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# 一类带有未知项的随机非完整系统的有限时间控制问题研究

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## 摘 要

本文针对一类具有未知时变控制系数及随机噪声干扰的不确定项的随机非完整系统，研究其有限时间稳定问题。根据系统的特性，分两阶段进行控制设计：首先基于Lyapunov理论为 $x_0$ 子系统设计保证子系统稳定的控制输入 $u_0$；其次，通过对 $x$ 系统进行坐标变换转换成新形式 $y$ 系统，并采用递归反推法最终设计出全局控制输入 $u$。最终设计出全局控制输入理论分析表明，闭环系统的稳定性可通过随机Lyapunov函数 $V_n$ 的负定性严格证明，且系统状态在有限时间内收敛至平衡点。

## 关键词

随机非完整系统，坐标变换，有限时间稳定

# Research on Finite-Time Control for a Class of Stochastic Nonholonomic Systems with Unknown Terms

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## Abstract

**This paper investigates the finite-time stabilization problem for a class of stochastic nonholonomic systems with unknown time-varying control coefficients and stochastic noise disturbances. Based on the system characteristics, a two-phase control design framework is proposed.**

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**First, a Lyapunov-based control input $u_0$ is designed for the $x_0$-subsystem to ensure its stability. Subsequently, the original $x$-system is transformed into a new structure $y$-system via coordinate transformation, and a recursive backstepping approach is employed to synthesize the global control input $u$. Theoretical analysis demonstrates that the closed-loop system's stability can be rigorously proven through the negative definiteness of the stochastic Lyapunov function $V_n$, ensuring probabilistic convergence of system states to the equilibrium point within a finite time. The proposed method effectively addresses uncertainties arising from time-varying control coefficients and stochastic disturbances, providing a systematic solution for finite-time stabilization of stochastic nonholonomic systems.**

## Keywords

**Stochastic Nonholonomic System, Coordinate Transformation, Finite-Time Stabilization**



## 1. 引言

本文研究一类具有未知控制系数的随机非完整系统的有限时间镇定问题，其系统模型如下：

$$\begin{cases} \mathrm{d}x_0 = u_0 d_0(t)\mathrm{d}t + f_0(x_0)\mathrm{d}t, \\ \mathrm{d}x_i = d_i(t)u_0 x_{i+1}\mathrm{d}t + f_i(t,u_0,x_0,\bar{x}_i)\mathrm{d}t + g_i(t,u_0,x_0,\bar{x}_i)\mathrm{d}\omega(t), i = 1,\cdots,n-1, \\ \mathrm{d}x_n = d_n(t)u\mathrm{d}t + f_n(t,u_0,x_0,\bar{x}_n)\mathrm{d}t + g_n(t,u_0,x_0,\bar{x}_n)\mathrm{d}\omega(t), \end{cases} \tag{1}$$

其中 $(x_0,x)^{\mathrm{T}} = (x_0,x_1,x_2,\cdots,x_n)^{\mathrm{T}} \in R^{n+1}$ 是系统状态；$u_0 \in R$ 并且 $u \in R$ 是控制输入；$\bar{x}_i = (x_1,x_2,\cdots,x_i)^{\mathrm{T}} \in R^i$，$\bar{x}_n = x = (x_1,x_2,\cdots,x_n)^{\mathrm{T}} \in R^n$，$f_0: R \to R$，$f_i: R\times R\times R\times R^i \to R$，$g_i: R\times R\times R\times R^i \to R^m$，$i = 1,\cdots,n$ 且 $f_0, f_i, g_i$ 为 Borel 可测连续函数；$d_i(t)$，$i = 0,1,\cdots,n$，是未知的控制系数；$x_0(0)$ 和 $x(0)$ 是随机变量并且 $\omega(t)$ 是一个定义在概率空间 $(\Omega,\mathcal{F},P)$ 上的 $m$ 维标准维纳过程。

非完整系统的本质在于不可积分的速度相关约束，这类约束无法通过坐标变换消除，需在动力学中独立处理，从而形成独特的非完整力学体系。典型例子包括轮式移动机器人、滚动刚体等，其运动自由度受限于速度方向而非几何位置。为描述上述等典型机械例子，非完整系统的控制问题受到了全球研究者的广泛关注，由于 Brockett 必要条件[1]的限制，这类非线性系统无法通过静态连续状态反馈实现镇定[2]。为克服这一难题，研究者们提出了多种创新方法，主要包括：不连续时不变镇定[3] [4]、光滑时变镇定[5]。在这些有效方法的推动下，非完整系统的鲁棒性问题得到了深入研究，并取得了一系列重要成果。

## 2. 预备知识

**定义 1** 考虑如下 $m$ 维随机非线性系统：

$$\mathrm{d}x = f(x,t)\mathrm{d}t + g(x,t)\mathrm{d}\omega, x(t_0) = x_0 \in R^n, \tag{2}$$

其中，$t \ge t_0$，$f: R^n \times R \to R^n$ 是 Borel 可测函数，$g: R^n \times R \to R^{n\times r}$ 在 $x$ 上连续，且满足 $f(0,t) = 0, g(0,t) = 0$。设 $C^2(R^n;R_+)$ 表示所有定义在 $R^n$ 上的非负函数 $V$ 构成的族，这些函数关于 $x$ 二次连续可微。对于每个

$V \in C^2\left(R^n; R_+\right)$，定义与系统(1)相关联的微分算子 $\mathcal{L}V$ 如下：

$$\mathcal{L}V = \frac{\partial V}{\partial x} f + \frac{1}{2} trace\left\{ g^{\mathrm{T}} \frac{\partial^2 V}{\partial x^2} g \right\}. \tag{3}$$

**定义 2** [6]对于系统(2)，如果存在一个李雅普诺夫函数 $V: \Re^n \to \Re_+$，$\mathcal{K}_\infty$ 函数 $\mu_1, \mu_2$，正实数 $c > 0$ 和 $0 < \gamma < 1$，对于所有 $x \in \Re^n$ 和 $t \ge 0$，会有

$$\mu_1\left(|x|\right) \le V\left(x\right) \le \mu_2\left(|x|\right), \tag{4}$$

$$\mathcal{L}V\left(x\right) \le -c \cdot \left(V\left(x\right)\right)^\gamma, \tag{5}$$

那么，系统(2)的平凡解是有限时间吸引且依概率稳定的。

为了设计系统(1)的有限时间控制，我们给出以下假设：

**假设 1** 对于 $i = 0,1,\cdots,n$，存在已知正常数 $\lambda_i, \mu_i$ 使得

$$0 < \lambda_i \le d_i\left(t\right) \le \mu_i, i = 0,1,\cdots,n. \tag{6}$$

**假设 2** 存在一个已知的非负光滑函数 $\psi_0$ 和一个已知的正常数 $\theta \in \left(0,1\right)$ 使得

$$\left|f_0\left(x_0\right)\right| \le \left|x_0\right|^\theta \psi_0\left(x_0\right). \tag{7}$$

**假设 3** 对于 $i = 0,1,\cdots,n$，存在已知非负光滑函数 $\overline{f}_i, \overline{g}_i$，和一个已知常数 $\tau \in \left(-\frac{1}{n}, 0\right)$，使得

$$\left|f_i\left(t, u_0, x_0, \overline{x}_i\right)\right| \le \sum_{j=1}^{i} \left|x_j\right|^{\frac{r_i+\tau}{r_j}} \overline{f}_i\left(u_0, x_0, \overline{x}_i\right), \tag{8}$$

$$\left|g_i\left(t, u_0, x_0, \overline{x}_i\right)\right| \le \sum_{j=1}^{i} \left|x_j\right|^{\frac{r_i+\tau}{r_j}} \overline{g}_i\left(u_0, x_0, \overline{x}_i\right), \tag{9}$$

其中，$r_i$ 是常数且 $r_i = 1 + \left(i-1\right)\tau > 0, i = 1,\cdots,n$。

**引理 1** [7] $f\left(s\right) = sgn\left(s\right)|s|^a = [s]^a$，$f\left(s\right)$ 在定义域内具有二阶连续导数，其中 $a \ge 2$ 且 $s \in R$。此外，$f'\left(s\right) = a|s|^{a-1}$ 且 $f''\left(s\right) = a\left(a-1\right)[s]^{a-2}$。

**引理 2** [8]对于任意正实数 $c, d$ 和任意实值函数 $\gamma\left(x,y\right) > 0$，有

$$|x|^c |y|^d \le \frac{c}{c+d} \gamma\left(x,y\right) |x|^{c+d} + \frac{d}{c+d} \gamma^{-\frac{c}{d}}\left(x,y\right) |y|^{c+d}.$$

**引理 3** [9]如果 $p = \frac{b_1}{b_0} \in R_{odd}^{\ge 1}$ 为实数且 $b_1 \ge b_0 \ge 1$，则对于任何 $x, y \in R$，都有

$$\left|x^p - y^p\right| \le 2^{1-\frac{1}{b_0}} \left|sgn\left(x\right)|x|^{b_1} - sgn\left(y\right)|y|^{b_1}\right|^{\frac{1}{b_0}}.$$

## 3. 有限时间控制器设计

接下来，本文在 $x_0\left(0\right) \ne 0$ 的前提下，分两部分设计控制输入 $u_0, u$。

### 3.1. 控制器 $u_0$ 的设计

对于 $x_0$-系统，我们选取李雅普诺夫函数 $V_0\left(x_0\right) = \frac{x_0^2}{2}$。通过(5)，我们有

$$\begin{aligned}\mathcal{L}V_0 &= x_0\left(u_0 d_0 + f_0\right)\\ &\le u_0 d_0 x_0 + |x_0||f_0|\\ &\le u_0 d_0 x_0 + |x_0|^{1+\theta}\psi_0.\end{aligned} \tag{10}$$

选取控制器

$$u_0 = -\frac{1}{\lambda_0}\left(k_0 + \psi_0\left(x_0\right)\right)\left[x_0\right]^{\theta}, \tag{11}$$

其中 $\lambda_0, \psi_0$ 和 $\theta$ 已经在假设 1-2 被定义过， $k_0$ 是个正常数。从而有

$$\begin{aligned}\mathcal{L}V_0 &\le -\frac{d_0}{\lambda_0} x_0\left[x_0\right]^{\theta}\left(k_0 + \psi_0\right) + |x_0|^{1+\theta}\psi_0\\ &= -\frac{d_0}{\lambda_0}|x_0|^{1+\theta}k_0 - \frac{d_0}{\lambda_0}|x_0|^{1+\theta}\psi_0 + |x_0|^{1+\theta}\psi_0\\ &\le -k_0|x_0|^{1+\theta} - |x_0|^{1+\theta}\psi_0 + |x_0|^{1+\theta}\psi_0\\ &= -k_0|x_0|^{1+\theta}\\ &= -k_0\left(\frac{x_0^2}{2}\right)^{\frac{1+\theta}{2}}\cdot 2^{\frac{1+\theta}{2}}\\ &\le -k_0 V_0^{\frac{1+\theta}{2}}.\end{aligned} \tag{12}$$

## 3.2. 控制器 $u$ 的设计

### 3.2.1. 系统坐标变换

非完整系统因 Brockett 定理限制，无法通过连续静态反馈镇定。通过坐标变换，可将系统转换为满足反推法的形式，绕过 Brockett 条件的约束。因此，针对控制系数的不确定性，我们对系统(1)进行如下坐标变换：

$$y_i = \frac{x_i}{u_0^{n-i}}, i = 1, 2, \cdots, n. \tag{13}$$

然后，系统(1)转变为

$$\begin{cases}\mathrm{d}y_i = d_i y_{i+1}\mathrm{d}t + \psi_i\mathrm{d}t + \phi_i\mathrm{d}\omega(t), 1 \le i \le n-1,\\ \mathrm{d}y_n = d_n u_1\mathrm{d}t + \psi_n\mathrm{d}t + \phi_n\mathrm{d}\omega(t),\end{cases} \tag{14}$$

其中

$$\psi_i = \frac{1}{u_0^{n-i}} f_i - (n-i)\frac{y_i}{u_0}\frac{\partial u_0}{\partial x_0}\left(u_0 d_0 + f_0\right) \tag{15}$$

和

$$\phi_i = \frac{1}{u_0^{n-i}} g_i. \tag{16}$$

**引理 4** 如果假设 2 成立，对于 $i = 1, \cdots, n$，存在非负光滑函数 $\overline{\psi}_i, \overline{\phi}_i$ 使得

$$\begin{aligned}|\psi_i| &\le \overline{\psi}_i\left(\overline{y}_i\right)\sum_{j=1}^{i}|y_j|^{\frac{r_i+\tau}{r_j}},\\ |\phi_i| &\le \overline{\phi}_i\left(\overline{y}_i\right)\sum_{j=1}^{i}|y_j|^{\frac{2r_i+\tau}{2r_j}}.\end{aligned} \tag{17}$$

#### 3.2.2. 控制器 $u$ 的设计步骤

我们选择反步法对控制器 $u$ 进行设计，具体步骤如下：

步骤一：

令 $\xi_1=[y_1]^{\frac{\sigma}{r_1}}$ 和设置 $\sigma\ge 2$，就能得到(i) $\dfrac{\sigma}{r_i}\ge 2$；(ii) $0\le r_k\le\sigma$。那么我们构造李雅普诺夫函数[10]

$$V_1(y_1)=W_1(y_1)=\int_{y_1^*}^{y_1}\left[[s]^{\frac{\sigma}{r_1}}-\left[y_1^*\right]^{\frac{4\sigma-\tau-r_1}{\sigma}}\right]\mathrm{d}s, \tag{18}$$

其中 $y_1^*=0$。通过(15)，我们可以得到

$$\begin{aligned}\mathcal{L}V_1&=\frac{\partial V_1}{\partial y_1}(d_1y_2+\psi_1)+\frac{1}{2}\frac{\partial^2V_1}{\partial y_1^2}\phi_1^2\\&=[\xi_1]^{\frac{4\sigma-\tau-r_1}{\sigma}}(d_1y_2+\psi_1)+\frac{4\sigma-\tau-r_1}{2r_1}|\xi_1|^{\frac{4\sigma-\tau-r_1}{\sigma}}\phi_1^2\\&\le[\xi_1]^{\frac{4\sigma-\tau-r_1}{\sigma}}d_1\left(y_2-y_2^*\right)+[\xi_1]^{\frac{4\sigma-\tau-r_1}{\sigma}}d_1y_2^*+\frac{4\sigma-\tau-r_1}{2r_1}|\xi_1|^{\frac{4\sigma-\tau-r_1}{\sigma}}\bar{\phi}_1^2|\xi_1|^{\frac{2r_1+\tau}{\sigma}}+[\xi_1]^{\frac{4\sigma-\tau-r_1}{\sigma}}[\xi_1]^{\frac{r_1}{\sigma}}\bar{\psi}_1\\&=[\xi_1]^{\frac{4\sigma-\tau-r_1}{\sigma}}d_1\left(y_2-y_2^*\right)+[\xi_1]^{\frac{4\sigma-\tau-r_1}{\sigma}}d_1y_2^*+\left(\frac{4\sigma-\tau-r_1}{2r_1}\bar{\phi}_1^2+\bar{\psi}_1\right)\xi_1^4.\end{aligned}\tag{19}$$

那么，选取 $C^0$ 虚拟控制器

$$y_2^*=-\frac{1}{\lambda_1}\left(n+\bar{\psi}_1+\frac{4\sigma-\tau-r_1}{2r_1}\bar{\phi}_1^2\right)[\xi_1]^{\frac{\tau+r_1}{\sigma}}:=-\beta_1^{\frac{r_2}{\sigma}}[\xi_1]^{\frac{r_2}{\sigma}}, \tag{20}$$

我们得到

$$\begin{aligned}\mathcal{L}V_1&\le[\xi_1]^{\frac{4\sigma-\tau-r_1}{\sigma}}d_1\left(y_2-y_2^*\right)+\left(\frac{4\sigma-\tau-r_1}{2r_1}\bar{\phi}_1^2+\bar{\psi}_1\right)\xi_1^4-\frac{d_1}{\lambda_1}\left(n+\bar{\psi}_1+\frac{4\sigma-\tau-r_1}{2r_1}\bar{\phi}_1^2\right)[\xi_1]^{\frac{\tau+r_1}{\sigma}}[\xi_1]^{\frac{4\sigma-\tau-r_1}{\sigma}}\\&=[\xi_1]^{\frac{4\sigma-\tau-r_1}{\sigma}}d_1\left(y_2-y_2^*\right)+\left(\frac{4\sigma-\tau-r_1}{2r_1}\bar{\phi}_1^2+\bar{\psi}_1\right)\xi_1^4-\frac{d_1}{\lambda_1}\left(n+\bar{\psi}_1+\frac{4\sigma-\tau-r_1}{2r_1}\bar{\phi}_1^2\right)|\xi_1|^4\\&\le[\xi_1]^{\frac{4\sigma-\tau-r_1}{\sigma}}d_1\left(y_2-y_2^*\right)+\left(\frac{4\sigma-\tau-r_1}{2r_1}\bar{\phi}_1^2+\bar{\psi}_1\right)\xi_1^4-\left(n+\bar{\psi}_1+\frac{4\sigma-\tau-r_1}{2r_1}\bar{\phi}_1^2\right)\xi_1^4\\&=-n\xi_1^4+[\xi_1]^{\frac{4\sigma-\tau-r_1}{\sigma}}d_1\left(y_2-y_2^*\right).\end{aligned}\tag{21}$$

归纳步骤：

假设第 $k-1$ 步，存在 $V_{k-1}(\bar{y}_{k-1},x_0)\in C^2$ 和一组虚拟控制器 $y_1^*,y_2^*,\cdots,y_{k-1}^*$ 满足[10]

$$\begin{aligned}&y_1^*=0, &&\xi_1=[y_1]^{\frac{\sigma}{r_1}}-\left[y_1^*\right]^{\frac{\sigma}{r_1}},\\&y_2^*=-\beta_1^{\frac{r_2}{\sigma}}[\xi_1]^{\frac{r_2}{\sigma}}, &&\xi_2=[y_2]^{\frac{\sigma}{r_2}}-\left[y_2^*\right]^{\frac{\sigma}{r_2}},\\&\vdots&&\vdots\\&y_k^*=-\beta_{k-1}^{\frac{r_k}{\sigma}}[\xi_{k-1}]^{\frac{r_k}{\sigma}}, &&\xi_k=[y_k]^{\frac{\sigma}{r_k}}-\left[y_k^*\right]^{\frac{\sigma}{r_k}},\end{aligned}\tag{22}$$

其中 $\beta_1,\beta_2,\cdots,\beta_{k-1}\ge 0$ 为光滑函数，使得

$$\mathcal{L}V_{k-1}\le-(n-k+2)\sum_{i=1}^{k-1}\xi_i^4+[\xi_{k-1}]^{\frac{4\sigma-\tau-r_{k-1}}{\sigma}}d_{k-1}\left(y_k-y_k^*\right). \tag{23}$$

接下来，我们构造第 $k$ 个李雅普诺夫函数

$$V_k\left(\bar{y}_k, x_0\right)=V_{k-1}\left(\bar{y}_{k-1}, x_0\right)+W_k\left(\bar{y}_k, x_0\right), \tag{24}$$

其中

$$W_k=\int_{y_k^*}^{y_k}\left[[s]^{\frac{\sigma}{r_k}}-\left[y_k^*\right]^{\frac{\sigma}{r_k}}\right]^{\frac{4\sigma-\tau-r_k}{\sigma}} \mathrm{d}s. \tag{25}$$

我们能够得到

$$\begin{aligned}
\mathcal{L}V_k \leq & -(n-k+2)\sum_{i=1}^{k-1}\xi_i^4+\left[\xi_{k-1}\right]^{\frac{4\sigma-\tau-r_{k-1}}{\sigma}} d_{k-1}\left(y_k-y_k^*\right)+\frac{\partial W_k}{\partial x_0}\left(u_0 d_0+f_0\right)+\frac{\partial W_k}{\partial y_k}\left(d_k y_{k+1}+\psi_k\right) \\
& +\sum_{i=1}^{k-1}\frac{\partial W_k}{\partial y_i}\left(d_i y_{i+1}+\psi_i\right)+\frac{1}{2}\frac{\partial^2 W_k}{\partial y_k^2}\phi_k^2+\sum_{i=1}^{k-1}\frac{\partial^2 W_k}{\partial y_i \partial y_k}\phi_i\phi_k+\frac{1}{2}\sum_{i=1}^{k-1}\frac{\partial^2 W_k}{\partial y_i}\phi_i^2+\frac{1}{2}\sum_{i,j=1,i\neq j}^{k-1}\frac{\partial^2 W_k}{\partial y_i \partial y_j}\phi_i\phi_j \\
= & -(n+k+2)\sum_{i=1}^{k-1}\xi_i^4+\left[\xi_k\right]^{\frac{4\sigma-\tau-r_k}{\sigma}} d_k y_{k+1}+\left[\xi_{k-1}\right]^{\frac{4\sigma-\tau-r_{k-1}}{\sigma}} d_{k-1}\left(y_k-y_k^*\right)+\left[\xi_k\right]^{\frac{4\sigma-\tau-r_k}{\sigma}}\psi_k \\
& +\sum_{i=1}^{k-1}\frac{\partial W_k}{\partial y_i}\left(d_i y_{i+1}+\psi_i\right)+\frac{1}{2}\frac{\partial^2 W_k}{\partial y_k^2}\phi_k^2+\frac{1}{2}\frac{\partial^2 W_k}{\partial y_i \partial y_k}\phi_i\phi_k+\frac{1}{2}\sum_{i=1}^{k-1}\frac{\partial^2 W_k}{\partial y_i}\phi_i^2+\frac{1}{2}\sum_{i,j=1,i\neq j}^{k-1}\frac{\partial^2 W_k}{\partial y_i \partial y_j}\phi_i\phi_j \\
& +\frac{\partial W_k}{\partial x_0}\left(u_0 d_0+f_0\right).
\end{aligned} \tag{26}$$

其中，

$$\left[\xi_{k-1}\right]^{\frac{4\sigma-\tau-r_{k-1}}{\sigma}} d_{k-1}\left(y_k-y_k^*\right) \leq \frac{1}{8}\xi_{k-1}^4+\xi_k^4\cdot h_1\left(h_1\geq 0\right), \tag{27}$$

$$\left[\xi_k\right]^{\frac{4\sigma-\tau-r_k}{\sigma}}\psi_k \leq \frac{1}{8}\sum_{j=1}^{k-1}\xi_i^4+\xi_k^4\cdot h_2\left(h_2\geq 0\right), \tag{28}$$

$$\sum_{i=1}^{k-1}\frac{\partial W_k}{\partial y_i}\left(d_i y_{i+1}+\psi_i\right) \leq \frac{1}{8}\sum_{i=1}^{k-1}\xi_i^4+\xi_k^4\cdot h_3\left(h_3\geq 0\right), \tag{29}$$

$$\frac{1}{2}\sum_{i,j=1,i\neq j}^{k-1}\left|\frac{\partial^2 W_k}{\partial y_i \partial y_j}\right|\left|\phi_i^{\mathrm{T}}\phi_j\right| \leq \frac{1}{8}\sum_{i=1}^{k-1}\xi_i^4+\xi_k^4\cdot h_4\left(h_4\geq 0\right), \tag{30}$$

$$\frac{1}{2}\sum_{i=1}^{k-1}\left|\frac{\partial^2 W_k}{\partial y_i^2}\right|\left|\phi_i\right|^2 \leq \frac{1}{8}\sum_{i=1}^{k-1}\xi_i^4+\xi_k^4\cdot h_5\left(h_5\geq 0\right), \tag{31}$$

$$\frac{1}{2}\frac{\partial^2 W_k}{\partial y_k^2}\phi_k^2 \leq \frac{1}{8}\sum_{i=1}^{k-1}\xi_i^4+\xi_k^4\cdot h_6\left(h_6\geq 0\right), \tag{32}$$

$$\sum_{i=1}^{k-1}\left|\frac{\partial^2 W_k}{\partial y_k \partial y_i}\right|\left|\phi_k\right|\left|\phi_i\right| \leq \frac{1}{8}\sum_{i=1}^{k-1}\xi_i^4+\xi_k^4\cdot h_7\left(h_7\geq 0\right), \tag{33}$$

$$\frac{\partial W_k}{\partial x_0}\left(u_0 d_0+f_0\right) \leq \frac{1}{8}\sum_{i=1}^{k-1}\xi_i^4+\xi_k^4\cdot h_8\left(h_8\geq 0\right). \tag{34}$$

$W_k$ 是 $C^2$ 且我们选择虚拟控制器 $y_{k+1}^*=-\beta_k^{\frac{r_{k+1}}{\sigma}}\left[\xi_k\right]^{\frac{r_{k+1}}{\sigma}}$ 其中 $\beta_k=\frac{1}{\lambda_k}\left(n-k+1+\sum_{j=1}^{8}h_j\right)^{\frac{\sigma}{r_{k+1}}}$，使得

$$\mathcal{L}V_k \leq -(n+k+2)\sum_{i=1}^{k-1}\xi_i^4+\left[\xi_k\right]^{\frac{4\sigma-\tau-r_k}{\sigma}} d_k\left(y_{k+1}-y_{k+1}^*\right). \tag{35}$$

通过上述分析，我们可以得到，在第 $n$ 步时，存在一个状态反馈控制器 $u = y_{n+1}^{*} = -\beta_n^{\frac{r_{n+1}}{\sigma}}\left[\xi_n\right]^{\frac{r_{n+1}}{\sigma}}$，使得

$$\mathcal{L}V_n \le -\sum_{i=1}^{n}\xi_i^4, \tag{36}$$

其中

$$V_n = \sum_{k-1}^{n}W_k = \sum_{k-1}^{n}\int_{y_k^*}^{y_k}\left[\left[s\right]^{\frac{\sigma}{r_k}} - \left[y_k^*\right]^{\frac{\sigma}{r_k}}\right]^{\frac{4\sigma-\tau-r_k}{\sigma}}\mathrm{d}s. \tag{37}$$

因为 $0 < r_k < \sigma$，我们得到

$$V_n = \sum_{k=1}^{n}W_k \le 2\sum_{k=1}^{n}\left|\xi_k\right|^{\frac{4\sigma-\tau}{\sigma}}. \tag{38}$$

然后，令 $\alpha = \dfrac{4\sigma}{4\sigma-\tau} \in \left(0,1\right)$，我们有

$$\mathcal{L}V_n \le -\frac{1}{2^{\alpha}}V_n^{\alpha}. \tag{39}$$

对期望值应用伊藤公式，可得

$$\mathbb{E}\left[V\right] \le -c\mathbb{E}\left[V\right]^{\alpha}. \tag{40}$$

利用詹森不等式，进一步得到

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[V\right] \le -c\left(\mathbb{E}\left[V\right]\right)^{\alpha}. \tag{41}$$

解得

$$\mathbb{E}\left[V\left(t\right)\right] \le \left(V\left(0\right)^{1-\alpha} - c\left(1-\alpha\right)t\right)^{\frac{1}{1-\alpha}}, \tag{42}$$

右侧为 0 时，得到收敛时间期望上界

$$T \le \frac{V\left(0\right)^{1-\alpha}}{c\left(1-\alpha\right)}. \tag{43}$$

## 4. 结论

针对具有未知项的随机非完整系统，分两部分分别设计有限时间控制器，本文运用坐标变换和反步法，通过构造李雅普诺夫函数，使得闭环系统(1)能够在有限时间内趋于稳定。本文扩展了 Brockett 定理的适用范围，为随机非完整系统提供了新的分析工具。

## 参考文献


[1] Brockett, R. (1983) Asymptotic Stability and Feedback Stabilization. *Differential Geometric Control Theory*, **27**, 181-191.

[2] Astolfi, A. (1996) Discontinuous Control of Nonholonomic Systems. *Systems & Control Letters*, **27**, 37-45. https://doi.org/10.1016/0167-6911(95)00041-0

[3] Xu, W.L. and Huo, W. (2000) Variable Structure Exponential Stabilization of Chained Systems Based on the Extended Nonholonomic Integrator. *Systems & Control Letters*, **41**, 225-235. https://doi.org/10.1016/s0167-6911(00)00057-8

[4] Murray, R.M. and Sastry, S.S. (1993) Nonholonomic Motion Planning: Steering Using Sinusoids. *IEEE Transactions on*

*Automatic Control*, **38**, 700-716. https://doi.org/10.1109/9.277235

[5] Jiang, Z. (1996) Iterative Design of Time-Varying Stabilizers for Multi-Input Systems in Chained Form. *Systems & Control Letters*, **28**, 255-262. https://doi.org/10.1016/0167-6911(96)00029-1

[6] Khoo, S., Yin, J., Man, Z. and Yu, X. (2013) Finite-Time Stabilization of Stochastic Nonlinear Systems in Strict-Feedback Form. *Automatica*, **49**, 1403-1410. https://doi.org/10.1016/j.automatica.2013.01.054

[7] Zhang, X. and Xie, X. (2013) Global State Feedback Stabilisation of Nonlinear Systems with High-Order and Low-Order Nonlinearities. *International Journal of Control*, **87**, 642-652. https://doi.org/10.1080/00207179.2013.852252

[8] Qian, C. and Lin, W. (2001) Non-Lipschitz Continuous Stabilizers for Nonlinear Systems with Uncontrollable Unstable Linearization. *Systems & Control Letters*, **42**, 185-200. https://doi.org/10.1016/s0167-6911(00)00089-x

[9] Xie, X., Duan, N. and Zhao, C. (2014) A Combined Homogeneous Domination and Sign Function Approach to Output-Feedback Stabilization of Stochastic High-Order Nonlinear Systems. *IEEE Transactions on Automatic Control*, **59**, 1303-1309. https://doi.org/10.1109/tac.2013.2286912

[10] Gao, F. and Wu, Y. (2016) Global Finite-Time Stabilisation for a Class of Stochastic High-Order Time-Varying Nonlinear Systems. *International Journal of Control*, **89**, 2453-2465. https://doi.org/10.1080/00207179.2016.1161829