ext_A^{1,*}(Z_p,Z_p)$ 的 Z_p 基由 $a_0 \in Ext_A^{1,1}(Z_p,Z_p)$,$h_i \in Ext_A^{1,p^iq}(Z_p,Z_p),(i \geq 0)$ 组成.

h_n 元素族与 $\tilde{\delta}$ 元素族在稳定同伦理论中均具有典型应用.h_n 元素族作为 Adams 谱序列一阶 Ext 生成元，不仅构成了 $\tilde{\alpha}$ 族等经典周期元的基本单元，而且参与了 $\tilde{\beta}$ 族、$\tilde{\gamma}$ 族的构造，并且决定了低维 Adams 微分形式. 而 $\tilde{\delta}$ 元素族作为高阶希腊字母元素，常出现在高维稳定茎中，并为构造非平凡同伦元素提供了素材. 二者乘积的平凡性，是计算球面稳定同伦群结构的重要内容. 于是，本文讨论乘积元素 $\tilde{\delta}_{s+4}h_0h_n$ 的平凡性, 得到主要定理如下:

定理 1. 当 $p \geq 11, n \geq 2, 0 \leq s \leq p-5, t = p^n+(s+4)p^3+(s+3)p^2+(s+2)p+(s+2), q = 2(p-1)$ 时，乘积元素 $0 \neq \tilde{\delta}_{s+4}h_0h_n \in Ext_A^{s+7,tq+s}(Z_p,Z_p)$.

下面给出 May 谱序列的相关知识，最后证明该定理.

2. 介绍

林金坤在 [4] 一书中指出，球面稳定同伦群的研究本质上分为两部分。首先是一个纯代数问题: 计算 $Ext_A^{*,*}(Z_p, Z_p)$, 其中 p 为素数，A 是模 p 的 Sreenrod 代数. 第二个问题更具几何性，即确定 $Ext_A^{*,*}(Z_p, Z_p)$ 中哪些元素能在 Adams 谱序列中留存至 E_∞-项并代表哪些球面之间的映射.

对于上述第一部分，目前计算 $Ext_A^{*,*}(Z_p, Z_p)$ 最成功的方法就是 May 谱序列 (MSS). 在这里我们主要介绍一些基础知识，如有不详可具体参见书 [5]

根据 [5], 存在 May 谱序列 { $E_r^{s,t\,*}, d_r$ }, 其收敛于 $Ext_A^{s,t}(Z_p, Z_p)$, 且其 E_1-项为

$$E_1^{*,*,*} = E(h_{m,i} \mid m > 0, i \geq 0) \otimes P(b_{m,i} \mid m > 0, i \geq 0) \otimes P(a_m \mid m \geq 0),$$

其中 E 是外代数，P 是多项式代数，并且对其生成元有

$$h_{m,i} \in E_1^{1,2(p^m-1)p^i,2m-1}, b_{m,i} \in E_2^{2,2(p^m-1)p^{i+1},p(2m-1)}, a_m \in E_1^{1,2p^m-1,2m+1},$$

$h_{m,i}, b_{m,i}, a_m$ 分别代表 $[\xi_m^{p^i}], \sum_{k=1}^{p-1} C_p^k/p[\xi_m^{kp^i} \mid \xi_m^{(p-k)p^i}]$ 和 $[\tau_m]$ 所在的同调类 $(A_* = E(\tau_0, \tau_1, \tau_2 \cdots) \otimes P(\xi_1, \xi_2, \xi_3, \cdots))$

对于 May 微分 $d_r : E_r^{s,t,M} \longrightarrow E_r^{s+r,t,M-r}$, 若 $x \in E_r^{s,t,M}, y \in E_r^{s',t',M'}$, 有

$$d_r(x \cdot y) = d_r(x) \cdot y + (-1)^s x \cdot d_r(y).$$

并且存在一种分次交换性

$$x \cdot y = (-1)^{(s+t)(s'+t')} y \cdot x$$

第一 May 微分 d_1 由

$$\begin{cases} d_1(h_{i,j}) = \sum_{0<k<i} h_{i-k,k+j} h_{k,j}, \\ d_1(a_i) = \sum_{0 \leq k<i} h_{i-k,k} a_k, \\ d_1(b_{i,j}) = 0 \end{cases}$$

给出.

对于给定的 $x \in E_1^{s,t,M}$, 我们分别定义 $dim(x) = s$,$deg(x) = t$,$M(x) = M$, 对于上述 $E_1^{*,*,*}$ 的生成元，我们有:

$$\begin{cases} dim(h_{i.j}) = dim(a_j) = 1, dim(b_{i.j}) = 2, \\ M(h_{i,j}) = 2i-1, M(a_i) = 2i+1, M(b_{i,j}) = (2i-1)p, \\ deg(h_{i,j}) = 2(p^i-1)p^j = (p^j + \cdots + p^{i+j-1})q, \\ deg(b_{i,j}) = 2(p^i-1)p^{j+1} = (p^{j+1} + \cdots + p^{i+j})q, \\ deg(a_i) = 2p^i - 1 = (1 + \cdots + p^{i-1})q + 1, \\ deg(a_0) = 1, \end{cases}$$

下面，我们使用文献 [6] 中给出的决定 $E_1^{*,*,*}$ 生成元的方法证明主要定理.

3. 主要定理的证明

引理 1. 设 $p \geq 11, n \geq 2, 0 \leq s \leq p-5, t = p^n + (s+4)p^3 + (s+3)p^2 + (s+2)p + (s+2), q = 2(p-1)$, 则 *May* 谱序列的 E_1-项 $E_1^{s+5,tq+s,*}$ 的生成元如下：

(1) 当 $n=2$ 时，没有生成元;

(2) 当 $n=3$ 时，没有生成元;

(3) 当 $n=4$ 时，有 12 个生成元:

$$\begin{cases} \left.\begin{array}{ll} g_1 = a_5 a_4^{s-1} h_{4,0} h_{3,1} h_{2,2} h_{1,3} h_{1,0}, & g_2 = a_4^{s-1} a_1 h_{5,0} h_{4,0} h_{3,1} h_{2,2} h_{1,3} \\ g_3 = a_4^s h_{4,1} h_{4,0} h_{2,2} h_{1,3} h_{1,0}, & g_4 = a_4^s h_{4,0} h_{3,2} h_{2,2} h_{2,0} h_{1,3} \\ g_5 = a_4^s h_{4,0} h_{3,0} h_{2,3} h_{2,2} h_{1,3}, & g_6 = a_4^s h_{5,0} h_{3,1} h_{2,2} h_{1,3} h_{1,0} \\ g_7 = a_4^s h_{4,0} h_{3,2} h_{3,1} h_{1,3} h_{1,0}, & g_8 = a_4^s h_{4,0} h_{3,2} h_{2,2} h_{2,0} h_{1,3} \\ g_9 = a_4^s h_{4,0} h_{3,1} h_{2,2} h_{2,3} h_{1,0}, & g_{10} = a_4^s h_{4,0} h_{3,0} h_{2,2} h_{2,3} h_{1,3} \end{array}\right\} (M_1 = 9s+19) \\ \left. g_{11} = a_4^s h_{5,0} h_{4,0} h_{1,3} b_{2,1} \right\} (M_2 = 9s+3p+17) \\ \left. g_{12} = a_4^s h_{5,0} h_{4,0} h_{2,2} b_{1,2} \right\} (M_3 = 9s+p+19) \end{cases}$$

(4) 当 $n \geq 5$ 且 $s = p-5$ 时，有 1 个生成元 $g_{13} = a_n^{p-5} h_{n,0} h_{n-2,2} h_{n-3,3} h_{n-4,4} h_{5,0}$ $(M_4 = 2np - 2n + p - 18)$.

证明. 由文献 [6] 介绍的方法，考虑生成元 $g = x_1 \cdots x_b y_1 \cdots y_m \in F_1^{\tilde{m}+b,tq+s,*}$, 其中 $x_i = a_i, y_i = h_{i_m,j_m}$. 由 $tq+s = q[p^n+(s+4)p^3+(s+3)p^2+(s+2)p+(s+2)]+s$ 知,$b = s$, 所以 $g = x_1 \cdots x_s y_1 \cdots y_m$, 且有以下方程组:

$$\begin{cases} x_{1,0} + \cdots + x_{s,0} + y_{1,0} + \cdots + y_{m,0} = (s+2) + k_1 p = c_0, \\ x_{1,1} + \cdots + x_{s,1} + y_{1,1} + \cdots + y_{m,1} = (s+2) + k_2 p - k_1 = c_1, \\ x_{1,2} + \cdots + x_{s,2} + y_{1,2} + \cdots + y_{m,2} = (s+3) + k_3 p - k_2 = c_2, \\ x_{1,3} + \cdots + x_{s,3} + y_{1,3} + \cdots + y_{m,3} = (s+4) + k_4 p - k_3 = c_3, \\ x_{1,4} + \cdots + x_{s,4} + y_{1,4} + \cdots + y_{m,4} = 0 + k_5 p - k_4 = c_4, \\ \vdots \\ x_{1,n-1} + \cdots + x_{s,n-1} + y_{1,n-1} + \cdots + y_{m,n-1} = 0 + k_n p - k_{n-1} = c_{n-1}, \\ x_{1,n} + \cdots + x_{s,n} + y_{1,n} + \cdots + y_{m,n} = 1_n - k_n = c_n, \end{cases}$$

上述线性方程组对应矩阵

$$\begin{array}{c} \quad A \qquad\qquad\qquad B \\ \left(\begin{array}{ccc|ccc} x_{1,0} & \cdots & x_{s,0} & y_{1,0} & \cdots & y_{m,0} \\ x_{1,1} & \cdots & x_{s,1} & y_{1,1} & \cdots & y_{m,1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{1,n} & \cdots & x_{s,n} & y_{1,n} & \cdots & y_{m,n} \end{array}\right) \begin{array}{c} c_0 \\ c_1 \\ \vdots \\ c_n \end{array} \end{array}$$

可以交换矩阵 A 部分和 B 部分的列，上面矩阵可以转化为如下形式的新矩阵

$$\begin{array}{c} A \qquad B \\ \left(\begin{array}{c|ccc} I_1 & & & \\ & I_2 & & \\ & & \ddots & \\ & & & I_3 \end{array}\right) \end{array}$$

其中 A 部分中的一列 $(\overbrace{1,\cdots,1}^{i},0,\cdots,0)^T$ 表示 a_i, B 部分中的一列 $(0,\cdots,0,1,\cdots,1,0,\cdots,0)^T$ $(j0s,i1s,0,\cdots,0)$ 表示 $h_{i,j}$.

按照文献 [6] 的方法得到 $F_1^{*,*,*}$ 的生成元后，需要进行替换操作 $h_{i,j+1}\to b_{i,j}$ 或分解操作 $h_{i,j}\to h_{i-k,j+k}h_{k,j}$, $a_i\to a_{i-j}h_{j,i-j}$. 经过分解和替换操作后就可得到 $E_1^{s,t,*}$ 的所有生成元.

情形 1 $n=2$

此时 $tq+s=q[(s+4)p^3+(s+4)p^2+(s+2)p+(s+2)]+s$, 且方程组化为

$$\begin{cases} (s+2)+k_1p & =c_0 \\ (s+2)+k_2p-k_1 & =c_1 \\ (s+4)+k_3p-k_2 & =c_2 \\ (s+4)-k_3 & =c_3 \end{cases}$$

由生成元的次数可知,$K=(k_1,k_2,k_3)$ 只能是 $K=(0,0,0)$, 则相对应的 $S=(s+2,s+2,s+4,s+4)$, 且 $\tilde{m}=4<5$, 因此得到矩阵

$$\left(\begin{array}{ccc|cccc} 1 & \cdots & 1 & 1 & 1 & 0 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 0 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 1 \end{array}\right) \begin{array}{c} s+2 \\ s+2 \\ s+4 \\ s+4 \end{array}$$

它决定元素 $a_4^s h_{4,0}^2 h_{2,2}^2 \in F_1^{s+4,*,*}$. 由次数原因可知，该元素需要进行一次分解或者替换操作, 但该元素无论进行一次分解, 还是进行一次替换都会出现 $h_{i,j}^2$, 这意味着在此情形下，生成元不存在.

情形 2 $n=3$

此时 $tq+s=q[(s+5)p^3+(s+3)p^2+(s+2)p+(s+2)]+s$, 且方程组化为

$$\begin{cases} (s+2)+k_1p & =c_0 \\ (s+2)+k_2p-k_1 & =c_1 \\ (s+3)+k_3p-k_2 & =c_2 \\ (s+5)-k_3 & =c_3 \end{cases}$$

由生成元的次数可知,$K=(k_1,k_2,k_3)$ 只能是 $K=(0,0,0)$, 则相对应的 $S=(s+2,s+2,s+3,s+5)$, 且 $\tilde{m}=5$, 因此得到矩阵

$$\left(\begin{array}{ccc|ccccc} 1 & \cdots & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right) \begin{array}{c} s+2 \\ s+2 \\ s+3 \\ s+5 \end{array}$$

它决定元素 $a_4^s h_{4,0}^2 h_{2,2} h_{1,3}^2 \in F_1^{s+5,*,*}$. 由次数原因可知, 该元素无需分解，于是有

$$0=a_4^s h_{4,0}^2 h_{2,2} h_{1,3}^2 \in E_1^{s+5,tq+s,*}.$$

情形 3 $n=4$

此时 $tq+s=q[p^4+(s+4)p^3+(s+3)p^2+(s+2)p+(s+2)]+s$, 且方程组化为

$$\begin{cases} (s+2)+k_1p & =c_0 \\ (s+2)+k_2p-k_1 & =c_1 \\ (s+3)+k_3p-k_2 & =c_2 \\ (s+4)+k_4p-k_3 & =c_3 \\ (s+5)-k_4 & =c_4 \end{cases}$$

由生成元的次数可知,$K=(k_1,k_2,k_3,k_4)=(0,0,0,0)$, 则相对应的 $S=(s+2,s+2,s+3,s+4,1)$, 且 $\tilde{m}=4<5$, 此时，得到矩阵的前 5 行可能为

$$\left(\begin{array}{ccc|cccc} 1 & \cdots & 1 & 1 & 1 & 0 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 0 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{c} s+2 \\ s+2 \\ s+3 \\ s+4 \\ \hline 1\cdots(1) \\ \cdots(2) \\ \cdots(3) \\ \cdots(4) \end{array}$$

第 5 行选择 (1), 可得到矩阵

$$\left(\begin{array}{ccc|cccc} 1 & \cdots & 1 & 1 & 1 & 0 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 0 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 1 \\ 1 & \cdots & 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{c} s+2 \\ s+2 \\ s+3 \\ s+4 \\ 1 \end{array}$$

它决定元素 $a_5a_4^{s-1}h_{4,0}^2h_{2,2}h_{1,3} \in F_1^{s+4,*,*}$.

类似地，第 5 行分别选择 $(2),(3),(4)$, 可以得到元素

$$\left\{ a_4^s h_{5,0}h_{4,0}h_{2,2}h_{1,3},\ \ a_4^s h_{4,0}^2 h_{3,2}h_{1,3},\ \ a_4^s h_{4,0}^2 h_{2,2}h_{2,3} \right\} \in F_1^{s+4,*,*}$$

对上述四个元素进行一次分解或者一次替换操作，得到 $E_1^{s+5,tq+s,*}$ 的非零生成元

$$\left\{ \begin{array}{l} a_5a_4^{s-1}h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,0},\ \ a_4^{s-1}a_1h_{5,0}h_{4,0}h_{3,1}h_{2,2}h_{1,3},\ \ a_4^s h_{4,1}h_{4,0}h_{2,2}h_{1,3}h_{1,0} \\ a_4^s h_{4,0}h_{3,2}h_{2,2}h_{2,0}h_{1,3},\ \ a_4^s h_{4,0}h_{3,0}h_{2,3}h_{2,2}h_{1,3},\ \ a_4^s h_{5,0}h_{3,1}h_{2,2}h_{1,3}h_{1,0}, \\ a_4^s h_{4,0}h_{3,2}h_{2,2}h_{2,0}h_{1,3}, \end{array} \right\} \in E_1^{s+5,*,*}$$

情形 4　$n \geq 5$

此时 $tq+s = q[p^n + (s+4)p^3 + (s+3)p^2 + (s+2)p + (s+2)] + s$, 且方程组化为

$$\begin{cases} (s+2) + k_1p & = c_0 \\ (s+2) + k_2p - k_1 & = c_1 \\ (s+3) + k_3p - k_2 & = c_2 \\ (s+4) + k_4p - k_3 & = c_3 \\ 0 + k_5p - k_4 & = c_4 \\ \vdots & \\ 0 + k_np - k_{n-1} & = c_{n-1} \\ 1 - k_n & = c_n \end{cases}$$

DOI: 10.12677/pm.2026.165128

此时, 序列 $K=(k_1,k_2,\cdots,k_n)$ 有如下可能

$$\begin{cases} K_1=(0,0,\cdots,0) \\ K_i=(0,0,\cdots,0,1^{(i)},1,\cdots,1) \end{cases}$$

其中 $1^{(i)}$ 表示序列 K_i 的第 i 项为 1.

对每个 K_i, 可以得到相对应的序列 $S=(c_0,c_1,\cdots,c_n)$ 为

$$\begin{cases} S_1=(s+2,s+2,s+3,s+4,0,\cdots,0,1) \\ S_i=(s+2,s+2,s+3,s+4,0\cdots,0,p^{(i)},p-1,\cdots,p-1,0) \end{cases}$$

情形 4.1 $K=K_1=(0,0,\cdots,0)$

此时, 相对应的序列 $S=S_1=(s+2,s+2,s+3,s+4,0,\cdots,0,1)$, $\tilde{m}=5$. 进而得到矩阵

$$\left(\begin{array}{ccc|ccccc} 1 & \cdots & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ & \cdots & & & & \cdots & & \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{c} s+2 \\ s+2 \\ s+3 \\ s+4 \\ 1 \\ \vdots \\ 0 \\ 1 \end{array}$$

它决定元素 $a_4^s h_{4,0}^2 h_{2,2} h_{1,3} h_{1,n}\in F_1^{s+5,*,*}$. 由次数原因可知, 该元素无需分解，于是有

$$0=a_4^s h_{4,0}^2 h_{2,2} h_{1,3} h_{1,n}\in E_1^{s+5,tq+s,*}.$$

情形 4.2 $K=K_i=(0,0,\cdots,0,1^{(i)},1,\cdots,1)(6\le i\le n)$

此时, 相对应的序列 $S=S_i=(s+2,s+2,s+3,s+4,0\cdots,0,p^{(i)},p-1,\cdots,p-1,0)$, $\tilde{m}=p+3>5$. 根据前面介绍的生成元确定方法知，此情形下生成元不存在.

情形 4.3 $K=K_5=(0,0,0,0,1,1,\cdots,1)$

此时, 相对应的序列 $S=S_5=(s+2,s+2,s+3,s+4,p,p-1,\cdots,p-1,0)$. 我们知道 $0<s<p-4$, 即 $0<s\le p-5$. 下面分别考虑 $0<s<p-5$ 和 $s=p-5$.

情形 4.3.1 $K=K_5=(0,0,0,0,1,1,\cdots,1)$ 且 $0<s<p-4$

此时, 相对应的序列 $S=S_5=(s+2,s+2,s+3,s+4,p,p-1,\cdots,p-1,0)$. 通过计算不难得到

$$\tilde{m}=m_0+m_1+\cdots+m_n=p-s>5$$

根据前面介绍的生成元确定方法知，此情形下生成元不存在.

情形 4.3.2 $K = K_5 = (0,0,0,0,1,1,\cdots,1)$ 且 $s = p-5$

此时有

$$S = S_5 = (s+2, s+2, s+3, s+4, s+5, s+4, \cdots, s+4, 0)$$

同样可计算得出 $\tilde{m} = 5$, 并且得到下面矩阵

$$\left(\begin{array}{cccc|ccccc} 1 & \cdots & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & \cdots & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & \cdots & 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right) \quad \begin{array}{r} s+2 \\ s+2 \\ s+3 \\ s+4 \\ p=s+5 \\ \hline p-1=s+4\cdots(1) \\ \cdots(2) \\ \cdots(3) \\ \cdots(4) \\ \cdots(5) \end{array}$$

第 6 行分别选择 $(1) \sim (5)$, 可以得到

$$\left\{\begin{array}{l} a_n^{p-5}h_{n,0}^2h_{n-2,2}h_{n-3,3}h_{1,4}, \quad a_n^{p-5}h_{n,0}^2h_{n-2,2}h_{2,3}h_{n-4,4}, \quad a_n^{p-5}h_{n,0}^2h_{3,2}h_{n-3,3}h_{n-4,4}, \\ a_n^{p-5}h_{n,0}h_{5,0}h_{n-2,2}h_{n-3,3}h_{n-4,4}, \quad a_n^{p-6}a_5h_{n,0}^2h_{n-2,2}h_{n-3,3}h_{n-4,4} \end{array}\right\} \in F_1^{s+5,*,*}$$

由次数原因可知, 上述元素无需分解，于是下面有 $E_1^{s+5,tq+s,*}$ 的非零生成元

$$\left\{ a_n^{p-5}h_{n,0}h_{5,0}h_{n-2,2}h_{n-3,3}h_{n-4,4} \right\} \in E_1^{s+5,tq+s,M_4}$$

其中,$M_4 = 2np - 2n + p - 18$.

综上，引理得证. □

引理 2. 当 $p \geq 11, n \geq 2, 0 \leq s \leq p-5$, 元素 $\tilde{\delta}_{s+4}h_0h_n \in Ext_A^{s+7,tq+s}(Z_p, Z_p)$ 可以被 *May* 谱序列中的元素 $a_4^sh_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,0}h_{1,n} \in E_1^{s+6,tq+s,9s+18}$ 所表示，其中 $t = p^n + (s+4)p^3 + (s+3)p^2 + (s+2)p + (s+2), q = 2(p-1), q = 2(p-1)$.

证明. 由次数原因，$h_{1,0}, h_{1,n}$ 在 May 谱序列中是永久循环的，并且非平凡的收敛到 h_0, h_n. 而由 [1] 可知，第四希腊字母族 $\tilde{\delta}_{s+4}$ 可由 May 谱序列中的 $a_4^sh_{4,0}h_{3,1}h_{2,2}h_{1,3} \in E_1^{s+4,t'q+s,*}$ 所表示，其中 $t' = (s+4)p^3 + (s+3)p^2 + (s+2)p + (s+1)$.

因此, 乘积元素 $\tilde{\delta}_{s+4}h_0h_n \in Ext_A^{s+7,tq+s}(Z_p, Z_p)$ 可由 May 谱序列中的 $a_4^sh_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,0}h_{1,n}$

$\in E_1^{s+6,tq+s,9s+18}$ 所表示. □

下面给出定理 1 的证明.

定理 1. 当 $p \geq 11, n \geq 2, 0 \leq s \leq p-5, t = p^n + (s+4)p^3 + (s+3)p^2 + (s+2)p + (s+2), q = 2(p-1)$ 时，乘积元素 $0 \neq \tilde{\delta}_{s+4}h_0h_n \in Ext_A^{s+7,tq+s}(Z_p, Z_p)$.

证明. 由引理 1 知 $\tilde{\delta}_{s+4}h_0h_n$ 可以被 May 谱序列中的中的 $a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,0}h_{1,n}$
$\in E_1^{s+6,tq+s,9s+18}$ 所表示.

接下来，我们需要证明乘积元素 $a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,0}h_{1,n}$ 不被 May 微分
$d_r : E_r^{s+5,tq+s,9s+18-r} \longrightarrow E_r^{s+6,tq+s,9s+18}(r \geq 1)$ 击中. 这意味着，我们需要处理 $E_1^{s+5,tq+s,M}$ 中那些 $M > 9s+18$ 的生成元. 根据引理 1,$E_1^{s+5,tq+s,*}$ 的生成元 $g_i(1 \leq i \leq 15)$ 均满足 $M > 9s+18$. 下面我们考虑 $E_1^{s+5,tq+s,*}$ 的所有生成元的第一 May 微分.

考虑生成元 $g_{!}$, 其中 $M(g_1) = 9s+19$, 由 May 微分公式得到下面微分形式:

$$
\begin{aligned}
d_1(g_1) &= d_1(a_5a_4^{s-1}h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,0}) \\
&= d_1(a_5)a_4^{s-1}h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,0} + (-1)a_5d_1(a_4^{s-1}h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,0}) \\
&= (h_{5,0}a_0 + h_{4,1}a_1 + h_{3,2}a_2 + h_{2,3}a_3 + h_{1,4}a_4)a_4^{s-1}h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,0} \\
&\quad - a_5d_1(a_4^{s-1})h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,0} - (-1)^{s-1}a_5a_4^{s-1}d_1(h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,0}) \\
&= (h_{5,0}a_0 + h_{4,1}a_1 + h_{3,2}a_2 + h_{2,3}a_3 + h_{1,4}a_4)a_4^{s-1}h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,0} \\
&\quad - a_5(s-1)a_4^{s-2}(\underline{h_{4,0}}a_0 + \underline{h_{3,1}}a_1 + \underline{h_{2,2}}a_2 + \underline{h_{1,3}}a_3)\underline{h_{4,0}h_{3,1}h_{2,2}h_{1,3}}h_{1,0} \\
&\quad - (-1)^{s-1}a_5a_4^{s-1}d_1(h_{4,0})h_{3,1}h_{2,2}h_{1,3}h_{1,0} - (-1)^{s}a_5a_4^{s-1}h_{4,0}d_1(h_{3,1}h_{2,2}h_{1,3}h_{1,0}) \\
&= (h_{5,0}a_0 + h_{4,1}a_1 + h_{3,2}a_2 + h_{2,3}a_3 + h_{1,4}a_4)a_4^{s-1}h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,0} \\
&\quad - (-1)^{s-1}a_5a_4^{s-1}(\underline{h_{3,1}}h_{1,0} + \underline{h_{2,2}}h_{2,0} + \underline{h_{1,3}}h_{3,0})\underline{h_{3,1}h_{2,2}h_{1,3}}h_{1,0} \\
&\quad - (-1)^{s}a_5a_4^{s-1}h_{4,0}d_1(h_{3,1})h_{2,2}h_{1,3}h_{1,0} + (-1)^{s}a_5a_4^{s-1}h_{4,0}h_{3,1}d_1(h_{2,2}h_{1,3}h_{1,0}) \\
&= (h_{5,0}a_0 + h_{4,1}a_1 + h_{3,2}a_2 + h_{2,3}a_3 + h_{1,4}a_4)a_4^{s-1}h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,0} \\
&\quad - (-1)^{s}a_5a_4^{s-1}h_{4,0}(\underline{h_{2,2}}h_{1,1} + \underline{h_{1,3}}h_{3,1})\underline{h_{2,2}h_{1,3}}h_{1,0} + (-1)^{s}a_5a_4^{s-1}h_{4,0}h_{3,1}d_1(h_{2,2})h_{1,3}h_{1,0} \\
&\quad + (-1)^{s+1}a_5a_4^{s-1}h_{4,0}h_{3,1}h_{2,2}\underline{d_1(h_{1,3}h_{1,0})} \\
&= (h_{5,0}a_0 + h_{4,1}a_1 + h_{3,2}a_2 + h_{2,3}a_3 + h_{1,4}a_4)a_4^{s-1}h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,0} \\
&\quad + (-1)^{s}a_5a_4^{s-1}h_{4,0}h_{3,1}(\underline{h_{1,3}}h_{1,2})\underline{h_{1,3}}h_{1,0} \\
&= (h_{5,0}a_0 + h_{4,1}a_1 + h_{3,2}a_2 + h_{2,3}a_3 + h_{1,4}a_4)a_4^{s-1}h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,0} \neq 0
\end{aligned}
$$

类似地, 考虑生成元 $g_i(2 \leq i \leq 10), M(g_i) = 9s+19$, 第一 May 微分 d_1 作用在生成元上的像

如下:

$$d_1(g_2) = (-1)^{s-1}a_4^{s-1}a_0h_{5,0}h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,0} + (-1)a_4^{s-1}a_1(h_{4,1}h_{1,0} + h_{3,2}h_{2,0} + h_{2,3}h_{3,0})h_{4,0}h_{3,1}h_{2,2}h_{1,3} \neq 0$$

$$d_1(g_3) = (-1)a_4^{s}(h_{3,2}h_{1,1} + h_{2,3}h_{2,1} + h_{1,4}h_{3,1})h_{4,0}h_{2,2}h_{1,3}h_{1,0} + (-1)^s a_4^{s-1}a_1h_{4,1}h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,0} \neq 0$$

$$d_1(g_4) = sa_4^{s-1}(h_{4,0}a_0 + h_{3,1}a_1)h_{4,0}h_{3,2}h_{2,2}h_{2,0}h_{1,3} + (-1)^s a_4^s h_{3,2}h_{3,1}h_{2,2}h_{2,0}h_{1,3}h_{1,0} + (-1)^{s+1}a_4^s h_{4,0}h_{2,3}h_{2,2}h_{2,0}h_{1,3}h_{1,2} + (-1)^{s+1}a_4^s h_{4,0}h_{3,2}h_{1,3}h_{1,1}h_{1,0} \neq 0$$

$$d_1(g_5) = sa_4^{s-1}a_1h_{4,0}h_{3,1}h_{3,0}h_{2,3}h_{2,2}h_{1,3} + (-1)^{s+1}a_4^s h_{4,0}(h_{2,1}h_{1,0} + h_{1,2}h_{2,0})h_{2,3}h_{2,2}h_{1,3} + (-1)^s a_4^s h_{3,1}h_{3,0}h_{2,3}h_{2,2}h_{1,3}h_{1,0} \neq 0$$

$$d_1(g_6) = (-1)^s a_4^s(h_{3,2}h_{2,0} + h_{2,3}h_{3,0} + h_{1,4}h_{4,0})h_{3,1}h_{2,2}h_{1,3}h_{1,0} + sa_4^{s-1}a_0h_{5,0}h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,0} \neq 0$$

$$d_1(g_7) = sa_4^{s-1}a_2h_{4,0}h_{3,2}h_{3,1}h_{2,2}h_{1,3}h_{1,0} + (-1)^{s+1}a_4^s h_{4,0}(h_{2,3}h_{1,2} + h_{1,4}h_{2,2})h_{3,1}h_{1,3}h_{1,0} + (-1)^s a_4^s h_{3,2}h_{3,1}h_{2,2}h_{2,0}h_{1,3}h_{1,0} + (-1)^s a_4^s h_{4,0}h_{3,2}h_{2,2}h_{1,3}h_{1,1}h_{1,0} \neq 0$$

$$d_1(g_8) = sa_4^{s-1}a_1h_{4,0}h_{3,2}h_{3,1}h_{2,2}h_{2,0}h_{1,3} + (-1)^s a_4^s h_{3,2}h_{3,1}h_{2,2}h_{2,0}h_{1,3}h_{1,0} + (-1)^{s+1}a_4^s h_{4,0}h_{2,3}h_{2,2}h_{2,0}h_{1,2}h_{1,3} + (-1)^{s+1}a_4^s h_{4,0}h_{3,2}h_{2,2}h_{1,3}h_{1,1}h_{1,0} \neq 0$$

$$d_1(g_9) = sa_4^{s-1}(h_{1,3}a_3)h_{4,0}h_{3,1}h_{2,2}h_{2,3}h_{1,0} + (-1)^s a_4^s(h_{1,3}h_{3,0})h_{3,1}h_{2,2}h_{2,3}h_{1,0} + (-1)^{s+1}a_4^s h_{4,0}(h_{1,3}h_{2,1})h_{2,2}h_{2,3}h_{1,0} + (-1)^s a_4^s h_{4,0}h_{3,1}(h_{1,3}h_{1,2})h_{2,3}h_{1,0} + (-1)^{s+1}a_4^s h_{4,0}h_{3,1}h_{2,2}(h_{1,4}h_{1,3})h_{1,0} \neq 0$$

$$d_1(g_{10}) = (-1)^{s+1}a_4^s h_{4,0}(h_{2,1}h_{1,0} + h_{1,2}h_{2,0})h_{2,2}h_{2,3}h_{1,3} + (-1)^s a_4^s(h_{3,1}h_{1,0})h_{3,0}h_{2,2}h_{2,3}h_{1,3} + sa_4^{s-1}(h_{3,1}a_1)h_{4,0}h_{3,0}h_{2,2}h_{2,3}h_{1,3} \neq 0$$

由上述计算易知，我们得到的生成元 $g_i(2 \leq i \leq 10)$ 的第一 May 微分非零，并且都至少含有一个其他生成元的第一 May 微分中所没有的项，可知这些生成元的第一 May 微分是线性无关的. 这意味着，当 $r \geq 2$ 时，$E_r^{s+5,tq+s,M(g_i)} = 0(1 \leq i \leq 11)$. 因此, 对 $r \geq 1$ 时，有

$$a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,0}h_{1,n} \notin d_r(E_r^{s+5,tq+s,M_1})$$

对于生成元 $g_{12} \in E_1^{s+5,tq+s,M_3}, g_{13} \in E_1^{s+5,tq+s,M_4}, g_{14} \in E_1^{s+5,tq+s,M_5}$ 有:

$$\begin{aligned} d_1(g_{11}) &= sa_4^{s-1}(h_{3,1}a_1 + h_{2,2}a_2)h_{5,0}h_{4,0}h_{1,3}b_{2,1} \\ &+ (-1)^s a_4^s(h_{4,1}h_{1,0} + h_{3,2}h_{2,0} + h_{2,3}h_{3,0})h_{4,0}h_{1,3}b_{2,1} \\ &+ (-1)^{s+1}a_4^s h_{5,0}(h_{3,1}h_{1,0} + h_{2,2}h_{2,0})h_{1,3}b_{2,1} \neq 0 \end{aligned}$$

$$\begin{aligned} d_1(g_{12}) &= sa_4^{s-1}(h_{3,1}h_{1,0} + h_{1,3}h_{3,0})h_{5,0}h_{4,0}h_{2,2}b_{1,2} \\ &+ (-1)^s a_4^s(h_{4,1}h_{1,0} + h_{3,2}h_{2,0} + h_{2,3}h_{3,0})h_{4,0}h_{1,3}b_{2,1} \\ &+ (-1)^{s+1}a_4^s h_{5,0}(h_{3,1}h_{1,0} + h_{1,3}h_{3,0})h_{2,2}b_{1,2} + (-1)^s a_4^s h_{5,0}h_{4,0}h_{1,3}h_{1,2}b_{1,2} \neq 0 \end{aligned}$$

$$\begin{aligned} d_1(g_{13}) &= d_1(a_n^{p-5}h_{n,0}h_{n-2,2}h_{n-3,3}h_{n-4,4}h_{5,0}) \\ &= (p-5)a_n^{p-6}(h_{n-1,1}a_1 + \sum_{k=5}^{n-6} h_{n-k,k}a_k \\ &\quad + \sum_{k=n-4}^{n-1} h_{n-k,k}a_k)h_{n,0}h_{n-2,2}h_{n-3,3}h_{n-4,4}h_{5,0} + \cdots \neq 0 \end{aligned}$$

同理，对 $r \geq 1$ 且 $2 \leq i \leq 4$ 有

$$a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,0}h_{1,n} \notin d_r(E_r^{s+5,tq+s,M_i})$$

由上述结果可知，当 $p \geq 11, n \geq 2, 0 \leq s \leq p-5$ 时,$a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,0}h_{1,n} \in E_1^{s+6,tq+s,9s+18}$ 不被任何 May 微分击中. 因此,$a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,0}h_{1,n}$ 在 MSS 中永久循环, 进而非平凡地收敛到 $\tilde{\delta}_{s+4}h_0h_n \in Ext_A^{s+7,tq+s}(Z_p, Z_p)$. 定理 1 得证. □

参考文献

[1] Adams, J.F. (1974) Stable Homotopy and Generalised Homology. University of Chicago Press.

[2] Wang, X. and Zheng, Q. (1998) The Convergence of $\tilde{\alpha}_s^n h_0 h_k$. *Science in China Series A: Mathematics*, **41**, 622-628. https://doi.org/10.1007/bf02876232

[3] Liulevicius, A. (1960) The Factorization of Cyclic Reduced Powers by Secondary Cohomology Operations. *Proceedings of the National Academy of Sciences*, **46**, 978-981. https://doi.org/10.1073/pnas.46.7.978

[4] 林金坤, 编著. Adams 谱序列和球面稳定同伦群 [M]. 北京: 科学出版社, 2007.

[5] Ravenel, D. (2003) Complex Cobordism and Stable Homotopy Groups of Spheres. American Mathematical Society. https://doi.org/10.1090/chel/347

[6] Zhong, L. and Wang, Y. (2013) Detection of a Nontrivial Product in the Stable Homotopy Groups of Spheres. *Algebraic Geometric Topology*, **13**, 3009-3029.
https://doi.org/10.2140/agt.2013.13.3009