

分段连续型随机微分方程平衡方法的均方收敛性与稳定性

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摘要

针对一类分段连续型随机微分方程, 证明了其采用全隐式平衡方法的情况下, 该方法具有良好的均方收敛性和稳定性, 并且证明了平衡方法的强收敛阶为 $1/2$, 同时通过数值分析实验验证了此类方程数学分析的正确以及合理性, 证明了强平衡隐式方法以及弱平衡隐式方法都是均方稳定的。

关键词

分段连续型随机微分方程, 平衡方法, 均方收敛性, 均方稳定性

The Mean Square Convergence and Stability of the Balanced Implicit Method of Stochastic Differential Equations with Piecewise Continuous Arguments

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Abstract

For a class of piecewise continuous stochastic differential equations, the mean square convergence

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and stability of the fully implicit method-balanced method are studied by using the balanced method. It is proved that the strong convergence order of the balanced method is $1/2$, and the rationality and correctness of the analysis are verified by numerical examples. It is shown that both the strong balanced implicit method and the weak balanced implicit method are mean square stable.

Keywords

Piecewise Continuous Stochastic Differential Equations, Balanced Method, Mean Square Convergence, Mean Square Stability

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1. 引言

随机微分方程[1]在经济学、控制理论、金融学、医学、生物学等各个领域[2] [3]都是很重要的数学工具并且都有较为广泛的应用, 比如在金融数学里可以用来期权定价、风险度量; 在物理学的统计力学方面有所体现, 生物学中关于种群增长模型等, 并且随机微分方程是一种考虑外部因素如白噪声对整个系统产生干扰的数学模型[4] [5], 其中本文研究的分段连续型随机微分方程, 该方程具有的解是一个局部光滑且连续的函数, 并在这些区间里满足该方程。该方程常用于神经网络模拟[6]、生物系统建模[7]等, 目前对分段连续型随机微分方程已经有诸多成果[8]-[14], 关于该方程全隐式方法的结论还没有找到有关文献。本篇文章提出分段连续型随机微分方程在满足全局 Lipschitz 条件下, 全隐式方法的数值方法, 全隐式方法是对干扰项都进行全隐式处理, 具有很好的稳定性, 并且给出了方程数值解的均方收敛性和均方稳定性[15]。

考虑如下分段连续型随机微分方程

$$dx(t) = f(x(t), x([t]))dt + g(x(t), x([t]))dW(t), t \in [0, T]. \quad (1.1)$$

其中初值为 $x(0) = x_0 \in \mathbb{R}^n$, 漂移项系数 $f: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, 扩散项系数 $g: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$, $W(t) = (W_1(t), W_2(t), \dots, W_d(t))^T$ 为 $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ 上的 d 维 Brownian 运动。

2. 平衡方法的收敛性

设 $|x|$ 表示向量的 Euclid 范数, 且 $x \in \mathbb{R}^n$ 。用 $[u]$ 表示 u 的整数部分。若 A 为一个向量或者矩阵, A^T 称为转置, $|A| = \sqrt{\text{tr}(A^T A)}$ 为迹范数。

与(1.1)等价的积分形式如下:

$$x(t) = x_0 + \int_0^t f(x(s), x([s]))ds + \int_0^t g(x(s), x([s]))dW(s) \quad (2.1)$$

下列两个方程(2.1)需要满足的条件, 我们假设 f, g 满足如下假设:

(a) 全局 Lipschitz 条件: 存在一个常数 $K > 0$, 使得对 $\forall x_i, y_i \in \mathbb{R}^d (i=1, 2)$ 有

$$|v(x_1, x_2) - v(y_1, y_2)|^2 \leq K(|x_1 - x_2|^2 + |y_1 - y_2|^2) \quad (2.2)$$

其中 $v = f, g$ 。

(b) 线性增长条件: 存在正常数 L 使得对所有 $x \in \mathbb{R}^d$ 有

$$|v(x_1, x_2)|^2 \leq L(1 + |x_1|^2 + |x_2|^2) \quad (2.3)$$

其中 $v = f, g$ 。

2.1. 数值格式

用等步长 h 将区间 $[0, T]$ 做均匀划分, 其中 $h = \frac{T}{J}, t_n = nh, n = 0, 1, \dots, J$ ($J \in \mathbb{Z}^+$, $\mathbb{Z}^+$ 为正整数)。将平衡方法应用到方程(2.1)得:

$$Y_{n+1} = Y_n + f(Y_n, \tilde{Y}_n)h + g(Y_n, \tilde{Y}_n)\Delta W_n + C(Y_n, \tilde{Y}_n)(Y_n - Y_{n+1}) \quad (2.4)$$

其中初值 $Y_0 = x(0)$, Y_n 是解 $x(t_n)$ 在分点 t_n 处的数值逼近值。 $t_n = nh$, $\Delta W_n = W(t_{n+1}) - W(t_n)$, 其中 $\tilde{Y}_n = Y_{\lfloor \frac{t_n}{h} \rfloor}$ 。

此处 $d \times d$ 值矩阵函数 $C(Y_n, \tilde{Y}_n)$ 用以下的形式表达:

$$C(Y_n, \tilde{Y}_n) = C_0(Y_n, \tilde{Y}_n)h + C_1(Y_n, \tilde{Y}_n)|\Delta W_n| = C_{0n}h + C_{1n}|\Delta W_n|. \quad (2.5)$$

此处 $d \times d$ 值函数 C_0, C_1 我们视为控制函数, 并且需要满足以下的假设:

若我们认为 C_0 和 C_1 具有一致有界性。之后假设对任意实数 $\alpha_0 \in [0, \bar{\alpha}], \alpha_1 \geq 0, \bar{\alpha} \geq h$ 以及 $x \in \mathbb{R}$, 矩阵函数 $M(x) = I + \alpha_0 C_0(x) + \alpha_1 C_1(x)$ 是可逆的, 且满足

$$\left| (M(x))^{-1} \right| \leq H < \infty \quad (2.6)$$

矩阵函数中的 I 为单位阵, H 是正常数。

根据以上过程我们把方程(2.4)改写为下列形式

$$Y_{n+1} = Y_n + [I + C(Y_n, \tilde{Y}_n)]^{-1} [f(Y_n, \tilde{Y}_n)h + g(Y_n, \tilde{Y}_n)\Delta W_n] \quad (2.7)$$

我们给出如下阶梯型函数

$$\overline{\Delta W}(t) = \sum_{k=0}^{\infty} |\Delta W_k| I_{[kh, (k+1)h)}(t), \quad \bar{h}(t) = \sum_{k=0}^{\infty} (t_{k+1} - t_k) I_{[kh, (k+1)h)}(t),$$

$$\bar{Y}_1(t) = \sum_{k=0}^{\infty} Y_k I_{[kh, (k+1)h)}(t), \quad \bar{Y}_2(t) = \sum_{k=0}^{\infty} \tilde{Y}_k I_{[kh, (k+1)h)}(t),$$

由此我们给出如下连续形式的数值解

$$\begin{aligned} Y^c(t) = Y(0) &+ \int_0^t (I + C_0(\bar{Y}_1(s), \bar{Y}_2(s))\bar{h}(s) + C_1(\bar{Y}_1(s), \bar{Y}_2(s))\overline{\Delta W}(s))^{-1} f(\bar{Y}_1(s), \bar{Y}_2(s)) ds \\ &+ \int_0^t (I + C_0(\bar{Y}_1(s), \bar{Y}_2(s))\bar{h}(s) + C_1(\bar{Y}_1(s), \bar{Y}_2(s))\overline{\Delta W}(s))^{-1} g(\bar{Y}_1(s), \bar{Y}_2(s)) dW(s) \end{aligned} \quad (2.8)$$

其中 $Y(0) = x_0$ 。可以容易得到 $Y^c(t_n) = Y_n$, 即 $Y^c(t)$ 与 Y_n 的节点在整数的情况下是一致的。

在证明数值格式(2.8)的强收敛性用另一种数值格式需要连续性表达, 其形式如下:

$$\begin{aligned} Y(t) = Y(0) &+ \int_0^t (I + D(s))^{-1} f(\bar{Y}_1(s), \bar{Y}_2(s)) ds \\ &+ \int_0^t (I + D(s))^{-1} g(\bar{Y}_1(s), \bar{Y}_2(s)) dW(s) \end{aligned} \quad (2.9)$$

其中初值 $Y(0) = x_0$ 。 $D(t) = C_0(Y_n, \tilde{Y}_n)(t - t_n) + C_1(Y_n, \tilde{Y}_n)|W(t) - W(t_n)|$, $t \in [t_n, t_{n+1})$ 。

2.2. 收敛性证明

引理 1. 若全局 Lipschitz 条件(2.2)和线性增长条件(2.3)成立, 则方程(2.1)的解满足:

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} |x(t)|^2 \right) \leq Q_1$$

其中 Q_1 与 h 是无关常数。

证明. 由(2.1)式和(2.3)式, 鞅的性质以及基本不等式 $|a + b + c|^2 \leq 3|a|^2 + 3|b|^2 + 3|c|^2$ 得

$$\begin{aligned} |x(t)|^2 &\leq 3|x(0)|^2 + 3 \left| \int_0^t f(x(s), x([s])) ds \right|^2 + 3 \left| \int_0^t g(x(s), x([s])) dW(s) \right|^2 \\ &\leq 3|x(0)|^2 + 3LT \int_0^t (1 + |x(s)|^2 + |x([s])|^2) ds + 3 \left| \int_0^t g(x(s), x([s])) dW(s) \right|^2 \\ \mathbb{E} \left(\sup_{0 \leq s \leq t} |x(s)|^2 \right) &\leq 3\mathbb{E}|x_0|^2 + 3LT \int_0^t (1 + \mathbb{E}|x(s)|^2 + \mathbb{E}|x([s])|^2) ds \\ &\quad + 3\mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^s g(x(s_1), x([s_1])) dW(s_1) \right|^2 \\ &\leq 3\mathbb{E}|x_0|^2 + 3L(T+4) \int_0^t (1 + \mathbb{E}|x(s)|^2 + \mathbb{E}|x([s])|^2) ds \end{aligned}$$

于是

$$\begin{aligned} 1 + \mathbb{E} \left(\sup_{0 \leq s \leq t} |x(s)|^2 \right) &\leq 1 + 3\mathbb{E}|x_0|^2 + 3L(T+4) \int_0^t (1 + 2\mathbb{E}|x(s)|^2) ds \\ &\leq 1 + 3\mathbb{E}|x_0|^2 + 6L(T+4) \int_0^t \left(1 + \mathbb{E} \left(\sup_{0 \leq r \leq s} |x(r)|^2 \right) \right) ds \end{aligned}$$

由 Gronwall 不等式得

$$1 + \mathbb{E} \left(\sup_{0 \leq s \leq t} |x(s)|^2 \right) \leq (1 + 3\mathbb{E}|x_0|^2) e^{6LT(T+4)}$$

因此

$$\mathbb{E} \left(\sup_{0 \leq s \leq t} |x(s)|^2 \right) \leq (1 + 3\mathbb{E}|x_0|^2) e^{6LT(T+4)}$$

证毕。

引理 2. 若条件(2.3)和条件(2.6)成立, 则有

$$\max_{0 \leq n \leq J} \mathbb{E}|Y_n|^2 \leq Q_2$$

证明. 根据 $\mathbb{E}|\Delta W_n|^2 = mh$, $\mathbb{E}\Delta W_n = 0$, 由(2.7)得

$$\begin{aligned}
\mathbb{E}|Y_{n+1}|^2 &= \mathbb{E}|Y_n|^2 + \mathbb{E}\left[\left[I + C(Y_n, \tilde{Y}_n)\right]^{-1} \left[f(Y_n, \tilde{Y}_n)h + g(Y_n, \tilde{Y}_n)\Delta W_n\right]\right]^2 \\
&\quad + 2h\mathbb{E}Y_n\mathbb{E}\left[\left[I + C(Y_n, \tilde{Y}_n)\right]^{-1} f(Y_n, \tilde{Y}_n)h\right] \\
&\leq \mathbb{E}|Y_n|^2 + 2\left[\left[I + C(Y_n, \tilde{Y}_n)\right]^{-2}\right] \cdot h^2 \cdot \mathbb{E}|f(Y_n, \tilde{Y}_n)|^2 + 2\left[I + C(Y_n, \tilde{Y}_n)\right]^{-2} \mathbb{E}|g(Y_n, \tilde{Y}_n)|^2 \cdot |\Delta W_n|^2 \\
&\quad + h\mathbb{E}\left(|Y_n|^2 + \left[\left[I + C(Y_n, \tilde{Y}_n)\right]^{-2}\right] \cdot |f(Y_n, \tilde{Y}_n)|^2\right) \\
&\leq \mathbb{E}|Y_n|^2 + 2H^2h^2L\mathbb{E}\left(1 + |Y_n|^2 + |\tilde{Y}_n|^2\right) + 2H^2mhL\mathbb{E}\left(1 + |Y_n|^2 + |\tilde{Y}_n|^2\right) \\
&\quad + h\mathbb{E}|Y_n|^2 + H^2hL\mathbb{E}\left(1 + |Y_n|^2 + |\tilde{Y}_n|^2\right) \\
&\leq (2H^2h^2L + 2H^2mhL + H^2hL) + (1 + h + 4H^2h^2L + 2H^2hL + 4H^2mhL) \max_{0 \leq i \leq J} \mathbb{E}|Y_i|^2 \\
&\leq h(3H^2L + 2H^2mL) + [1 + h(1 + 6H^2L + 4H^2mL)] \max_{0 \leq i \leq J} \mathbb{E}|Y_i|^2
\end{aligned}$$

设 $P = 3H^2L + 2H^2mL$, $Q = 1 + 6H^2L + 4H^2mL$, $S_n = \max_{0 \leq i \leq J} \mathbb{E}|Y_i|^2$, 上式可得:

$$\begin{aligned}
S_{n+1} &\leq Ph + (1 + Qh)S_n \\
&= Ph + Ph(1 + Qh) + Ph(1 + Qh)^2 + \cdots + (1 + Qh)^{n+1}S_0 \\
&= \frac{P}{Q}[(1 + Qh)^{n+1} - 1] + (1 + Qh)^{n+1}S_0 \\
&\leq \frac{P}{Q}[(1 + Qh)^{J+1} - 1] + (1 + Qh)^{J+1}S_0 \\
&= \frac{P}{Q}\left\{\left[(1 + Qh)^{\frac{1}{Qh}}\right]^{Qh(J+1)}\right\} + \left[(1 + Qh)^{\frac{1}{Qh}}\right]^{Qh(J+1)}\mathbb{E}|Y_0|^2
\end{aligned}$$

当 $h \rightarrow 0$ 时, 收敛于 $\frac{P}{Q}(e^{2QT} - 1) + e^{2QT}x_0^2$ 。

对 $k = 0, 1, 2, \dots, n$, $S_k \leq \frac{P}{Q}(e^{2QT} - 1) + e^{2QT}x_0^2$ 。

当 $k = 0, -1, -2, \dots$, $Y_k = x_0^2$ 。

取 $Q_2 = \frac{P}{Q}(e^{2QT} - 1) + e^{2QT}x_0^2$, 故 $\mathbb{E}|Y_n|^2 \leq Q_2$ 。

引理 3. 若条件(2.3)和条件(2.6)成立, 则有

$$\mathbb{E} \sup_{0 \leq t \leq T} |Y(t) - \bar{Y}_1(t)|^2 \leq Q_3 h$$

其中 Q_3 与 h 是无关常数。

证明. 设 $n_t = [t/h] \leq J \in Z^+$, 由 $|a + b + c|^2 \leq 3|a|^2 + 3|b|^2 + 3|c|^2$ 得

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq t \leq T} |Y(t) - \bar{Y}_1(t)|^2 \\
& \leq 3\mathbb{E} \sup_{0 \leq t \leq T} \left| \sum_{i=0}^{n_t-1} \int_{t_i}^{t_{i+1}} (I + D(s))^{-1} f(\bar{Y}_1(s), \bar{Y}_2(s)) ds - \sum_{i=0}^{n_t-1} (I + C(Y_i, \tilde{Y}_i))^{-1} f(Y_i, \tilde{Y}_i) h \right|^2 \\
& \quad + 3\mathbb{E} \sup_{0 \leq t \leq T} \left| \sum_{i=0}^{n_t-1} \int_{t_i}^{t_{i+1}} (I + D(s))^{-1} g(\bar{Y}_1(s), \bar{Y}_2(s)) dW(s) - \sum_{i=0}^{n_t-1} (I + C(Y_i, \tilde{Y}_i))^{-1} g(Y_i, \tilde{Y}_i) \Delta W_i \right|^2 \\
& \quad + 3\mathbb{E} \sup_{0 \leq t \leq T} \left| \int_{t_{n_t}}^t (I + D(s))^{-1} f(\bar{Y}_1(s), \bar{Y}_2(s)) ds + \int_{t_{n_t}}^t (I + D(s))^{-1} g(\bar{Y}_1(s), \bar{Y}_2(s)) dW(s) \right|^2
\end{aligned} \tag{2.10}$$

注意到 $C_0(Y_i, \tilde{Y}_i), C_1(Y_i, \tilde{Y}_i)$ 一致有界, $\exists Q_0$ 正常数使得 $|C_j(Y_i, \tilde{Y}_i)| \leq Q_0$ ($j=1, 2, \dots; i=0, 1, 2, \dots$)。由基本不等式 $\sum_{i=0}^k |a_i|^2 \leq (k+1) \sum_{i=0}^k |a_i|^2$, (2.3), (2.6), 引理 2 和 $\mathbb{E} |\Delta W_n|^2 = mh$, 我们可以推出

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq t \leq T} \left| \sum_{i=0}^{n_t-1} \left[\int_{t_i}^{t_{i+1}} (I + D(s))^{-1} f(\bar{Y}_1(s), \bar{Y}_2(s)) ds - (I + C(Y_i, \tilde{Y}_i))^{-1} f(Y_i, \tilde{Y}_i) h \right] \right|^2 \\
& = \mathbb{E} \sup_{0 \leq t \leq T} \left| \sum_{i=0}^{n_t-1} \left[\int_{t_i}^{t_{i+1}} (-D(s))(I + D(s))^{-1} f(\bar{Y}_1(s), \bar{Y}_2(s)) ds \right. \right. \\
& \quad \left. \left. + \int_{t_i}^{t_{i+1}} f(\bar{Y}_1(s), \bar{Y}_2(s)) - (I + C(Y_i, \tilde{Y}_i))^{-1} f(Y_i, \tilde{Y}_i) h \right] \right|^2 \\
& \leq 2J \sup_{0 \leq t \leq T} \sum_{i=0}^{n_t-1} \left| \int_{t_i}^{t_{i+1}} (I + D(s))^{-1} f(\bar{Y}_1(s), \bar{Y}_2(s)) ds \right|^2 + 2\mathbb{E} \sup_{0 \leq t \leq T} \left| \sum_{i=0}^{n_t-1} \left[I - (I + C(Y_i, \tilde{Y}_i))^{-1} f(Y_i, \tilde{Y}_i) h \right] \right|^2 \\
& \leq 2J\mathbb{E} \sup_{0 \leq t \leq T} \sum_{i=0}^{n_t-1} \left[\int_{t_i}^{t_{i+1}} |(-D(s))(I + D(s))^{-1}|^2 ds \int_{t_i}^{t_{i+1}} |f(\bar{Y}_1(s), \bar{Y}_2(s))|^2 ds \right] \\
& \quad + 2Jh^2 \mathbb{E} \sup_{0 \leq t \leq T} \sum_{i=0}^{n_t-1} \left| \left[C(Y_i, \tilde{Y}_i)(I + C(Y_i, \tilde{Y}_i)) \right]^{-1} f(Y_i, \tilde{Y}_i) \right|^2 \\
& = 2J\mathbb{E} \sup_{0 \leq t \leq T} \sum_{i=0}^{n_t-1} \left[\int_{t_i}^{t_{i+1}} \left| C_0(Y_i, \tilde{Y}_i)(s - t_i) + C_1(Y_i, \tilde{Y}_i) |W(s) - W(t_i)| (I + D(s))^{-1} \right|^2 ds \int_{t_i}^{t_{i+1}} |f(\bar{Y}_1(s), \bar{Y}_2(s))|^2 ds \right] \\
& \quad + 2Jh^2 \mathbb{E} \sup_{0 \leq t \leq T} \sum_{i=0}^{n_t-1} \left[\int_{t_i}^{t_{i+1}} \left| \left(C_0(Y_i, \tilde{Y}_i)h + C_1(Y_i, \tilde{Y}_i) |\Delta W_i| \right) \cdot (I + C(Y_i, \tilde{Y}_i))^{-1} f(Y_i, \tilde{Y}_i) \right|^2 ds \right] \\
& \leq 4JH^2 Q_0^2 (T+m) h^2 L \sum_{i=0}^{J-1} \int_{t_i}^{t_{i+1}} \left(1 + \mathbb{E} |\bar{Y}_1(s)|^2 + \mathbb{E} |\bar{Y}_2(s)|^2 \right) ds + 4JH^2 Q_0^2 (T+m) h^3 L \sum_{i=0}^{J-1} \mathbb{E} \left(1 + |Y_n|^2 + |\tilde{Y}_n|^2 \right) \\
& \leq 8H^2 Q_0^2 L T^2 (T+m) (1+2Q_2) h
\end{aligned} \tag{2.11}$$

同理可得

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq t \leq T} \left| \sum_{i=0}^{n_t-1} \int_{t_i}^{t_{i+1}} (I + D(s))^{-1} g(\bar{Y}_1(s), \bar{Y}_2(s)) dW(s) - \sum_{i=0}^{n_t-1} (I + C(Y_i, \tilde{Y}_i))^{-1} g(Y_i, \tilde{Y}_i) \Delta W_i \right|^2 \\
& \leq 2\mathbb{E} \sup_{0 \leq t \leq T} \left| \sum_{i=0}^{n_t-1} \left[\int_{t_i}^{t_{i+1}} (-D(s))(I + D(s))^{-1} g(\bar{Y}_1(s), \bar{Y}_2(s)) dW(s) \right] \right|^2 \\
& \quad + 2\mathbb{E} \sup_{0 \leq t \leq T} \left| \sum_{i=0}^{n_t-1} \left[\int_{t_i}^{t_{i+1}} g(\bar{Y}_1(s), \bar{Y}_2(s)) dW(s) - (I + C(Y_i, \tilde{Y}_i))^{-1} g(Y_i, \tilde{Y}_i) \Delta W_i \right] \right|^2
\end{aligned} \tag{2.12}$$

注意到对任意的 $J \in \mathbb{Z}^+$, 离散化

$$\zeta_n = \left\{ \sum_{i=0}^{n-1} \left[\int_{t_i}^{t_{i+1}} (-D(s))(I+D(s))^{-1} g(\bar{Y}_1(s), \bar{Y}_2(s)) dW(s) \right] \right\}$$

是 $\{\mathcal{F}_{t_n} : 0 \leq n \leq J\}$ 可测的。由 Doob 鞅不等式, $W(t)$ 的性质, $\mathbb{E}|\Delta W_n|^2 = mh$, $\mathbb{E}|\Delta W_n|^4 = 3m^2h^2$, 条件(2.3)和条件(2.6), 可得

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq t \leq T} \left| \sum_{i=0}^{n-1} \left[\int_{t_i}^{t_{i+1}} (-D(s))(I+D(s))^{-1} g(\bar{Y}_1(s), \bar{Y}_2(s)) dW(s) \right] \right|^2 \\ & \leq 4\mathbb{E} \left| \sum_{i=0}^{J-1} \left[\int_{t_i}^{t_{i+1}} (-D(s))(I+D(s))^{-1} g(\bar{Y}_1(s), \bar{Y}_2(s)) dW(s) \right] \right|^2 \\ & = 4 \sum_{i=0}^{J-1} \left[\int_{t_i}^{t_{i+1}} \mathbb{E} \left| (-D(s))(I+D(s))^{-1} g(\bar{Y}_1(s), \bar{Y}_2(s)) dW(s) \right|^2 ds \right] \\ & \leq 8H^2Q_0^2LT(T+m)(1+2Q_2)h \end{aligned} \quad (2.13)$$

利用 $\mathbb{E} \left[\int_{t_i}^{t_{i+1}} g(s) dW(s) \int_{t_j}^{t_{j+1}} g(s) dW(s) \right] = 0$, $\mathbb{E} [g(Y_i) \Delta W_i g(Y_j) \Delta W_j] = 0 (i \neq j)$, 可导出

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq t \leq T} \left| \sum_{i=0}^{n-1} \left[\int_{t_i}^{t_{i+1}} g(\bar{Y}_1(s), \bar{Y}_2(s)) dW(s) - (I+C(Y_i, \tilde{Y}_i))^{-1} g(Y_i, \tilde{Y}_i) \Delta W_i \right] \right|^2 \\ & \leq \sum_{i=0}^{J-1} \mathbb{E} \left| \left(I - (I+C(Y_i, \tilde{Y}_i))^{-1} \right) g(Y_i, \tilde{Y}_i) \Delta W_i \right|^2 \\ & = \sum_{i=0}^{J-1} \mathbb{E} \left| C(Y_i, \tilde{Y}_i) (I+C(Y_i, \tilde{Y}_i))^{-1} g(Y_i, \tilde{Y}_i) \Delta W_i \right|^2 \\ & \leq 2H^2Q_0^2LT(mT+3m^2)(1+2Q_2)h \end{aligned} \quad (2.14)$$

将(2.13)和(2.14)代入(2.12)得

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq t \leq T} \left| \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (I+D(s))^{-1} g(\bar{Y}_1(s), \bar{Y}_2(s)) dW(s) - \sum_{i=0}^{n-1} (I+C(Y_i, \tilde{Y}_i))^{-1} g(Y_i, \tilde{Y}_i) \Delta W_i \right|^2 \\ & \leq H^2Q_0^2LT \left[8(T+m) + 2(mT+3m^2) \right] (1+2Q_2)h \end{aligned} \quad (2.15)$$

不难得到

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_{t_n}^t (I+D(s))^{-1} f(\bar{Y}_1(s), \bar{Y}_2(s)) ds + \int_{t_n}^t (I+D(s))^{-1} g(\bar{Y}_1(s), \bar{Y}_2(s)) dW(s) \right|^2 \\ & = \max_{0 \leq i \leq J-1} \mathbb{E} \sup_{t_i \leq t \leq t_{i+1}} \left| \int_{t_i}^t (I+D(s))^{-1} f(\bar{Y}_1(s), \bar{Y}_2(s)) ds + \int_{t_n}^t (I+D(s))^{-1} g(\bar{Y}_1(s), \bar{Y}_2(s)) dW(s) \right|^2 \\ & \leq 2 \max_{0 \leq i \leq J-1} H^2h \int_{t_i}^{t_{i+1}} \mathbb{E} |f(\bar{Y}_1(s), \bar{Y}_2(s))|^2 ds + 8 \max_{0 \leq i \leq J-1} H^2 \mathbb{E} |g(\bar{Y}_1(s), \bar{Y}_2(s))|^2 ds \\ & \leq 2H^2L(1+2Q_2)(T+4)h \end{aligned} \quad (2.16)$$

将(2.11), (2.15), (2.16)带代入(2.10)可得

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq t \leq T} |Y(t) - \bar{Y}_1(t)|^2 \\ & \leq 6H^2L(1+2Q_2) \left\{ (T+4) + 4T^2Q_0^2(T+m) + 2T^2Q_0^2 \left[4(T+m) + (mT+3m^2) \right] \right\} \\ & = Q_3h \end{aligned}$$

引理 4. 若线性增长条件(2.3)和条件(2.6)成立, 则有

$$\mathbb{E} \sup_{0 \leq t \leq T} |Y([t]) - \bar{Y}_2(t)|^2 \leq Q_4 h$$

其中 Q_4 与 h 是无关常数。

证明. 对任给 $t \in [0, T]$, 存在某个 k , 使得 $t \in [kh, (k+1)h)$, 有

$$Y([t]) - \bar{Y}_2(t) = Y([t]) - \tilde{Y}_k = Y([t]) - Y\left(\left[\frac{[t]}{h}\right]h\right) = Y([t]) - Y^C\left(\left(\left[\frac{[t]}{h}\right], h\right)\right)$$

其中 $t-1 \leq [t] \leq t, t-1-h \leq \left(\left[\frac{[t]}{h}\right]-1\right)h \leq \left[\frac{[t]}{h}\right] \cdot h \leq t$ 。

这里只有一种情况, 就是当 $[t] \geq \left[\frac{[t]}{h}\right] \cdot h \geq 0$,

$$[t] - \left[\frac{[t]}{h}\right] \cdot h \leq t - t + 1 + h = 1 + h \leq (\gamma + 1)h$$

这里的 γ 是大于等于 1 的常数, 且与 Q_4 与 h 无关。

记 $n_t = [t/h] \leq J$

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq t \leq T} |Y([t]) - \bar{Y}_2(t)|^2 \\ &= \mathbb{E} \sup_{0 \leq t \leq T} |Y([t]) - Y(n_t \cdot h) + Y(n_t \cdot h) - Y^C(n_t \cdot h)|^2 \\ &\leq 2\mathbb{E} \sup_{0 \leq t \leq T} |Y([t]) - Y(n_t \cdot h)|^2 + 2\mathbb{E} \sup_{0 \leq t \leq T} |Y(n_t \cdot h) - Y^C(n_t \cdot h)|^2 \\ &\leq 2\mathbb{E} \sup_{0 \leq t \leq T} \left| \int_{n_t \cdot h}^{[t]} (I + D(s))^{-1} f(\bar{Y}_1(s), \bar{Y}_2(s)) ds + \int_{n_t \cdot h}^{[t]} (I + D(s))^{-1} g(\bar{Y}_1(s), \bar{Y}_2(s)) dW(s) \right|^2 \\ &\quad + 4\mathbb{E} \sup_{0 \leq t \leq T} \left| \sum_{i=0}^{n_t-1} \int_{t_i}^{t_{i+1}} (I + D(s))^{-1} f(\bar{Y}_1(s), \bar{Y}_2(s)) ds - \sum_{i=0}^{n_t-1} (I + C(Y_i, \tilde{Y}_i))^{-1} f(Y_i, \tilde{Y}_i) h \right|^2 \\ &\quad + 4\mathbb{E} \sup_{0 \leq t \leq T} \left| \sum_{i=0}^{n_t-1} \int_{t_i}^{t_{i+1}} (I + D(s))^{-1} g(\bar{Y}_1(s), \bar{Y}_2(s)) dW(s) - \sum_{i=0}^{n_t-1} (I + C(Y_i, \tilde{Y}_i))^{-1} g(Y_i, \tilde{Y}_i) \Delta W_i \right|^2 \end{aligned} \quad (2.17)$$

可以推出

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_{n_t \cdot h}^{[t]} (I + D(s))^{-1} f(\bar{Y}_1(s), \bar{Y}_2(s)) ds + \int_{n_t \cdot h}^{[t]} (I + D(s))^{-1} g(\bar{Y}_1(s), \bar{Y}_2(s)) dW(s) \right|^2 \\ &= \max_{0 \leq i \leq J-1} \mathbb{E} \sup_{t-1 \leq [t] \leq t} \left| \int_{n_t \cdot h}^{[t]} (I + D(s))^{-1} f(\bar{Y}_1(s), \bar{Y}_2(s)) ds + \int_{n_t \cdot h}^{[t]} (I + D(s))^{-1} g(\bar{Y}_1(s), \bar{Y}_2(s)) dW(s) \right|^2 \\ &\leq 2H^2(\gamma+1)h \int_{n_t \cdot h}^t \mathbb{E} |f(\bar{Y}_1(s), \bar{Y}_2(s))|^2 ds + 8H^2(\gamma+1)h \int_{n_t \cdot h}^t \mathbb{E} |g(\bar{Y}_1(s), \bar{Y}_2(s))|^2 ds \\ &\leq 2H^2(\gamma+1)hL \int_{n_t \cdot h}^t \left(1 + \mathbb{E} |\bar{Y}_1(s)|^2 + \mathbb{E} |\bar{Y}_2(s)|^2\right) ds + 8H^2L \int_{n_t \cdot h}^t \left(1 + \mathbb{E} |\bar{Y}_1(s)|^2 + \mathbb{E} |\bar{Y}_2(s)|^2\right) ds \\ &\leq 2H^2(\gamma+1)hL(1+2Q_2)[(\gamma+1)h+4]h \end{aligned} \quad (2.18)$$

由(2.11)得

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq t \leq T} \left| \sum_{i=0}^{n_i-1} \left[\int_{t_i}^{t_{i+1}} (I + D(s))^{-1} f(\bar{Y}_1(s), \bar{Y}_2(s)) ds - (I + C(Y_i, \tilde{Y}_i))^{-1} f(Y_i, \tilde{Y}_i) h \right] \right|^2 \\ & \leq 8H^2 Q_0^2 L T^2 (T+m)(1+2Q_2)h \end{aligned} \quad (2.19)$$

由(2.15)得

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq t \leq T} \left| \sum_{i=0}^{n_i-1} \int_{t_i}^{t_{i+1}} (I + D(s))^{-1} g(\bar{Y}_1(s), \bar{Y}_2(s)) dW(s) - \sum_{i=0}^{n_i-1} (I + C(Y_i, \tilde{Y}_i))^{-1} g(Y_i, \tilde{Y}_i) \Delta W_i \right|^2 \\ & \leq H^2 Q_0^2 L T \left[8(T+m) + 2(mT + 3m^2) \right] (1+2Q_2)h \end{aligned} \quad (2.20)$$

将(2.18), (2.19), (2.20)代入(2.17)可得

$$\mathbb{E} \sup_{0 \leq t \leq T} |Y([t]) - \bar{Y}_2(t)|^2 \leq Q_4 h$$

引理 5. 若条件(2.2)~(2.6)成立, 则数值解(2.9)满足

$$\mathbb{E} \sup_{0 \leq t \leq T} |Y(t) - x(t)|^2 \leq Q_5 h \quad (2.21)$$

其中 Q_5 与 h 是无关常数。

证明. 对于任意的 $0 \leq t_1 \leq T$, 有

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq t \leq t_1} |Y(t) - x(t)|^2 \\ & \leq 4\mathbb{E} \sup_{0 \leq t \leq t_1} \left| \int_0^t (I + D(s))^{-1} [f(\bar{Y}_1(s), \bar{Y}_2(s)) - f(x(s), x([s]))] ds \right|^2 \\ & \quad + 4\mathbb{E} \sup_{0 \leq t \leq t_1} \left| \int_0^t (I + D(s))^{-1} [g(\bar{Y}_1(s), \bar{Y}_2(s)) - g(x(s), x([s]))] dW(s) \right|^2 \\ & \quad + 4\mathbb{E} \sup_{0 \leq t \leq t_1} \left| \int_0^t [(I + D(s))^{-1} - I] f(x(s), x([s])) ds \right|^2 \\ & \quad + 4\mathbb{E} \sup_{0 \leq t \leq t_1} \left| \int_0^t [(I + D(s))^{-1} - I] g(x(s), x([s])) dW(s) \right|^2 \end{aligned} \quad (2.22)$$

第一项:

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq t \leq t_1} \left| \int_0^t (I + D(s))^{-1} [f(\bar{Y}_1(s), \bar{Y}_2(s)) - f(x(s), x([s]))] ds \right|^2 \\ & \leq H^2 T \mathbb{E} \int_0^{t_1} |f(\bar{Y}_1(s), \bar{Y}_2(s)) - f(x(s), x([s]))|^2 ds \\ & \leq H^2 T K \mathbb{E} \int_0^{t_1} (|\bar{Y}_1(s) - x(s)|^2 + |\bar{Y}_2(s) - x([s])|^2) ds \\ & = H^2 T K \mathbb{E} \int_0^{t_1} |\bar{Y}_1(s) - Y(s) + Y(s) - x(s)|^2 + |\bar{Y}_2(s) - Y([s]) + Y([s]) - x([s])|^2 ds \\ & \leq 2H^2 T K \int_0^{t_1} \mathbb{E} |\bar{Y}_1(s) - Y(s)|^2 + \mathbb{E} |Y(s) - x(s)|^2 + \mathbb{E} |\bar{Y}_2(s) - Y([s])|^2 + \mathbb{E} |Y([s]) - x([s])|^2 ds \\ & \leq 2H^2 T^2 K (Q_3 + Q_4)h + 4H^2 T K \mathbb{E} \int_0^{t_1} \sup_{0 \leq \delta \leq s} |Y(\delta) - x(\delta)|^2 ds \end{aligned} \quad (2.23)$$

第二项: 类似(2.22)得

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq t \leq t_1} \left| \int_0^t (I + D(s))^{-1} \left[g(\bar{Y}_1(s), \bar{Y}_2(s)) - g(x(s), x([s])) \right] dW(s) \right|^2 \\ & \leq 8H^2 TK(Q_3 + Q_4)h + 16H^2 TK \mathbb{E} \int_0^{t_1} \sup_{0 \leq \delta \leq s} |Y(\delta) - x(\delta)|^2 ds \end{aligned} \quad (2.24)$$

第三项:

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq t \leq t_1} \left| \int_0^t \left[(I + D(s))^{-1} - I \right] f(x(s), x([s])) ds \right|^2 \\ & \leq 2H^2 T Q_0^2 (T + m) h \mathbb{E} \int_0^{t_1} \left| f(x(s), x([s])) \right|^2 ds \\ & \leq 2H^2 T Q_0^2 (T + m) h \mathbb{E} \int_0^{t_1} \left(1 + |x(s)|^2 + |x([s])|^2 \right) ds \\ & \leq 2H^2 T^2 L Q_0^2 (T + m) (1 + 2Q_1) h \end{aligned} \quad (2.25)$$

第四项: 类似于(2.24)得

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq t \leq t_1} \left| \int_0^t \left[(I + D(s))^{-1} - I \right] g(x(s), x([s])) dW(s) \right|^2 \\ & \leq 8H^2 T^2 L Q_0^2 (T + m) (1 + 2Q_1) h \end{aligned} \quad (2.26)$$

把(2.23), (2.24), (2.25)和(2.26)代入(2.22), 由 Gronwall 不等式得到

$$\mathbb{E} \sup_{0 \leq t \leq t_1} |Y(t) - x(t)|^2 \leq Q_5 h$$

下面给出本章节主要定理:

定理 1. 若条件(2.2)~(2.6)成立, 则数值解(2.9)满足

$$\mathbb{E} \sup_{0 \leq t \leq T} |Y^C(t) - x(t)|^2 \leq Q_6 h \quad (2.27)$$

其中 Q_6 与 h 是无关常数.

证明. 由(2.3), (2.6), (2.9)以及引理 2 和引理 3 得

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq t \leq T} |Y(t) - Y^C(t)|^2 \\ & = \mathbb{E} \sup_{0 \leq t \leq T} |Y(t) - \bar{Y}_1(t) + \bar{Y}_1(t) - Y^C(t)|^2 \\ & \leq 2\mathbb{E} \sup_{0 \leq t \leq T} |Y(t) - \bar{Y}_1(t)|^2 \\ & \quad + 2\mathbb{E} \sup_{0 \leq t \leq T} \left| \left[I + C_0(Y_{n_t}, \tilde{Y}_{n_t})h + C_1(Y_{n_t}, \tilde{Y}_{n_t})[\Delta W_{n_t}] \right]^{-1} f(Y_{n_t}, \tilde{Y}_{n_t})(t - t_{n_t}) \right. \\ & \quad \left. + \left[I + C_0(Y_{n_t}, \tilde{Y}_{n_t})h + C_1(Y_{n_t}, \tilde{Y}_{n_t})[\Delta W_{n_t}] \right]^{-1} g(Y_{n_t}, \tilde{Y}_{n_t})(W(t) - W(t_{n_t})) \right|^2 \\ & \leq 2Q_3 h + 4H^2 \mathbb{E} \sup_{0 \leq t \leq T} \left| (t - t_{n_t}) f(Y_{n_t}, \tilde{Y}_{n_t}) \right|^2 + 4H^2 \mathbb{E} \sup_{0 \leq t \leq T} \left| (W(t) - W(t_{n_t})) g(Y_{n_t}, \tilde{Y}_{n_t}) \right|^2 \\ & \leq 2Q_3 h + 4H^2 h^2 \mathbb{E} \sup_{0 \leq t \leq T} \left| f(Y_{n_t}, \tilde{Y}_{n_t}) \right|^2 + 4H^2 m h \mathbb{E} \sup_{0 \leq t \leq T} \left| g(Y_{n_t}, \tilde{Y}_{n_t}) \right|^2 \\ & \leq 2Q_3 h + 4H^2 h^2 L (1 + 2Q_2) + 4H^2 m h L (1 + 2Q_2) \\ & \leq [2Q_3 + 4H^2 T L (1 + 2Q_2) + 4H^2 m L (1 + 2Q_2)] h \end{aligned}$$

所以

$$\begin{aligned}
 & \mathbb{E} \sup_{0 \leq t \leq T} |Y^C(t) - x(t)|^2 \\
 &= \mathbb{E} \sup_{0 \leq t \leq T} |Y^C(t) - Y(t) + Y(t) - x(t)|^2 \\
 &\leq 2\mathbb{E} \sup_{0 \leq t \leq T} |Y^C(t) - Y(t)|^2 + 2\mathbb{E} \sup_{0 \leq t \leq T} |Y(t) - x(t)|^2 \\
 &\leq 2[2Q_3 + 4H^2TL(1 + 2Q_2) + 4H^2mL(1 + 2Q_2)]h + 2Q_3h \\
 &\leq Q_6h
 \end{aligned}$$

证毕。

3. 平衡方法的均方稳定性

对于方程(1.1)，我们考虑其线性标量的形式：

$$\begin{cases} dx(t) = [a_1x(t) + a_2x([t])]dt + [b_1x(t) + b_2x([t])]dW(t) \\ x(0) = x_0 \end{cases} \quad (3.1)$$

引理 6. [16]若方程(3.1)满足如下条件：

$$2a_1 + b_1^2 + b_2^2 + 2|a_2 + b_1b_2| < 0 \quad (3.2)$$

成立，则方程(3.1)的解析解为均方稳定的。

3.1. 强平衡隐式方法的均方稳定性

定义 1. 在给定的序列 $\{a_1, a_2, b_1, b_2\}$ 以及步长 h ，对于任意 Y_0 ，将平衡隐式方法应用于方程(3.1)得到的数值解 Y_n 满足

$$\lim_{n \rightarrow \infty} \mathbb{E}(Y_n)^2 = 0,$$

则称平衡隐式方法为均方稳定。

定理 2. 若条件(2.6)和(3.2)成立，则数值方法是均方稳定的。

证明. (2.4)应用于(3.1)得

$$Y_{n+1} = Y_n + (a_1Y_n + a_2\tilde{Y}_n)h + (b_1Y_n + b_2\tilde{Y}_n)\Delta W_n + C_n(Y_n - Y_{n+1}) \quad (3.3)$$

其中 $C_n = C_{0n}h + C_{1n}|\Delta W_n|$ 。我们只考虑 $C_{0n}, C_{1n} \in \mathbb{R} (n = 0, 1, \dots)$ ，将(3.3)式改写为以下形式：

$$Y_{n+1} = \left[1 + (a_1h + b_1\Delta W_n)(1 + C_n)^{-1}\right]Y_n + \left[(a_2h + b_2\Delta W_n)(1 + C_n)^{-1}\right]\tilde{Y}_n \quad (3.4)$$

对(3.4)式两边平方并取期望得：

$$\begin{aligned}
 \mathbb{E}Y_{n+1}^2 &= \mathbb{E}Y_n^2 \mathbb{E}\left[1 + (a_1h + b_1\Delta W_n)(1 + C_n)^{-1}\right]^2 + \mathbb{E}\tilde{Y}_n^2 \mathbb{E}\left[(a_2h + b_2\Delta W_n)(1 + C_n)^{-1}\right]^2 \\
 &\quad + 2\mathbb{E}Y_n\mathbb{E}\tilde{Y}_n \mathbb{E}\left\{\left[1 + (a_1h + b_1\Delta W_n)(1 + C_n)^{-1}\right] \cdot \left[(a_2h + b_2\Delta W_n)(1 + C_n)^{-1}\right]\right\} \\
 &= \mathbb{E}Y_n^2 \mathbb{E}\alpha_1^2 + \mathbb{E}\tilde{Y}_n^2 \mathbb{E}\alpha_2^2 + 2\mathbb{E}Y_n\mathbb{E}\tilde{Y}_n \mathbb{E}(\alpha_1\alpha_2) \\
 &\leq \left(\mathbb{E}\alpha_1^2 + \mathbb{E}\alpha_2^2 + 2|\mathbb{E}(\alpha_1\alpha_2)|\right) \max_{0 \leq i \leq n} \mathbb{E}Y_i^2
 \end{aligned} \quad (3.5)$$

其中 $\alpha_1 = 1 + (a_1h + b_1\Delta W_n)(1 + C_n)^{-1}$ ， $\alpha_2 = (a_2h + b_2\Delta W_n)(1 + C_n)^{-1}$ 于是

$$\begin{aligned}
& \mathbb{E}\alpha_1^2 + \mathbb{E}\alpha_2^2 + 2|\mathbb{E}(\alpha_1\alpha_2)| \\
&= 1 + 2a_1h\mathbb{E}(1+C_n)^{-1} + 2b_2\mathbb{E}\left[\Delta W_n(1+C_n)^{-1}\right] + (a_1^2h^2 + a_2^2h^2)\mathbb{E}(1+C_n)^{-2} \\
&\quad + (b_1^2 + b_2^2)\mathbb{E}\left[\Delta W_n^2(1+C_n)^{-2}\right] + (2a_1b_1h + 2a_2b_2h)\mathbb{E}\left[\Delta W_n(1+C_n)^{-2}\right] \\
&\quad + 2\left[a_2h\mathbb{E}(1+C_n)^{-1} + b_2\mathbb{E}\left[\Delta W_n(1+C_n)^{-1}\right] + a_1a_2h^2\mathbb{E}(1+C_n)^{-2}\right. \\
&\quad \left.+ (a_1b_2h + a_2b_1h)\mathbb{E}\left[\Delta W_n(1+C_n)^{-2}\right] + b_1b_2 + \mathbb{E}\left[\Delta W_n^2(1+C_n)^{-2}\right]\right]
\end{aligned} \tag{3.6}$$

设 ξ 为服从标准正态分布随机变量, 则

$$\begin{aligned}
& \mathbb{E}\left[\Delta W_n(1+C_n)^{-1}\right] \\
&= \mathbb{E}\left(\sqrt{h}\xi(1+C_{0n}h+C_{1n}\sqrt{h}|\xi|)^{-1}\right) \\
&= \frac{\sqrt{h}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} x(1+C_{0n}h+C_{1n}\sqrt{h}|x|)^{-1} dx \\
&= 0
\end{aligned} \tag{3.7}$$

类似可得

$$E\left[\Delta W_n(1+C_n)^{-2}\right] = 0 \tag{3.8}$$

而

$$\begin{aligned}
& (b_1^2 + b_2^2)\mathbb{E}\left[\Delta W_n^2(1+C_n)^{-2}\right] \\
&= (b_1^2 + b_2^2)\mathbb{E}\left[\Delta W_n^2\left(1 - \frac{2C_n}{1+C_n} + \frac{C_n^2}{(1+C_n)^2}\right)\right] \\
&= (b_1^2 + b_2^2)h - 2(b_1^2 + b_2^2)\mathbb{E}\left[\Delta W_n^2 \frac{C_n}{1+C_n}\right] + (b_1^2 + b_2^2)\mathbb{E}\left[\Delta W_n^2 \frac{C_n^2}{(1+C_n)^2}\right].
\end{aligned} \tag{3.9}$$

其中

$$\begin{aligned}
& \left| -2(b_1^2 + b_2^2)\mathbb{E}\left[\Delta W_n^2 \frac{C_n}{1+C_n}\right] \right| \\
&\leq 2(b_1^2 + b_2^2)\mathbb{E}\left|\Delta W_n^2 \frac{C_n}{1+C_n}\right| \\
&\leq 2(b_1^2 + b_2^2)H\mathbb{E}\left[\Delta W_n^2 |C_{0n}h + C_{1n}|\Delta W_n|\right] \\
&= O\left(h^{\frac{3}{2}}\right),
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
& (b_1^2 + b_2^2)\mathbb{E}\left[\Delta W_n^2 \frac{C_n^2}{(1+C_n)^2}\right] \\
&\leq (b_1^2 + b_2^2)H^2\mathbb{E}\left[\Delta W_n^2 |C_{0n}h + C_{1n}|\Delta W_n|^2\right] \\
&= O(h^2)
\end{aligned} \tag{3.11}$$

将(3.10), (3.11)代入(3.9)得

$$(b_1^2 + b_2^2) \mathbb{E} \left[\Delta W_n^2 (1 + C_n)^{-2} \right] = (b_1^2 + b_2^2) h + O(h) \quad (3.12)$$

同理可以得到

$$\mathbb{E} \left[(1 + C_n)^{-2} \right] = \mathbb{E} \left[1 - \frac{2C_n}{1 + C_n} + \frac{C_n^2}{(1 + C_n)^2} \right] = 1 + O\left(h^{\frac{1}{2}}\right) \quad (3.13)$$

$$\mathbb{E} \left[(1 + C_n)^{-1} \right] = \mathbb{E} \left[1 - \frac{C_n}{1 + C_n} \right] = 1 + O\left(h^{\frac{1}{2}}\right) \quad (3.14)$$

现在将(3.7), (3.8), (3.12), (3.13), (3.14)代入(3.6)得

$$\begin{aligned} & \mathbb{E} \alpha_1^2 + \mathbb{E} \alpha_2^2 + 2 \left| \mathbb{E}(\alpha_1 \alpha_2) \right| \\ &= 1 + 2a_1 h + (a_1^2 h^2 + a_2^2 h^2) + (b_1^2 + b_2^2) h + 2 \left| a_2 h + a_1 a_2 h^2 + b_1 b_2 h \right| + o(h) \\ &\leq 1 + \left[2a_1 + b_1^2 + b_2^2 + 2 \left| a_2 + b_1 b_2 \right| \right] h + o(h) \end{aligned} \quad (3.15)$$

当 $h \rightarrow 0$ 时, 可得结论强平衡方法均方稳定。

3.2. 弱平衡隐式方法的均方稳定性

考虑弱平衡方法

$$Y_{n+1} = Y_n + (a_1 Y_n + a_2 \tilde{Y}_n) h + (b_1 Y_n + b_2 \tilde{Y}_n) \overline{\Delta W_n} + \overline{C_n} (Y_n - Y_{n+1}) \quad (3.16)$$

其中 $Y_0 = x(0)$, $\overline{C_n} = C_{0n} h + C_{1n} |\overline{\Delta W_n}|$. $P(\overline{\Delta W_n} = \sqrt{h}) = P(\overline{\Delta W_n} - \sqrt{h}) = \frac{1}{2}$, 不难证明 $\mathbb{E}(\overline{\Delta W_n}) = 0$, $\mathbb{E}(\overline{\Delta W_n}^2) = h$.

定理 3. 条件(2.6)和(3.2)成立, 则弱平衡隐式方法(3.16)是均方稳定的。

证明. 类似于定理 2, 可得

$$\mathbb{E} Y_{n+1}^2 \leq \left(\mathbb{E} \overline{\alpha_1}^2 + \mathbb{E} \overline{\alpha_2}^2 + 2 \left| \mathbb{E}(\overline{\alpha_1 \alpha_2}) \right| \right) \max_{0 \leq i \leq n} \mathbb{E} Y_i^2$$

其中 $\overline{\alpha_1} = 1 + (a_1 h + b_1 \overline{\Delta W_n})(1 + \overline{C_n})^{-1}$, $\overline{\alpha_2} = (a_2 h + b_2 \overline{\Delta W_n})(1 + \overline{C_n})^{-1}$.

$$\begin{aligned} & \mathbb{E} \overline{\alpha_1}^2 + \mathbb{E} \overline{\alpha_2}^2 + 2 \left| \mathbb{E}(\overline{\alpha_1 \alpha_2}) \right| \\ &= 1 + 2a_1 h \mathbb{E} (1 + \overline{C_n})^{-1} + 2b_2 \mathbb{E} \left[\overline{\Delta W_n} (1 + \overline{C_n})^{-1} \right] + (a_1^2 h^2 + a_2^2 h^2) \mathbb{E} (1 + \overline{C_n})^{-2} \\ & \quad + (b_1^2 + b_2^2) \mathbb{E} \left[\overline{\Delta W_n}^2 (1 + \overline{C_n})^{-2} \right] + (2a_1 b_1 h + 2a_2 b_2 h) \mathbb{E} \left[\overline{\Delta W_n} (1 + \overline{C_n})^{-2} \right] \\ & \quad + 2 \left| a_2 h \mathbb{E} (1 + \overline{C_n})^{-1} + b_2 \mathbb{E} \left[\overline{\Delta W_n} (1 + \overline{C_n})^{-1} \right] + a_1 a_2 h^2 \mathbb{E} (1 + \overline{C_n})^{-2} \right. \\ & \quad \left. + (a_1 b_2 h + a_2 b_1 h) \mathbb{E} \left[\overline{\Delta W_n} (1 + \overline{C_n})^{-2} \right] + b_1 b_2 + \mathbb{E} \left[\overline{\Delta W_n}^2 (1 + \overline{C_n})^{-2} \right] \right| \end{aligned} \quad (3.17)$$

因 $P(\overline{\Delta W_n} = \sqrt{h}) = P(\overline{\Delta W_n} - \sqrt{h}) = \frac{1}{2}$, 可以推出

$$\begin{aligned}
& \mathbb{E} \left[\overline{\Delta W_n} (1 + \overline{C_n})^{-1} \right] \\
&= \mathbb{E} \left(\overline{\Delta W_n} (1 + C_{0n}h + C_{1n} |\overline{\Delta W_n}|)^{-1} \right) \\
&= \frac{1}{2} \left[\sqrt{h} (1 + C_{0n}h + C_{1n}\sqrt{h}) \right]^{-1} + \frac{1}{2} \left[\sqrt{h} (1 + C_{0n}h + C_{1n}\sqrt{h}) \right]^{-1} \\
&= 0
\end{aligned} \tag{3.18}$$

类似的有

$$\mathbb{E} \left[\overline{\Delta W_n} (1 + \overline{C_n})^{-2} \right] = 0 \tag{3.19}$$

注意到

$$\begin{aligned}
& \mathbb{E} \left[\overline{\Delta W_n}^2 (1 + \overline{C_n})^{-2} \right] \\
&= \mathbb{E} \left(\overline{\Delta W_n}^2 (1 + C_{0n}h + C_{1n} |\overline{\Delta W_n}|)^{-2} \right) \\
&= h (1 + C_{0n}h + C_{1n}\sqrt{h})^{-2}
\end{aligned}$$

那么一定存在 $\hat{h}^* = \min \{1/(|C_0|^2 + |C_1|^2), 1\}$ 。对任意的 $h \in (0, \hat{h}^*)$ ，使得 $|C_{0n}h + C_{1n}\sqrt{h}| < 1$ 。由泰勒展开得

$$(1 + C_{0n}h + C_{1n}\sqrt{h})^{-2} = 1 - 2C_{1n}\sqrt{h} + o(\sqrt{h}).$$

因此可以得到

$$\mathbb{E} \left[\overline{\Delta W_n}^2 (1 + \overline{C_n})^{-2} \right] = h (1 - 2C_{1n}\sqrt{h} + o(\sqrt{h})) = h + o(h). \tag{3.20}$$

类似可以得到

$$\mathbb{E} (1 + \overline{C_n})^{-2} = \mathbb{E} (1 + \overline{C_n})^{-1} = 1 + O(\sqrt{h}). \tag{3.21}$$

将(3.18), (3.19), (3.20)和(3.21)代入(3.17)得

$$\begin{aligned}
& \mathbb{E} \overline{\alpha_1}^2 + \mathbb{E} \overline{\alpha_2}^2 + 2 \left| \mathbb{E} (\overline{\alpha_1 \alpha_2}) \right| \\
&= 1 + 2a_1h + (a_1^2h^2 + a_2^2h^2) + (b_1^2 + b_2^2)h + 2|a_2h + a_1a_2h^2 + b_1b_2h| + o(h)
\end{aligned}$$

同理证得弱平衡方法均方稳定。

4. 数值算例

首先，验证平衡方法的收敛性，我们考虑其线性标量的形式：

$$dx(t) = [a_1x(t) + a_2x([t])]dt + [b_1x(t) + b_2x([t])]dW(t) \tag{4.1}$$

其中初值 $x(0) = 1$ 。

选取如下三组系数:

算例 1. $a_1 = 1; a_2 = 0.1; b_1 = 0.3; b_2 = 0.2$ 。

算例 2. $a_1 = -4; a_2 = 0.3; b_1 = 0.3; b_2 = 0.2$ 。

算例 3. $a_1 = -2; a_2 = 0.1; b_1 = 0.4; b_2 = 0.1$ 。

选择算例 1、2、3 作为方程(4.1)的系数, 同时为了方便, 令 $C_{0j} = C_{1j} = 1$ ($j = 0, 1, \dots$), 并选择步长 $h = 8\Delta t, 16\Delta t, 32\Delta t, 64\Delta t$ 。 $\Delta t = h$, 均方误差 $\varepsilon^r = \varepsilon_{2^{3+r}\Delta t}^r$, $r = 0, 1, 2, 3$ 均在终点时刻 $T = 1$ 处的由以下方式估计, 选取 2000 个样本来模拟。

$$\varepsilon^r \cong \frac{1}{2000} \sum_{j=1}^{2000} \left| x_{r,T}(\omega_j) - Y_{r,i}(\omega_j) \right|^2.$$

在图 1 以对数为底的坐标系中画出 $\sqrt{\varepsilon^r}$, 表明平衡方法是均方收敛的。

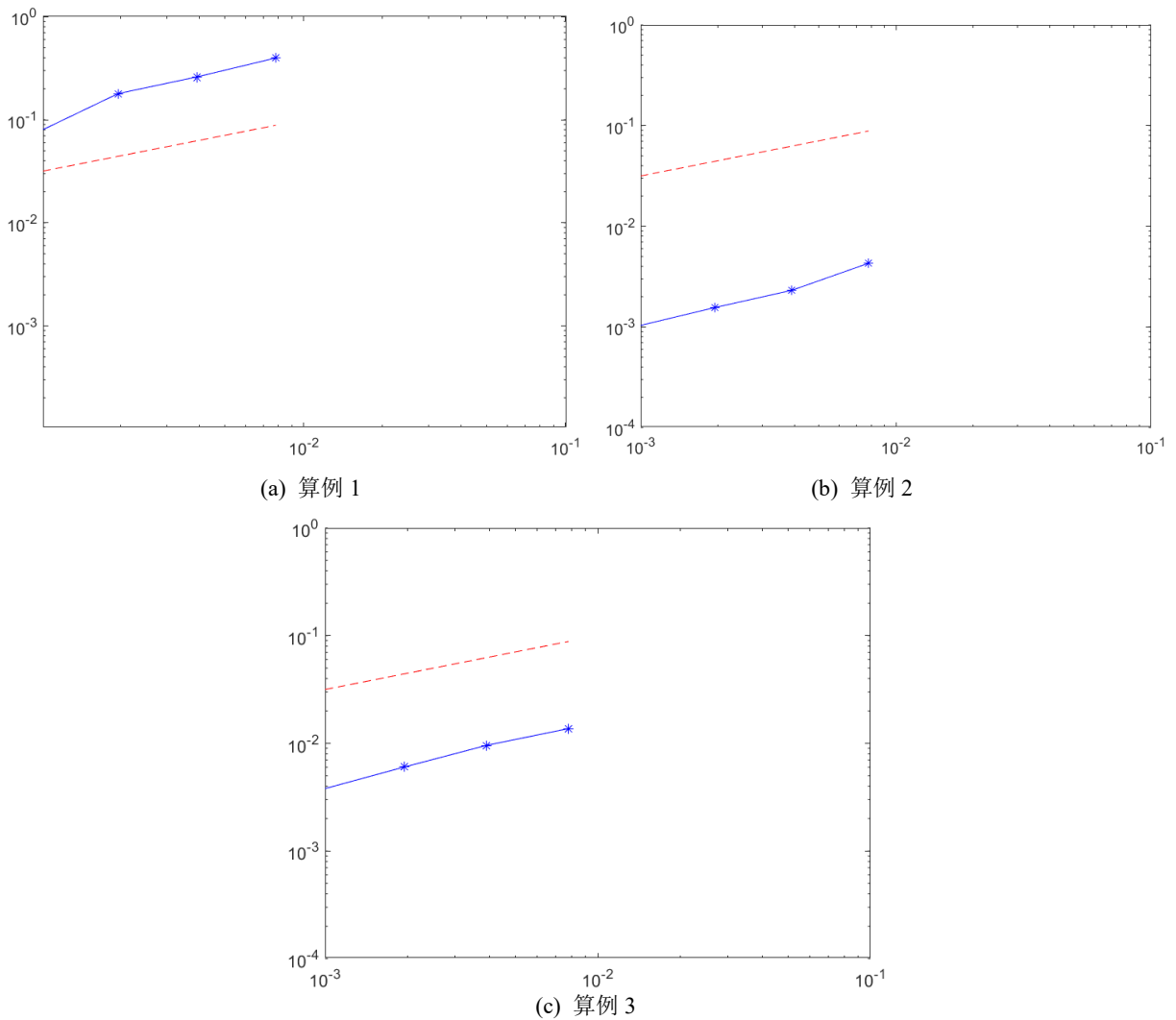


Figure 1. The convergence of the strong balance method
图 1. 强平衡方法的收敛性

其次验证平衡方法的均方稳定性, 这里取 $a_1 = -2; a_2 = 1; b_1 = 0.1; b_2 = 1$,

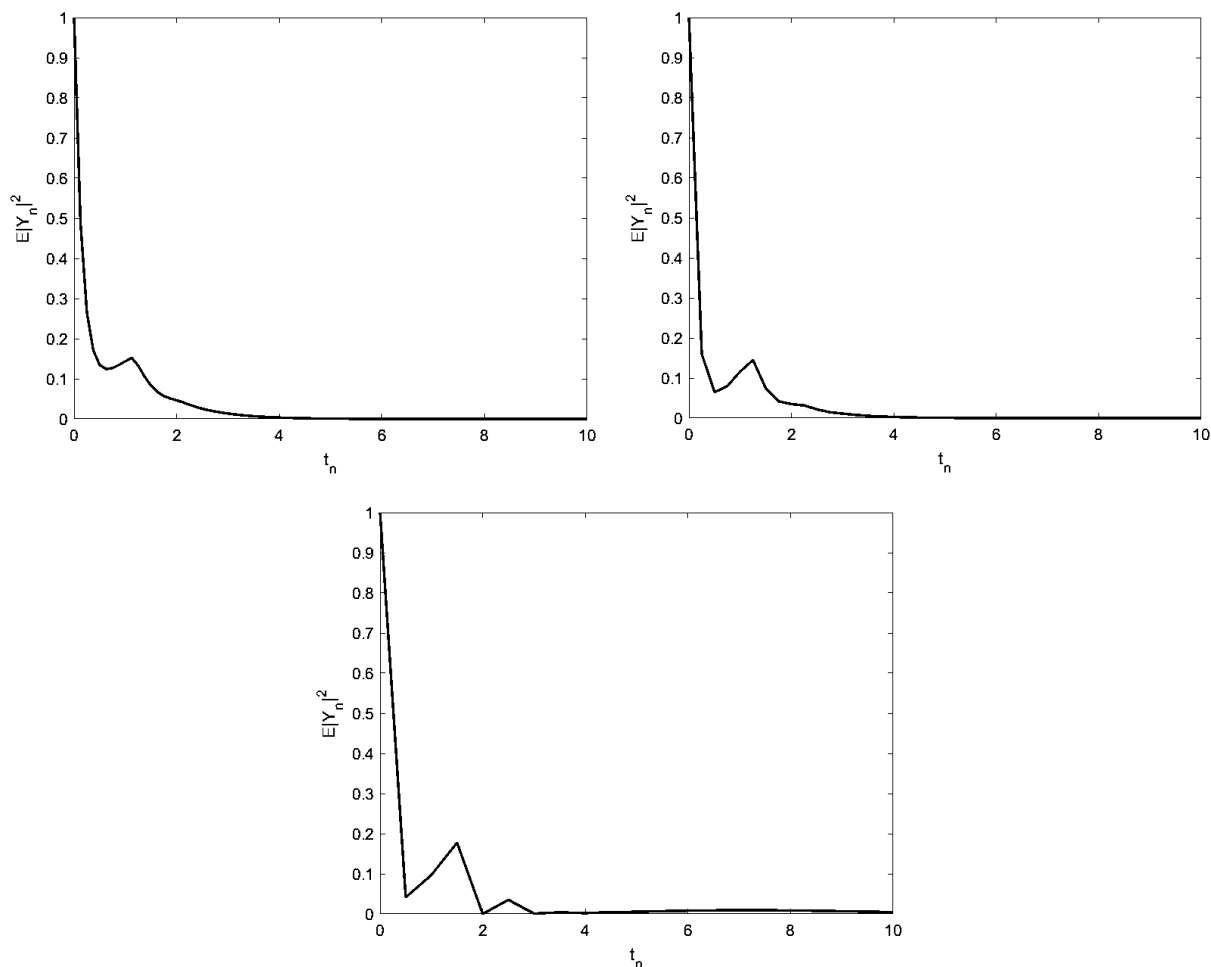


Figure 2. Mean-square stability of balanced implicit methods
图 2. 平衡方法的均方稳定性

如图 2, 当步长 $h = 0.125, 0.25, 0.5$ 时, 所有的曲线趋势都趋于零, 因此平衡方法是均方稳定的。

5. 结论

本文针对满足全局 Lipschitz 条件的分段连续型随机微分方程, 研究了平衡方法的均方收敛性, 在此基础上, 文章还构建了线性标量形式的随机微分方程, 得到了强平衡方法有均方稳定性, 弱平衡方法也具有均方稳定性。最终通过一系列数值实验分析证明了理论结果的可行合理性。此外, 若本文放宽对满足全局 Lipschitz 条件的要求, 满足关于其中一个变量的单边 Lipschitz 条件也是一个研究方向, 讨论精确解是否有界, 并研究其收敛性和稳定性。

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